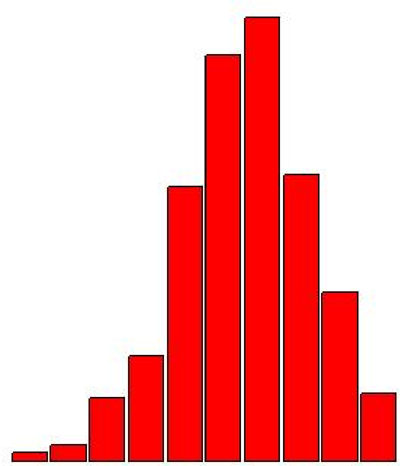
University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, DECEMBER, 2008**
First Year - CHE, CIV, IND, LME, MEC, MMS
MAT186H1F - CALCULUS I
Exam Type: A

General Comments: with a term mark average of 78.0% going into the final exam, this exam was deliberately set to be on the challenging side. To this end 15% of the exam consisted of questions unlike any on previous exams of the last few years. Nevertheless it was questions right out of the book that caused the most problems of all!

1. Questions 1 and 2 were very badly done. Many students seemed confused about the concept of an asymptote.
2. The two limits in Question 7 were very badly done, although they are completely routine questions of the type found in Sections 4.8 or 4.9 of the textbook.
3. Question 8(b) was supposed to be a challenge, but many students did very well on this.
4. Question 10 was supposed to be a challenge, but it was actually right out of the book.
5. Question 12, which was done very badly, was right out of the homework, Section 3.3 of the book.
6. For some reason part (b) of Question 13 was very poorly done, whereas part (a) was done very well.

Breakdown of Results: 436 students wrote this exam. The marks ranged from 4% to 100%, and the average was 60.25%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	3.7%
A	12.8%	80-89%	9.1%
B	15.6%	70-79%	15.6%
C	24.1%	60-69%	24.1%
D	22.0 %	50-59%	22.0%
F	25.5%	40-49%	14.9%
		30-39%	5.7%
		20-29%	3.5%
		10-19%	0.9%
		0-9%	0.5%



1. Suppose the function $f(x)$ is continuous for all values of x. What is the greatest possible number of asymptotes to the graph of f?

(a) 1

(b) 2

(c) 3

(d) 4

Solution: there could be at most one horizontal or slant asymptote as $x \to \infty$; there could be at most one horizontal or slant asymptote as $x \to -\infty$. There can be no vertical asymptotes since $f(x)$ is continuous for all x.

The answer is (b).

2. How many vertical asymptotes are there to the graph of $f(x) = \dfrac{\sin(x^2-1)}{x^4-5x^2+4}$?

(a) 1

(b) 2

(c) 3

(d) 4

Solution:

$$f(x) = \frac{\sin(x^2-1)}{x^4-5x^2+4} = \frac{\sin(x^2-1)}{(x^2-1)(x^2-4)}$$

There are removable discontinuities at $x = \pm 1$; there are infinite limits as $x \to \pm 2^+$ and as $x \to \pm 2^-$. So only $x = \pm 2$ are vertical asymptotes.

The answer is (b).

3. The arc length of the curve $f(x) = \tan x$ for $0 \le x \le 1$ is given by

(a) $\displaystyle\int_0^1 \sqrt{1+\tan^4 x}\, dx$

(b) $\displaystyle\int_0^1 \sec x\, dx$

(c) $\displaystyle\int_0^1 \sqrt{1+\sec^2 x}\, dx$

(d) $\displaystyle\int_0^1 \sqrt{1+\sec^4 x}\, dx$

Solution:

$$\int_0^1 \sqrt{1+(f'(x))^2}\, dx = \int_0^1 \sqrt{1+(\sec^2 x)^2}\, dx$$

The answer is (d).

4. Which of the following is equal to $\sin(3A)$, for all values of A?

(a) $3\sin A - 4\sin^3 A$

(b) $3\sin A\cos A$

(c) $4\sin A - 3\sin^3 A$

(d) $3\sin A + 4\sin^3 A$

Solution: Try $A = \dfrac{\pi}{2}$; then $\sin(3A) = -1$. Check: choice (a) also gives -1 but choice (b) gives 0, choice (c) gives 1, and choice (d) gives 7.

The answer is (a).

5. The value of $\displaystyle\int_0^{\ln 2} e^x \ln(e^x+1)\,dx$ is given by

(a) $\displaystyle\int_0^{\ln 2} \ln u\,du$

(b) $\displaystyle\int_1^{2} \ln u\,du$

(c) $\displaystyle\int_2^{3} \ln u\,du$

(d) $\displaystyle\int_0^{\ln 2} \ln(u+1)\,du$

Solution: Let $u = e^x + 1$; then $du = e^x\,dx$. Then

$$\int_0^{\ln 2} e^x \ln(e^x+1)\,dx = \int_2^3 \ln u\,du$$

since $u = 2$ when $x = 0$, and $u = 2 + 1 = 3$ when $x = \ln 2$.

The answer is (c).

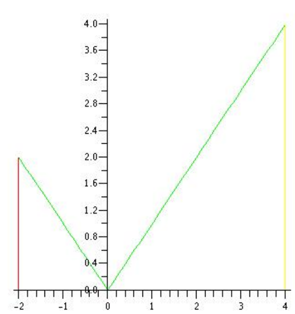
6. $\displaystyle\int_{-2}^{4} |x|\,dx =$

(a) 10

(b) 12

(c) 8

(d) 14



Solution: The combined area of the two triangles is $2 + 8 = 10$.

The answer is (a).

7. Find the following limits:

(a) $\lim_{x\to 0} \dfrac{2x - \sin^{-1}(2x)}{x^3}$.

The limit is in the $\frac{0}{0}$ form; use L'Hopital's rule:

$$\begin{aligned}
\lim_{x\to 0} \frac{2x - \sin^{-1}(2x)}{x^3} &= \lim_{x\to 0} \frac{\left(2 - \dfrac{2}{\sqrt{1-4x^2}}\right)}{3x^2}, \text{ still } \frac{0}{0} \text{ form} \\
&= \frac{2}{3} \lim_{x\to 0} \frac{\dfrac{1}{2}\dfrac{-8x}{(1-4x^2)^{3/2}}}{2x}, \text{ by L'Hopital's rule} \\
&= -\frac{4}{3} \lim_{x\to 0} \frac{1}{(1-4x^2)^{3/2}} \\
&= -\frac{4}{3}\frac{1}{(1-0)^{3/2}} \\
&= -\frac{4}{3}
\end{aligned}$$

(b) $\lim_{x\to\infty} \left(1 + \dfrac{\pi}{2} - \tan^{-1} x\right)^x$

Solution: Since

$$\lim_{x\to\infty} \left(\frac{\pi}{2} - \tan^{-1} x\right) = 0,$$

the given limit is in the 1^∞ form; let the limit be L, and evaluate the limit of $\ln L$ by using L'Hopital's Rule:

$$\begin{aligned}
\ln L &= \lim_{x\to\infty} x \ln\left(1 + \frac{\pi}{2} - \tan^{-1} x\right), \text{ which is in the } \infty \cdot 0 \text{ form} \\
&= \lim_{x\to\infty} \frac{\ln\left(1 + \dfrac{\pi}{2} - \tan^{-1} x\right)}{\dfrac{1}{x}}, \text{ which is in the } \frac{0}{0} \text{ form} \\
&= \lim_{x\to\infty} \frac{\dfrac{1}{1+\frac{\pi}{2} - \tan^{-1} x}\left(\dfrac{-1}{1+x^2}\right)}{-\dfrac{1}{x^2}}, \text{ by L'Hopital's rule} \\
&= \lim_{x\to\infty} \frac{x^2}{1+x^2}, \text{ since } \lim_{x\to\infty} \frac{1}{1+\frac{\pi}{2} - \tan^{-1} x} = 1 \\
&= 1 \\
\Rightarrow L &= e
\end{aligned}$$

8(a) Find $F'(4)$ if $F(x) = \int_{\pi}^{\sqrt{x}} e^{-t^2}\,dt$.

Soluton: By the Fundamental Theorem of Calculus, and the chain rule,

$$F'(x) = e^{-(\sqrt{x})^2}\frac{1}{2}\frac{1}{\sqrt{x}} = \frac{1}{2}\frac{1}{\sqrt{x}}e^{-x}.$$

So

$$F'(4) = \frac{1}{2}\frac{1}{\sqrt{4}}e^{-4} = \frac{1}{4}e^{-4}.$$

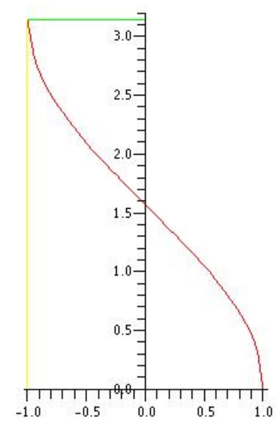
8(b) Show that $\int_{-1}^{1} \cos^{-1} x\,dx = \pi$. (Hint: draw a graph.)

Solution: using symmetry.

$$\int_{-1}^{1} \cos^{-1} x\,dx$$

is the area under $y = \cos^{-1} x$. From the diagram you can see that this area is the area of the rectangle with base 1 and height π, *excluding* the area A_1 of the region above $y = \cos^{-1} x$, below $y = \pi$, *including* the area A_2 under $y = \cos^{-1} x$ for $x = 0$ to $x = 1$. By symmetry, $A_1 = A_2$. Thus

$$\int_{-1}^{1} \cos^{-1} x\,dx = \pi - A_1 + A_2 = \pi.$$



Alternate Solution: more formally, using integrals, to calculate A_1 and A_2.

$$\begin{aligned}
\int_{-1}^{1} \cos^{-1} x\,dx &= \pi - \int_{-1}^{0} \left(\pi - \cos^{-1} x\right)\,dx + \int_{0}^{1} \cos^{-1} x\,dx \\
\text{(in terms of } y\text{)} &= \pi - \int_{\pi/2}^{\pi} (0 - \cos y)\,dy + \int_{0}^{\pi/2} \cos y\,dy \\
&= \pi + \int_{\pi/2}^{\pi} \cos y\,dy + \int_{0}^{\pi/2} \cos y\,dy \\
&= \pi + [\sin y]_{\pi/2}^{\pi} + [\sin y]_{0}^{\pi/2} \\
&= \pi + \sin\pi - \sin(\pi/2) + \sin(\pi/2) - \sin 0 \\
&= \pi + 0 - 1 + 1 - 0 \\
&= \pi
\end{aligned}$$

9. Suppose the velocity of a particle at time t is given by $v = 3t^2 - 6t$, for $0 \le t \le 4$. Find the following:

(a) the net change in position of the particle from $t = 0$ to $t = 4$.

Solution:

$$\begin{aligned} \text{net change in position} = \int_0^4 v\,dt &= \int_0^4 \left(3t^2 - 6t\right)\,dt \\ &= \left[t^3 - 3t^2\right]_0^4 \\ &= 64 - 48 \\ &= 16 \end{aligned}$$

(b) the total distance travelled by the particle for $0 \le t \le 4$.

Solution: total distance travelled is given by $\displaystyle\int_0^4 |v|\,dt$.

Since $v = 3t(t-2)$,

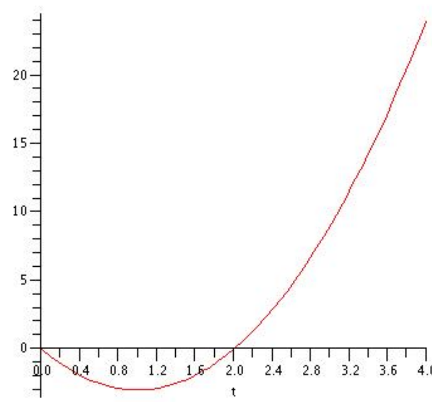
$$v \le 0 \text{ for } 0 \le t \le 2$$

and

$$v \ge 0 \text{ for } 2 \le t \le 4.$$

Therefore the distance travelled is

$$\int_0^4 |v|\,dt = \int_0^2 (-v)\,dt + \int_2^4 v\,dt.$$



Calculate:

$$\begin{aligned} \int_0^4 |v|\,dt &= \int_0^2 (6t - 3t^2)\,dt + \int_2^4 (3t^2 - 6t)\,dt \\ &= \left[3t^2 - t^3\right]_0^2 + \left[t^3 - 3t^2\right]_2^4 \\ &= (12 - 8) + (64 - 48 - 8 + 12) \\ &= 24 \end{aligned}$$

10. The Average Value Theorem, as stated in our text book, says

If f is continuous on $[a, b]$, then

$$f(\overline{x}) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

for some number $\overline{x}$ in $[a, b]$.

(a) Illustrate this theorem with a suitable picture, and indicate why, to quote Edwards and Penney, it means:

> Every continuous function on a closed interval attains its average value at some point of the interval.

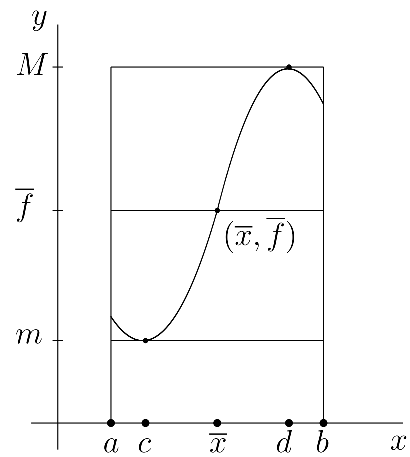
Solution: the average value of f on $[a, b]$ is

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The theorem says there is $\overline{x} \in [a, b]$ such that

$$f(\overline{x}) = \overline{f};$$

that is, $\overline{f}$ is in the range of f.



(b) Use the Intermediate Value Property to prove the Average Value Theorem.
Solution: Since f is continuous on $[a, b]$, there are numbers $c, d \in [a, b]$ such that

$$m = f(c) \le f(x) \le f(d) = M$$

for all $x \in [a, b]$. Thus

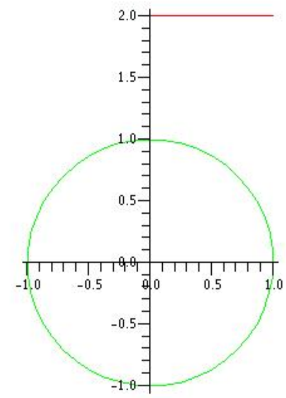
$$\begin{aligned} \int_a^b m\,dx \le & \int_a^b f(x)\,dx && \le \int_a^b M\,dx \\ \Rightarrow m(b-a) \le & \int_a^b f(x)\,dx && \le M(b-a) \\ \Rightarrow m \le & \frac{1}{b-a}\int_a^b f(x)\,dx && \le M \end{aligned}$$

So $\overline{f}$ is an intermediate value between m and M. By the Intermediate Value Property there is an $\overline{x} \in [c, d] \subset [a, b]$ such that

$$f(\overline{x}) = \overline{f}.$$

11. A storage tank, full of water with density ρ, is in the shape of a sphere with radius 1 m. How much work is done in emptying the tank by pumping all the water up to a transfer pipe 1 m above the top of the tank?

Solution: a side view of the tank is to the right. The origin is chosen as the centre of the circle; its equation then is $x^2 + y^2 = 1$.



The cross-sectional area of the tank at height y is given by

$$A(y) = \pi x^2 = \pi(1 - y^2).$$

The height of the transfer pipe is $h = 2$; the bottom of the tank is at $a = -1$ and the top at $b = 1$.

The work done in emptying the tank by pumping all the water up to the transfer pipe is

$$\begin{aligned} W &= \int_a^b \rho g A(y)(h - y)\,dy \\ &= \int_{-1}^{1} \rho g \pi (1 - y^2)(2 - y)\,dy \\ &= \rho g \pi \int_{-1}^{1} \left(y^3 - 2y^2 - y + 2\right)\,dy \\ &= \rho g \pi \left[\frac{1}{4}y^4 - \frac{2}{3}y^3 - \frac{1}{2}y^2 + 2y\right]_{-1}^{1} \\ &= \frac{8}{3}\rho g \pi \end{aligned}$$

Alternate Solution: if the origin is chosen to be at the bottom of the circle, then its equation is $x^2 + (y - 1)^2 = 1$; and $A(y) = \pi x^2 = \pi(2y - y^2), h = 3, a = 0, b = 2$. So

$$W = \int_0^2 \rho g \pi (2y - y^2)(3 - y)\,dy.$$

12. A spherical snowball is melting in such a way that the rate of decrease of its volume is proportional to its surface area. At 9AM its volume is 500 cc and at 10AM its volume is 250 cc. When does the snowball finish melting?

Solution: use

$$V = \frac{4}{3}\pi r^3 \text{ and } SA = 4\pi r^2.$$

It is given that

$$\frac{dV}{dt} = kSA = 4k\pi r^2, \text{ for some constant } k.$$

On the other hand, differentiating V with respect to t gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Comparing these two expressions for $\frac{dV}{dt}$ gives

$$\begin{aligned} 4\pi r^2 \frac{dr}{dt} = 4k\pi r^2 \quad &\Rightarrow \quad \frac{dr}{dt} = k \\ &\Rightarrow \quad r = \int k\,dt \\ &\Rightarrow \quad r = k\,t + c, \text{ for constants } k, c \end{aligned}$$

Let t be measured in hours since 9 AM, and use the given data to find k and c :

At 9 AM: $t = 0, V = 500$:

$$\begin{aligned} \frac{4}{3}\pi r^3 = 500 \quad &\Rightarrow \quad r = \left(\frac{375}{\pi}\right)^{1/3} \\ &\Rightarrow \quad c = \left(\frac{375}{\pi}\right)^{1/3} \end{aligned}$$

At 10 AM: $t = 1, V = 250$:

$$\begin{aligned} \frac{4}{3}\pi r^3 = 250 \quad &\Rightarrow \quad r = \left(\frac{375}{2\pi}\right)^{1/3} \\ &\Rightarrow \quad k + c = \left(\frac{375}{2\pi}\right)^{1/3} \\ &\Rightarrow \quad k = \left(\frac{375}{2\pi}\right)^{1/3} - \left(\frac{375}{\pi}\right)^{1/3} \end{aligned}$$

Finally, the snowball has finished melting when $r = 0$:

$$r = 0 \Leftrightarrow t = -\frac{c}{k} = -\frac{\left(\frac{375}{\pi}\right)^{1/3}}{\left(\frac{375}{2\pi}\right)^{1/3} - \left(\frac{375}{\pi}\right)^{1/3}} = \frac{1}{1 - 2^{-1/3}} \simeq 4.85.$$

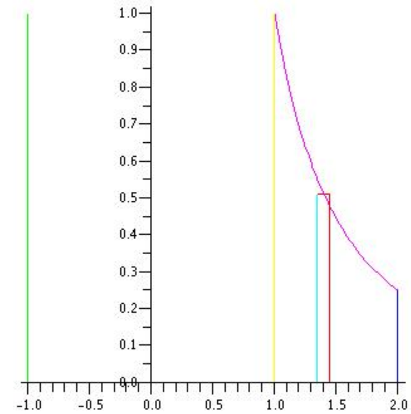
The snowball will be finished melting at approximately 1:51 PM.

13. Let R be the region in the plane bounded by $x = 1, x = 2, y = 0$ and $y = \dfrac{1}{x^2}$.

(a) [6 marks] Find the volume of the solid of revolution obtained by revolving R about the line $x = -1$.

Solution: using method of shells.

$$\begin{aligned} V &= \int_1^2 2\pi(x+1)\frac{1}{x^2}\,dx \\ &= 2\pi \int_1^2 \left(\frac{1}{x} + \frac{1}{x^2}\right) dx \\ &= 2\pi \left[\ln x - \frac{1}{x}\right]_1^2 \\ &= 2\pi \ln 2 - \pi + 2\pi \\ &= 2\pi \ln 2 + \pi \text{ or } \pi(1 + \ln 4) \end{aligned}$$



(b) [6 marks] What is the volume of the solid of revolution obtained by revolving R about the line $y = 2$?

Solution: using method of discs.

$$\begin{aligned} V &= \int_1^2 \pi\left(2^2 - \left(2 - \frac{1}{x^2}\right)^2\right) dx \\ &= \pi \int_1^2 \left(4 - 4 + \frac{4}{x^2} - \frac{1}{x^4}\right) dx \\ &= \pi \int_1^2 \left(\frac{4}{x^2} - \frac{1}{x^4}\right) dx \\ &= \pi \left[-\frac{4}{x} + \frac{1}{3}\frac{1}{x^3}\right]_1^2 \\ &= \pi\left(-2 + \frac{1}{24} + 4 - \frac{1}{3}\right) \\ &= \frac{41}{24}\pi \end{aligned}$$