University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, DECEMBER, 2010**
First Year - CHE, CIV, IND, LME, MEC, MMS
MAT186H1F - CALCULUS I
Exam Type: A

General Comments:

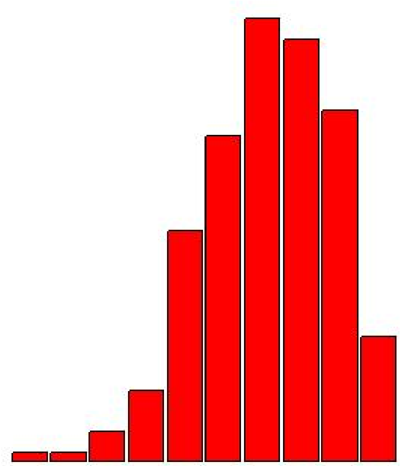
1. In 1(c), some students were confused between 'distance travelled' and 'displacement,' to use the terminology of the book. See page 377.
2. In 1(d) many students turned an essentially short question into an incredibly messy, convoluted–and invariably incorrect–calculation.
3. In 2(a) many students forgot to use the chain rule; in 2(b) many students tried incorrectly to reduce the problem to $\sin^{-1} u$.
4. Questions 3, 4, 5, 6 and 7 covered basic applications of the derivative. These questions should all have been aced! Probably they could have been done in high school.
5. The fact that in Question 8(a)

$$\int_1^2 2\pi x\,(x^2-1)\,dx = \int_1^2 2\pi x\,\Big(3-(x^2-1)\Big)\,dx$$

is just a coincidence! So some students got the correct answer for the wrong reason.

Breakdown of Results: 482 students wrote the exam. The marks ranged from 2% to 98%, and the average was 65.9%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	6.2%
A	23.6%	80-89%	17.4%
B	21.0%	70-79%	21.0%
C	22.0%	60-69%	22.0%
D	16.2%	50-59%	16.2%
F	17.2%	40-49%	11.4%
		30-39%	3.5%
		20-29%	1.5%
		10-19%	0.4%
		0-9%	0.4%



1. Find the following:

(a) [6 marks] $\displaystyle\int \left(\frac{1}{1+x^2} + 3\sin x + \frac{1}{x}\right) dx$

Solution:

$$\begin{aligned}
\int \left(\frac{1}{1+x^2} + 3\sin x + \frac{1}{x}\right) dx &= \int \frac{dx}{1+x^2} + 3\int \sin x \, dx + \int \frac{dx}{x} \\
&= \tan^{-1} x - 3\cos x + \ln|x| + C
\end{aligned}$$

(b) [6 marks] the average value of $f(x) = e^{-2x}$ on the interval $[0, 4]$.

Solution: use the formula for average.

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-0}\int_0^4 e^{-2x}dx = \frac{1}{4}\left[-\frac{e^{-2x}}{2}\right]_0^4 = -\frac{1}{8}(e^{-8}-1) = \frac{1}{8}\left(1-\frac{1}{e^8}\right)$$

(c) [6 marks] the distance travelled by a particle for $0 \le t \le 4$ if its velocity at time t is given by $v = \sqrt{t} - 1$.

Solution: observe that $v \le 0$ if $0 \le t \le 1$.

$$\int_0^4 |v|\,dt = \int_0^1 (1-\sqrt{t})\,dt + \int_1^4 (\sqrt{t}-1)\,dt = \left[t - \frac{2t^{3/2}}{3}\right]_0^1 + \left[\frac{2t^{3/2}}{3} - t\right]_1^4 = \frac{1}{3} + \frac{5}{3} = 2$$

(d) [6 marks] $\displaystyle\lim_{x\to 1} (4x-3)^{\tan(\pi x/2)}$

Solution: the limit is in the 1^∞ form. Let the limit be L.

$$\begin{aligned}
\ln L &= \lim_{x\to 1} \tan(\pi x/2)\ln(4x-3) \\
&= \lim_{x\to 1} \frac{\ln(4x-3)}{\cot(\pi x/2)} \\
\text{(L'H)} \quad &= \lim_{x\to 1} \frac{\frac{4}{4x-3}}{-\csc^2(\pi x/2)\,\pi/2} \\
&= -\frac{8}{\pi} \\
\Rightarrow L &= e^{-8/\pi}
\end{aligned}$$

2. Find the following:

(a) [5 marks] $F'(2)$ if $F(x) = \int_2^{x^2} e^{\sqrt{t}}\,dt$.

Soluton: By the Fundamental Theorem of Calculus, and the chain rule,

$$F'(x) = 2x\,e^{\sqrt{x^2}} = 2x\,e^{|x|}.$$

So

$$F'(2) = 4\,e^2.$$

(b) [5 marks] $\displaystyle\int_0^1 \frac{x}{\sqrt{4-3x^2}}\,dx$.

Solution: let $u = 4 - 3x^2$. Then $du = -6\,dx$ and

$$\begin{aligned}
\int_0^1 \frac{x}{\sqrt{4-3x^2}}\,dx &= -\frac{1}{6}\int_4^1 \frac{du}{\sqrt{u}} \\
&= \frac{1}{6}\int_1^4 \frac{du}{\sqrt{u}} \\
&= \frac{1}{6}\left[2\sqrt{u}\right]_1^4 \\
&= \frac{1}{6}(4-2) \\
&= \frac{1}{3}
\end{aligned}$$

3. [10 marks] Let $f(x) = 4x^{1/3} - x^{4/3}$, for which you may assume

$$f''(x) = -\frac{4}{9}\frac{(2+x)}{x^{5/3}}.$$

Find the open intervals on which f is concave up and the open intervals on which f is concave down. Find all the inflection points of f, if any.

Solution: $f''(x) = -\frac{4}{9}\frac{(2+x)}{x^{5/3}}$. Then

$$f''(x) > 0 \Leftrightarrow \frac{2+x}{x^{5/3}} < 0 \Leftrightarrow -2 < x < 0$$

and

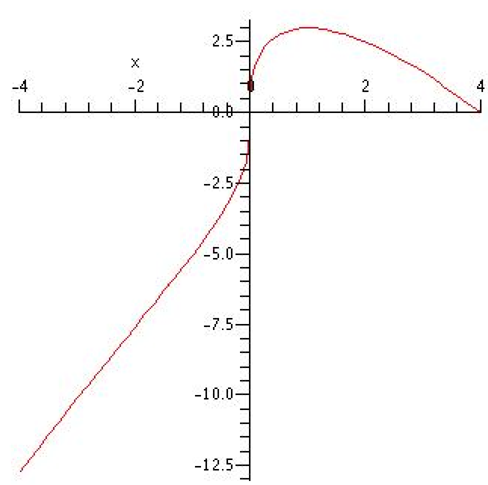
$$f''(x) < 0 \Leftrightarrow \frac{2+x}{x^{5/3}} > 0 \Leftrightarrow x < -2 \text{ or } x > 0.$$

So f is concave up on the interval $(-2, 0)$;

and f is concave down on the intervals $(-\infty, -2)$ and $(0, \infty)$.

Consequently there are inflection points at $(0, 0)$ and $(-2, -6 \cdot 2^{1/3})$.

For interest, the graph of f is shown below:



4. [12 marks] Consider $f(x) = \sin^2 x - \cos x$ on the open interval $(-\pi, 3\pi)$. Find the open intervals on which f is increasing and the open intervals on which f is decreasing. Find all the critical points of f in the interval $(-\pi, 3\pi)$ and determine if they are relative maximum or relative minimum points.

Solution: $f'(x) = 2\sin x \cos x + \sin x = \sin x\,(2\cos x + 1)$. Then

$$f'(x) = 0 \Leftrightarrow \sin x = 0 \text{ or } \cos x = -\frac{1}{2} \Rightarrow x = 0, \pi, 2\pi \text{ or } x = -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$$

for x in $(-\pi, 3\pi)$. The sign of the derivative switches at each critical point; that is

$$f'(x) > 0 \Leftrightarrow -\pi < x < -\frac{2\pi}{3} \text{ or } 0 < x < \frac{2\pi}{3} \text{ or } \pi < x < \frac{4\pi}{3} \text{ or } 2\pi < x < \frac{8\pi}{3}.$$

So f is increasing on the intervals

$$\left(-\pi, -\frac{2\pi}{3}\right), \left(0, \frac{2\pi}{3}\right), \left(\pi, \frac{4\pi}{3}\right) \text{ and } \left(2\pi, \frac{8\pi}{3}\right);$$

and f is decreasing on the intervals

$$\left(-\frac{2\pi}{3}, 0\right), \left(\frac{2\pi}{3}, \pi\right), \left(\frac{4\pi}{3}, 2\pi\right) \text{ and } \left(\frac{8\pi}{3}, 3\pi\right).$$

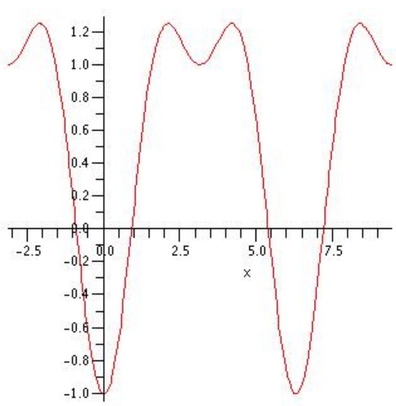
The four maximum points are

$$\left(-\frac{2\pi}{3}, \frac{5}{4}\right), \left(\frac{2\pi}{3}, \frac{5}{4}\right), \left(\frac{4\pi}{3}, \frac{5}{4}\right) \text{ and } \left(\frac{8\pi}{3}, \frac{5}{4}\right)$$

and the three minimum points are

$$(0, -1), (\pi, 1) \text{ and } (2\pi, -1)$$

For interest, the graph of f is shown below:



5. [10 marks] Find all the asymptotes to the graph of $f(x) = \dfrac{x^2}{x-1}$, and sketch its graph. You may assume that

$$f'(x) = \frac{x(x-2)}{(x-1)^2} \text{ and } f''(x) = \frac{2}{(x-1)^3}.$$

Label the asymptotes, the critical points, and the inflection points, if any.

Solution:

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$$

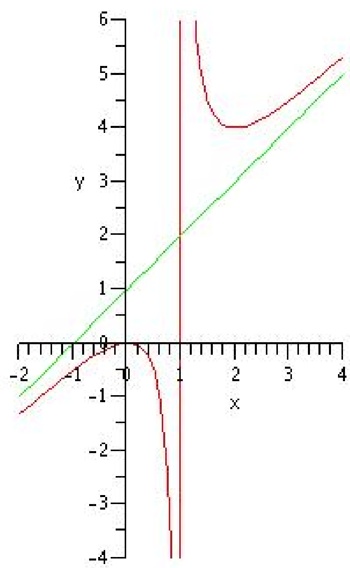
so $y = x + 1$ is a slant asymptote to the graph of f; and $x = 1$ is a vertical asymptote since

$$\lim_{x\to 1^-} \frac{x^2}{x-1} = -\infty \text{ and } \lim_{x\to 1^+} \frac{x^2}{x-1} = \infty.$$

$$f'(x) = 0 \Rightarrow x = 0 \text{ or } 2;$$

$$f''(0) = -2 < 0,\ f''(2) = 2 > 0;$$

By the second derivative test, $(0, 0)$ is a maximum point, and $(2, 4)$ is a minimum point. The graph of f is concave up if $x > 1$, concave down if $x < 1$. There are no infleciton points.



6. [12 marks] An open box is to be made from a 1 m by 3 m rectangular piece of sheet metal by cutting out squares of equal size from the four corners and bending up the sides. Find the maximum volume that the box can have.

Solution: let the dimensions of each corner square be $a \times a$. Then the dimensions of the box are $(3-2a) \times (1-2a) \times a$ and the volume of the box is

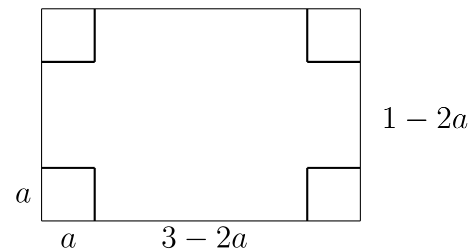
$$V = (3-2a)(1-2a)a = 4a^3 - 8a^2 + 3a.$$

Since lengths must be positive, $a < 1/2$. The problem is: find the maximum value of V on the open interval $(0, 1/2)$. Compute:

$$\frac{dV}{da} = 12a^2 - 16a + 3$$

and

$$\frac{d^2V}{da^2} = 24a - 16.$$



Then

$$\frac{dV}{da} = 0 \Rightarrow a = \frac{16 \pm \sqrt{256-144}}{24} = \frac{4 \pm \sqrt{7}}{6}.$$

Of these two, only

$$a = \frac{4-\sqrt{7}}{6} \simeq 0.2257 \text{ is in } \left(0, \frac{1}{2}\right).$$

At this point

$$\frac{d^2V}{da^2} = 4(4-\sqrt{7}) - 16 = -4\sqrt{7} < 0;$$

Indeed,

$$\frac{d^2V}{da^2} < 0 \text{ for all } a \text{ in } \left(0, \frac{1}{2}\right)$$

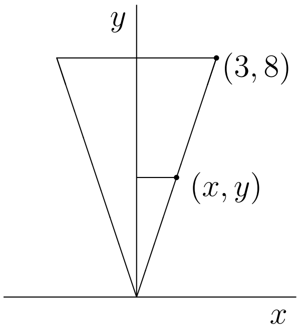
so V does attain its maximum value at $a = (4-\sqrt{7})/6$. The maximum value is

$$4\left(\frac{4-\sqrt{7}}{6}\right)^3 - 8\left(\frac{4-\sqrt{7}}{6}\right)^2 + 3\left(\frac{4-\sqrt{7}}{6}\right) = \frac{1}{27}\left(7\sqrt{7}-10\right) \simeq 0.316$$

cubic metres.

7. [10 marks] A conical water tank with vertex down has a radius of 3 m at the top and is 8 m high. If water flows into the tank at a rate of 1 m^3/min how fast is the depth of the water increasing when the water is 5 m deep?

Solution: Let the radius of the cone at height y be x and consider the side view of the cone, below. By similar triangles,



$$\frac{x}{y} = \frac{3}{8} \Leftrightarrow x = \frac{3y}{8}.$$

In terms of the depth of the water, y, the volume of the water is

$$V = \frac{\pi}{3}\left(\frac{3y}{8}\right)^2 y = \frac{3\pi y^3}{64}.$$

By the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = \frac{9\pi\, y^2}{64}\,\frac{dy}{dt}.$$

Substitute

$$\frac{dV}{dt} = 1,\ \ y = 5$$

and solve for $\frac{dy}{dt}$:

$$\frac{dy}{dt} = \frac{64}{9\pi}\frac{1}{5^2} = \frac{64}{225\pi}.$$

Thus when the depth of the water in the tank is 5 m, the depth of the water is increasing at a rate of

$$\frac{64}{225\pi} \simeq 0.09$$

m per min.

8. Let $f(x) = x^2 - 1$ for $1 \le x \le 2$. Find the following:

(a) [6 marks] the volume of the solid generated by revolving the region between the curves $y = 3$ and $y = f(x)$ for $1 \le x \le 2$ about the y-axis.

Solution: use the method of shells.

$$\begin{aligned} V = \int_1^2 2\pi\, x\, \big(3 - (x^2 - 1)\big)\, dx &= \int_1^2 2\pi\, x\, (4 - x^2)\, dx \\ &= 2\,\pi \int_1^2 (4x - x^3)\, dx \\ &= 2\,\pi \left[2x^2 - \frac{x^4}{4}\right]_1^2 \\ &= \frac{9\pi}{2} \end{aligned}$$

Alternatively: discs with respect to y. $V = \displaystyle\int_0^3 \pi \left(\left(\sqrt{y+1}\right)^2 - 1^2 \right) dy = \frac{9\pi}{2}$,

(b) [6 marks] the area of the surface generated by revolving the curve $y = f(x)$ about the y-axis for $1 \le x \le 2$.

Solution: integrating with respect to x.

$$y = x^2 - 1 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + 4x^2}\, dx.$$

$$\begin{aligned} SA &= \int_1^2 2\,\pi\, x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \pi \int_1^2 2x\, \sqrt{1 + 4x^2}\, dx \\ (\text{Let } u = 1 + 4x^2) \quad &= \frac{\pi}{4} \int_5^{17} \sqrt{u}\, du \\ &= \frac{\pi}{4} \left[\frac{2}{3} u^{3/2}\right]_5^{17} \\ &= \frac{\pi}{6} \left(17^{3/2} - 5^{3/2}\right) \end{aligned}$$

Alternatively: integrate with respect to y, with $x = \sqrt{y+1}$. Then

$$SA = \int_0^3 2\,\pi\, x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^3 \pi \sqrt{4y + 5}\, dy = \frac{\pi}{6} \left(17^{3/2} - 5^{3/2}\right).$$