University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, DECEMBER, 2009**
First Year - CHE, CIV, IND, LME, MEC, MMS
MAT186H1F - CALCULUS I
Exam Type: A

General Comments:

1. This exam consisted almost entirely of routine, standard problems. The only exception was the very last question. Two of the asymptotes are straightforward to find, but the calculation of $\lim_{x\to\infty} f(x)$ is quite messy. You have to use L'Hopital's Rule at least twice.

2. In Question 9, the most common error was to miss the inflection point at $x = 0$.

3. Question 11 was done very badly! This is surprising on two accounts: it was based on a WileyPlus homework question, and the question itself is high school level.

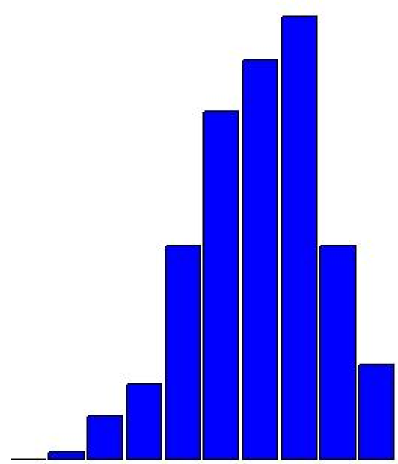
4. Both parts of Question 12 could be done with either method, discs or shells, but both questions are most appropriately done by integrating with respect to x.

5. In Question 13(a), a lot of students had no idea what the correct formula for surface area is.

6. It was absolutely astounding how many students thought the square root function (to mention only one) is linear. So in Question 11 many students simplified the equation of the ellipse incorrectly to $3x + 4y = 12$.

Breakdown of Results: 466 students wrote this exam. The marks ranged from 18% to 98%, and the average was 64.2%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	5.2%
A	17.0%	80-89%	11.8%
B	24.0%	70-79%	24.0%
C	21.7%	60-69%	21.7%
D	18.9%	50-59%	18.9%
F	18.4%	40-49%	11.5%
		30-39%	4.1 %
		20-29%	2.4%
		10-19%	0.4%
		0-9%	0.0%



1. Let $f(x) = x^2$. What is the average value of f on the interval $[-1, 1]$?

(a) $\frac{1}{4}$

(b) $\frac{1}{3}$

(c) $\frac{1}{2}$

(d) $\frac{2}{3}$

Solution:

$$f_{\text{ave}} = \frac{1}{1-(-1)} \int_{-1}^{1} f(x)\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3}$$

The answer is (b).

2. If Newton's method is used to approximate a solution to the equation $x^3 - x - 7 = 0$ and x_0 is chosen to be 2, then the value of x_2 is

(a) 2

(b) $2.09090909\ldots$

(c) $2.08675431\ldots$

(d) $2.45454545\ldots$

Solution:

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 7}{3x_n^2 - 1}$$

$$x_0 = 2 \Rightarrow x_1 = 2.09090909... \Rightarrow x_2 = 2.08675431\ldots$$

The answer is (c).

3. Let $f(x) = \dfrac{x^2+5}{x+2}$. Which of the following statements is a true statement about the function f on the interval $[-8, -2)$?

(a) The absolute maximum value of f on the interval $[-8, -2)$ is -10 and the absolute minimum value of f on the interval $[-8, -2)$ is -11.5

(b) The absolute maximum value of f on the interval $[-8, -2)$ is -5 and the absolute minimum value of f on the interval $[-8, -2)$ is -8.

(c) The absolute maximum value of f on the interval $[-8, -2)$ is -5; there is no absolute minimum value of f on the interval $[-8, -2)$.

(d) The absolute maximum value of f on the interval $[-8, -2)$ is -10; there is no absolute minimum value of f on the interval $[-8, -2)$.

Solution:

$$f(x) = \frac{x^2+5}{x+2} = x - 2 + \frac{9}{x+2};\ f'(x) = 1 - \frac{9}{(x+2)^2} = 0 \Rightarrow x = -5 \text{ or } 1$$

$$f(-8) = -11.5; f(-5) = -10;\ \text{and} \lim_{x\to -2^-} f(x) = -\infty$$

The answer is (d).

4. The area of the region between the two curves with equations $y = 4x^2$ and $y = x^3$ for $0 \le x \le 6$ is given by

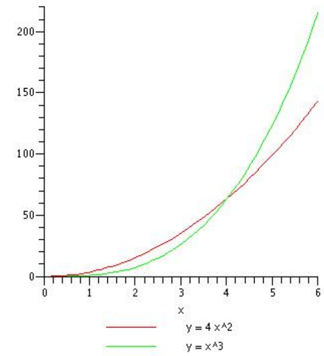
(a) $\displaystyle\int_0^6 (x^3 - 4x^2)\,dx$

(b) $\displaystyle\int_0^6 (4x^2 - x^3)\,dx$

(c) $\displaystyle\int_0^4 (x^3 - 4x^2)\,dx + \int_4^6 (4x^2 - x^3)\,dx$

(d) $\displaystyle\int_0^4 (4x^2 - x^3)\,dx + \int_4^6 (x^3 - 4x^2)\,dx$

Solution: $4x^2 = x^3 \Leftrightarrow x = 0$ or $x = 4$



The answer is (d).

5. The value of $\displaystyle\int_1^9 \frac{dx}{\sqrt{x}\,(x+1)^2}$ is given by

(a) $\displaystyle\int_1^3 \frac{2\,du}{(u^2+1)^2}$

(b) $\displaystyle\int_1^9 \frac{2\,du}{(u^2+1)^2}$

(c) $\displaystyle\int_1^3 \frac{du}{2(u^2+1)^2}$

(d) $\displaystyle\int_1^9 \frac{du}{2(u^2+1)^2}$

Solution: Let $u = \sqrt{x}$; then $du = \dfrac{dx}{2\sqrt{x}}$. So

$$\int_1^9 \frac{dx}{\sqrt{x}\,(x+1)^2} = \int_1^3 \frac{2\,du}{(u^2+1)^2}$$

since $u = 1$ when $x = 1$, and $u = 3$ when $x = 9$.

The answer is (a).

6. Suppose the position of a particle on the x-axis at time t is given by $x = -4t^2 + 8t - 5$. For which values of t is the particle slowing down?

(a) $t > 1$

(b) $t < 1$

(c) $t > 2$

(d) $t < 2$

Solution:

$$v = \frac{dx}{dt} = -8t + 8; \;\; a = \frac{dv}{dt} = -8 < 0.$$

The particle is slowing down if

$$av < 0 \Leftrightarrow v > 0 \Leftrightarrow 8 > 8t \Leftrightarrow 1 > t$$

The answer is (b).

7(a) [5 marks] Find $F'(2)$ if $F(x) = \displaystyle\int_4^{x^2} \sqrt{t^2+9}\,dt$.

Soluton: By the Fundamental Theorem of Calculus, and the chain rule,

$$F'(x) = 2x\sqrt{(x^2)^2+9} = 2x\sqrt{x^4+9}.$$

So

$$F'(2) = 4\sqrt{16+9} = 20.$$

7(b) [5 marks] Find $\displaystyle\int_1^3 \left(\frac{x^2+1}{x}\right) dx$.

Solution:

$$\begin{aligned}
\int_1^3 \left(\frac{x^2+1}{x}\right) dx &= \int_1^3 \left(x + \frac{1}{x}\right) dx \\
&= \left[\frac{x^2}{2} + \ln|x|\right]_1^3 \\
&= \frac{9}{2} + \ln 3 - \frac{1}{2} - \ln 1 \\
&= 4 + \ln 3
\end{aligned}$$

8. Let $f(x) = xe^{-x^2}$ for which $f'(x) = e^{-x^2} - 2x^2e^{-x^2}$. Find the open intervals on which f is increasing and the open intervals on which f is decreasing. Find the critical points of f and determine if they are relative maximum or relative minimum points.

Solution: $f'(x) = e^{-x^2}(1 - 2x^2)$. Then

$$f'(x) > 0 \Leftrightarrow 1 - 2x^2 > 0 \Leftrightarrow x^2 < \frac{1}{2} \Leftrightarrow -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

and

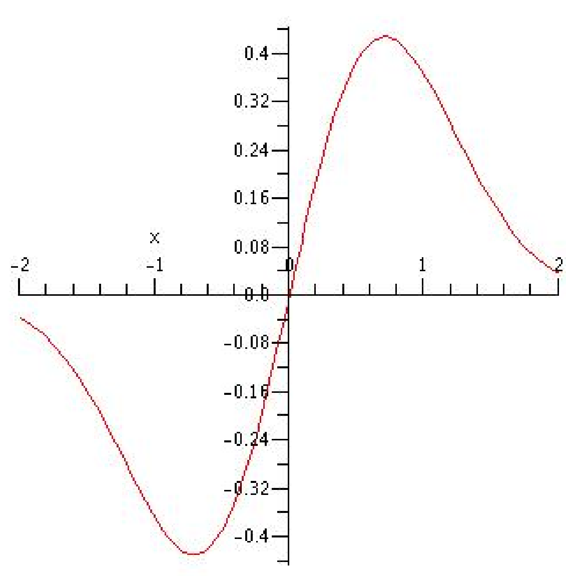
$$f'(x) < 0 \Leftrightarrow 1 - 2x^2 < 0 \Leftrightarrow x^2 > \frac{1}{2} \Leftrightarrow x < -\frac{1}{\sqrt{2}} \text{ or } x > \frac{1}{\sqrt{2}}.$$

So f is increasing on the interval $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$;

and f is decreasing on the intervals $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$.

By the first derivative test, f has a relative minimum point at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$ and a relative maximum point at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$.

For interest, the graph of f is shown below:



9. Let $f(x) = 6x^{1/3} + 3x^{4/3}$ for which

$$f''(x) = -\frac{4}{3}x^{-5/3} + \frac{4}{3}x^{-2/3}.$$

Find the open intervals on which f is concave up and the open intervals on which f is concave down. Find all the inflection points of f.

Solution: $f''(x) = -\frac{4}{3}\frac{(1-x)}{x^{5/3}}$. Then

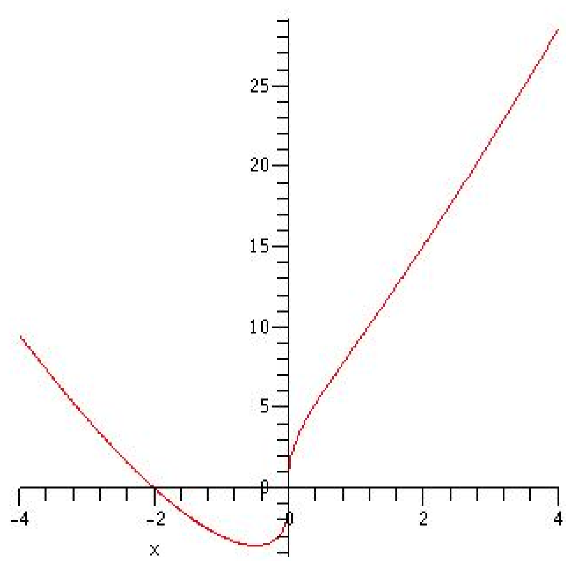
$$f''(x) > 0 \Leftrightarrow \frac{1-x}{x^{5/3}} < 0 \Leftrightarrow x < 0 \text{ or } x > 1$$

and

$$f''(x) < 0 \Leftrightarrow \frac{1-x}{x^{5/3}} > 0 \Leftrightarrow 0 < x < 1.$$

So f is concave up on the intervals $(-\infty, 0)$ and $(1, \infty)$; and f is concave down on the interval $(0, 1)$. The two inflection points are $(0, 0)$ and $(1, 9)$.

For interest, the graph of f is shown below:



10. Suppose the velocity of a particle at time t is given by $v = 6t - 2t^2$, for $0 \leq t \leq 4$. Find the following:

(a) [4 marks] the average acceleration of the particle for $0 \leq t \leq 4$.

Solution:

$$\begin{aligned} a_{\text{ave}} &= \frac{1}{4}\int_0^4 a\,dt \\ &= \frac{1}{4}\left[v\right]_0^4 \\ &= \frac{1}{4}(24 - 32 - 0) \\ &= -2 \end{aligned}$$

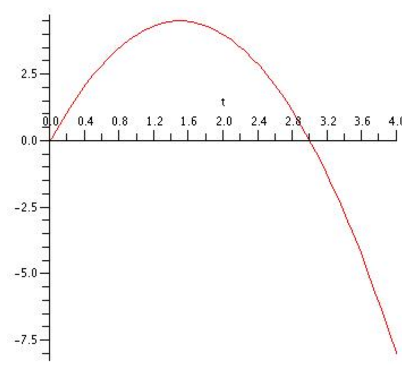
(b) [6 marks] the average speed of the particle for $0 \leq t \leq 4$.

Solution: $v = 2t(3-t)$. So

$$v \geq 0 \text{ for } 0 \leq t \leq 3$$

and

$$v \leq 0 \text{ for } 3 \leq t \leq 4.$$



The speed of the particle at time t is $|v|$ and the average speed is given by

$$|v|_{\text{avg}} = \frac{1}{4}\int_0^4 |v|\,dt.$$

Calculate:

$$\begin{aligned} \frac{1}{4}\int_0^4 |v|\,dt &= \frac{1}{4}\left(\int_0^3 v\,dt + \int_3^4 (-v)\,dt\right) \\ &= \frac{1}{4}\left[3t^2 - \frac{2}{3}t^3\right]_0^3 + \frac{1}{4}\left[\frac{2}{3}t^3 - 3t^2\right]_3^4 \\ &= \frac{9}{4} + \frac{11}{12} \\ &= \frac{19}{6} \end{aligned}$$

11. Find the dimensions of the rectangle (centered at the origin) with maximum area that can be inscribed inside the ellipse with equation $\dfrac{x^2}{4^2}+\dfrac{y^2}{3^2}=1$.

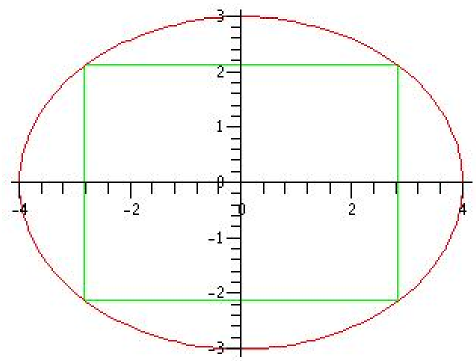
Solution: let the vertex of the rectangle in the first quadrant be (a,b), $a\ge 0, b\ge 0$.

Then the area of the rectangle is

$$A = 4ab$$

and the problem is to find the maximum value of A such that

$$\frac{a^2}{16}+\frac{b^2}{9}=1.$$



Solve for b in terms of a :

$$b^2 = 9\left(1-\frac{a^2}{16}\right) \Rightarrow b = 3\sqrt{1-\frac{a^2}{16}}.$$

Thus as a function of a,

$$A = 12a\sqrt{1-\frac{a^2}{16}} = 3a\sqrt{16-a^2}.$$

The problem is to maximize A on the interval $[0,4]$. At both endpoints $a=0, a=4$, $A=0$. So the maximum value of A must be at a critical point in the interval $(0,4)$.

$$\frac{dA}{da} = 3\sqrt{16-a^2} - \frac{3a^2}{\sqrt{16-a^2}} = \frac{48-6a^2}{\sqrt{16-a^2}} = 0 \Rightarrow a^2 = 8 \Rightarrow a = \sqrt{8} \text{ or } 2\sqrt{2}.$$

The dimensions of the rectangle with maximum area are

$$2a \times 2b = 2\sqrt{8}\times 2\left(\frac{3}{\sqrt{2}}\right) = 2\sqrt{8}\times 3\sqrt{2}$$

(and the maximum area is 24.)

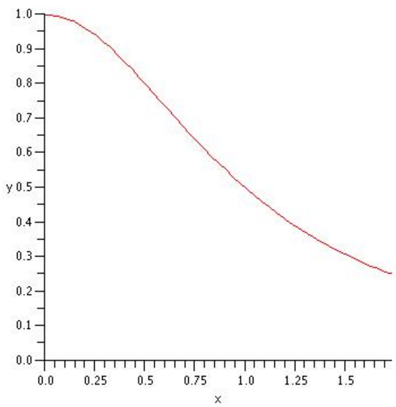
Alternate Simplifying Calculations:

$$A^2 = 9a^2(16-a^2) = 144a^2 - 9a^4 \Rightarrow 2A\frac{dA}{da} = 288a - 36a^3 = 0 \Leftrightarrow a = 0 \text{ or } a^2 = 8;$$

$$\text{or } x = 4\cos t, y = 3\sin t \Rightarrow A = 4\cdot 4\cos t\cdot 3\sin t = 24\sin(2t) = 24 \Leftrightarrow t = \pi/4.$$

12(a) [6 marks] Find the volume of the solid of revolution obtained by revolving the region enclosed by $x = 0, x = \sqrt{3}, y = 0$ and $y = \dfrac{1}{1+x^2}$ about the y-axis.

Solution: use the method of cylindrical shells.

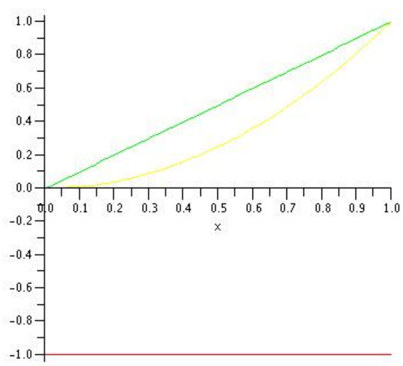


$$
\begin{aligned}
V &= \int_0^{\sqrt{3}} 2\pi x\, y\, dx \\
&= \pi \int_0^{\sqrt{3}} \frac{2x}{x^2+1}\, dx \\
&= \pi \left[\ln(x^2+1)\right]_0^{\sqrt{3}} \\
&= \pi\,(\ln 4 - \ln 1) \\
&= \pi\, \ln 4
\end{aligned}
$$

Alternatively with discs: $V = \displaystyle\int_0^{1/4} \pi(\sqrt{3})^2\, dy + \int_{1/4}^{1} \pi\left(\frac{1}{y} - 1\right)\, dy$

12(b) [6 marks] Find the volume of the solid of revolution obtained by revolving the region enclosed by the curves $y = x, y = x^2, x = 0$ and $x = 1$ about the line with equation $y = -1$.

Solution: use the method of discs (or washers.)



$$
\begin{aligned}
V &= \int_0^1 \left(\pi(x+1)^2 - \pi(x^2+1)^2\right)\, dx \\
&= \pi \int_0^1 \left(x^2 + 2x + 1 - x^4 - 2x^2 - 1\right)\, dx \\
&= \pi \int_0^1 \left(2x - x^2 - x^4\right)\, dx \\
&= \pi \left[x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 \\
&= \pi\left(1 - \frac{1}{3} - \frac{1}{5}\right) \\
&= \frac{7\pi}{15}
\end{aligned}
$$

Alternatively with shells: $V = \displaystyle\int_0^1 2\pi(y+1)(\sqrt{y} - y)\, dy$

13(a) [6 marks] Find the area of the surface generated by revolving the curve with equation $y = 2\sqrt{1-x}$, for $-1 \le x \le 0$, about the x-axis.

Solution:

$$\begin{aligned} SA = \int_{-1}^{0} 2\pi\, y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx &= \int_{-1}^{0} 2\pi\, y \sqrt{1 + \left(\frac{-1}{\sqrt{1-x}}\right)^2}\, dx \\ &= 2\,\pi \int_{-1}^{0} 2\sqrt{1-x}\,\sqrt{\frac{2-x}{1-x}}\, dx \\ &= 4\,\pi \int_{-1}^{0} \sqrt{2-x}\, dx \\ &= 4\,\pi \left[-\frac{2}{3}\,(2-x)^{3/2}\right]_{-1}^{0} = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right) \end{aligned}$$

Alternatively with respect to y : $SA = \displaystyle\int_{2}^{2\sqrt{2}} 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$, with $x = 1 - \dfrac{y^2}{4}$.

13(b) [6 marks] Find all the asymptotes to the graph of $f(x) = \left(\dfrac{1}{x^2}\right)^{\pi/2 - \tan^{-1} x}$.

Solution: $y = 0$ and $x = 0$ are obvious asymptotes, since

$$\lim_{x\to-\infty} f(x) \text{ is of the form } 0^{\pi}, \text{ and } \lim_{x\to 0} f(x) \text{ is of the form } \infty^{\pi/2}.$$

$L = \displaystyle\lim_{x\to\infty} f(x)$ is in the 0^0 form and turns out to be 1, but it is quite messy to calculate:

$$\begin{aligned} \ln L = \lim_{x\to\infty}\left(\frac{\pi}{2} - \tan^{-1} x\right)(-\ln x^2) &= -\lim_{x\to\infty}(\pi - 2\tan^{-1} x)\ln x = -\lim_{x\to\infty}\frac{\ln x}{(\pi - 2\tan^{-1} x)^{-1}} \\ \text{(using L'Hopital's Rule)} &= -\lim_{x\to\infty}\frac{1/x}{-(\pi - 2\tan^{-1} x)^{-2}\,(-2/(1+x^2))} \\ &= -\frac{1}{2}\lim_{x\to\infty}\frac{(\pi - 2\tan^{-1} x)^2}{x/(1+x^2)} \\ \text{(using L'Hopital's Rule)} &= 2\lim_{x\to\infty}\frac{(\pi - 2\tan^{-1} x)/(1+x^2)}{(1-x^2)/(1+x^2)^2} \\ &= 2\lim_{x\to\infty}(\pi - 2\tan^{-1} x)\ \lim_{x\to\infty}\frac{1+x^2}{1-x^2} \\ &= 2\cdot 0\cdot(-1) = 0 \end{aligned}$$

So $\ln L = 0 \Leftrightarrow L = e^0 = 1$, which means $y = 1$ is also an asymptote to the graph of f.