University of Toronto

FACULTY OF APPLIED SCIENCE AND ENGINEERING

Solutions to **FINAL EXAMINATION, DECEMBER, 2007**

First Year - CHE, CIV, IND, LME, MEC, MMS

MAT 186H1F - CALCULUS I

Exam Type: A

SURNAME: ____________________

GIVEN NAMES: ____________________

STUDENT NUMBER: ____________________

SIGNATURE: ____________________

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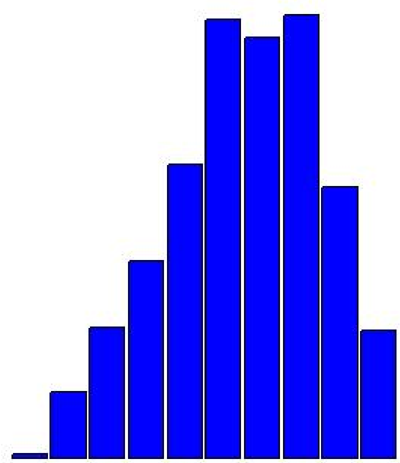
Breakdown of Results: 546 students wrote this exam. The marks ranged from 6% to 99%, and the average was 59.9%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	5.3%
A	16.7%	80-89%	11.4%
B	18.5%	70-79%	18.5%
C	17.6%	60-69%	17.6%
D	18.3%	50-59%	18.3%
F	28.9%	40-49%	12.3%
		30-39%	8.2%
		20-29%	5.5%
		10-19%	2.7%
		0-9%	0.2 %



1. What is the equation of the tangent line to the graph of $f(x) = e^x$ at the point $(x, y) = (0, 1)$?

   (a) $y = x$
   (b) $y = x + 1$
   (c) $y = x - 1$
   (d) $y = x + e$

   Solution: $y - 1 = f'(0)(x - 0) \Leftrightarrow y = e^0 x + 1 = x + 1$. The answer is (b).

2. The arc length of the curve $f(x) = 2\cos x$ for $0 \le x \le \pi/2$ is given by

   (a) $\displaystyle\int_0^{\pi/2} \sqrt{1 - \sin^2 x}\, dx$
   (b) $\displaystyle\int_0^{\pi/2} \sqrt{1 + \sin^2 x}\, dx$
   (c) $\displaystyle\int_0^{\pi/2} \sqrt{1 - 4\sin^2 x}\, dx$
   (d) $\displaystyle\int_0^{\pi/2} \sqrt{1 + 4\sin^2 x}\, dx$

   Solution:
   $$\int_0^{\pi/2} \sqrt{1 + (f'(x))^2}\, dx = \int_0^{\pi/2} \sqrt{1 + (-2\sin x)^2}\, dx = \int_0^{\pi/2} \sqrt{1 + 4\sin^2 x}\, dx.$$
   The answer is (d).

3. How many asymptotes – horizontal, vertical, or slant – are there to the graph of
   $$y = x + 2 - \frac{2x - 2}{x^2 + 2x - 3}?$$

   (a) none
   (b) one
   (c) two
   (d) three

   Solution:
   $$y = x + 2 - \frac{2x - 2}{x^2 + 2x - 3} = x + 2 - \frac{2}{x + 3}, \quad \text{for } x \neq 1.$$
   The line $y = x + 2$ is a slant asymptote, and the line $x = -3$ is a vertical asymptote. The answer is (c).

4. If $\sin\theta = \dfrac{1}{4}$ and $\cos\theta < 0$, then the exact value of $\cos(2\theta)$ is

(a) $\dfrac{\sqrt{15}}{8}$

(b) $\dfrac{7}{8}$

(c) $-\dfrac{\sqrt{15}}{8}$

(d) $-\dfrac{7}{8}$

Solution: $\cos(2\theta) = 1 - 2\sin^2\theta = 1 - 2\left(\dfrac{1}{16}\right) = \dfrac{7}{8}$. The answer is (b).

5. $\displaystyle\int_1^{e^3} \frac{(4+\ln x)^2}{x}\,dx =$

(a) $\displaystyle\int_1^{e^3} u^2\,du$

(b) $\displaystyle\int_1^{3} u^2\,du$

(c) $\displaystyle\int_4^{7} u^2\,du$

(d) $\displaystyle\int_1^{7} u^2\,du$

Solution: Let $u = 4 + \ln x$. Then $du = \dfrac{1}{x}\,dx$. When $x = 1, u = 4$; when $x = e^3, u = 7$. So

$$\int_1^{e^3} \frac{(4+\ln x)^2}{x}\,dx = \int_4^7 u^2\,du.$$

The answer is (c).

6. $\displaystyle\lim_{x\to 0} \frac{e^x - x - 1}{x^2} =$

(a) 0

(b) $-\dfrac{1}{2}$

(c) $\dfrac{1}{2}$

(d) ∞

Soluton:

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-x-1}{x^2} &= \lim_{x\to 0}\frac{e^x-1}{2x}\\ &= \lim_{x\to 0}\frac{e^x}{2}\\ &= \frac{1}{2}\end{aligned}$$

The answer is (c).

7. Find the following limits:

(a) $\lim_{x\to\infty} x\left(\frac{\pi}{2} - \tan^{-1} x\right)$.

Solution: Since
$$\lim_{x\to\infty}\left(\frac{\pi}{2} - \tan^{-1} x\right) = 0,$$
the given limit is in the $\infty \cdot 0$ form. Rewrite it in the $\dfrac{0}{0}$ form and use L'Hopital's Rule:

$$\begin{aligned}
\lim_{x\to\infty} \frac{\frac{\pi}{2} - \tan^{-1} x}{\frac{1}{x}} &= \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{x^2}{1+x^2} \\
&= \lim_{x\to\infty} \frac{1}{1/x^2+1} \\
&= 1
\end{aligned}$$

(b) $\lim_{x\to 0^+} (1+\sin^{-1}(3x))^{\csc(2x)}$

Solution: The limit is in the 1^∞ form; let the limit be L, and evaluate the limit of $\ln L$ by using L'Hopital's Rule:

$$\begin{aligned}
\ln L &= \lim_{x\to 0^+} \csc(2x)\ln(1+\sin^{-1}(3x)) \\
&= \lim_{x\to 0^+} \frac{\ln(1+\sin^{-1}(3x))}{\sin(2x)}, \text{ which is in the } \frac{0}{0} \text{ form} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{1+\sin^{-1}(3x)}\frac{3}{\sqrt{1-9x^2}}}{2\cos(2x)} \\
&= \frac{3}{2} \\
\Rightarrow L &= e^{3/2}
\end{aligned}$$

8. Find the following:

(a) $F'(\pi/2)$, if $F(x) = \int_0^{\cos x} e^{t^2}\, dt$.

Soluton: By the Fundamental Theorem of Calculus, and the chain rule,

$$F'(x) = (-\sin x)\, e^{\cos^2 x}.$$

So

$$F'(\pi/2) = (-1)e^0 = -1.$$

(b) $\int_0^1 \frac{x+1}{x^2+1}\, dx$.

Solution:

$$\begin{aligned} \int_0^1 \frac{x+1}{x^2+1}\, dx &= \int_0^1 \frac{x}{x^2+1}\, dx + \int_0^1 \frac{1}{x^2+1}\, dx \\ &= \left[\frac{1}{2}\ln(x^2+1)\right]_0^1 + \left[\tan^{-1} x\right]_0^1 \\ &= \frac{1}{2}\ln 2 + \frac{\pi}{4} \end{aligned}$$

9. Suppose the velocity of a particle at time t is given by $v = 2t - t^2$, for $0 \leq t \leq 3$. Find the following:

(a) the average velocity of the particle for $0 \leq t \leq 3$.

Solution:

$$\begin{aligned}
\text{average velocity} &= \frac{1}{3-0}\int_0^3 \left(2t - t^2\right)\, dt \\
&= \frac{1}{3}\left[t^2 - \frac{1}{3}t^3\right]_0^3 \\
&= \frac{1}{3}\left(9 - \frac{1}{3}(27)\right) \\
&= 0
\end{aligned}$$

(b) the average speed of the particle for $0 \leq t \leq 3$.

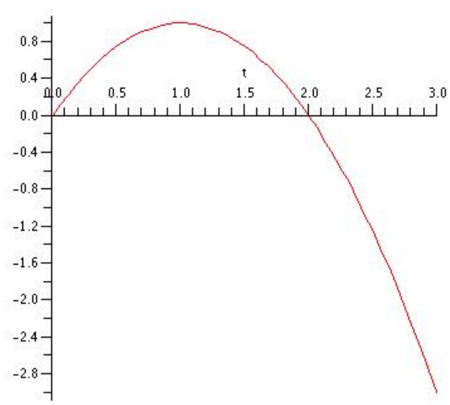
Solution:

$$\text{average speed} = \frac{1}{3}\int_0^3 |v|\, dt \text{ and}$$

$$v \geq 0 \Leftrightarrow 0 \leq t \leq 2.$$

So

$$\int_0^3 |v|\, dt = \int_0^2 v\, dt + \int_2^3 -v\, dt.$$



Hence

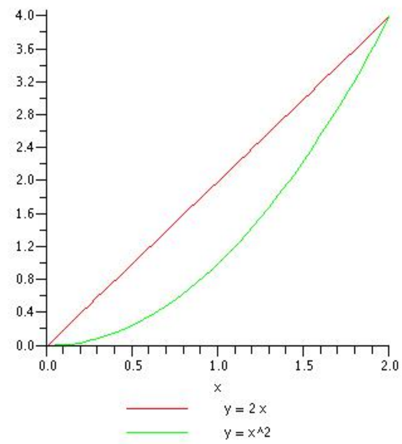
$$\begin{aligned}
\frac{1}{3}\int_0^3 |v|\, dt &= \frac{1}{3}\int_0^2 (2t - t^2)\, dt - \frac{1}{3}\int_2^3 (2t - t^2)\, dt \\
&= \frac{1}{3}\left[t^2 - \frac{1}{3}t^3\right]_0^2 - \frac{1}{3}\left[t^2 - \frac{1}{3}t^3\right]_2^3 \\
&= \frac{1}{3}\left(4 - \frac{8}{3}\right) - \frac{1}{3}\left(9 - \frac{1}{3}(27) - 4 + \frac{8}{3}\right) \\
&= \frac{8}{9}
\end{aligned}$$

10. Let R be the region in the plane bounded by $x = 0, x = 2, y = 2x$ and $y = x^2$. Find the following:

   (a) The volume of the solid obtained by revolving the region R about the x-axis.

   Solution: using method of discs.

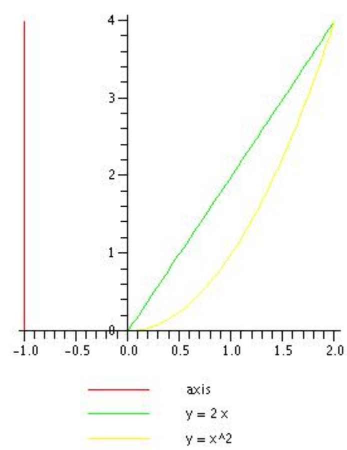
$$\begin{aligned} V &= \int_0^2 \pi((2x)^2 - (x^2)^2)\,dx \\ &= \int_0^2 \pi(4x^2 - x^4)\,dx \\ &= \pi\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 \\ &= \frac{64}{15}\pi \end{aligned}$$



   (b) The volume of the solid obtained by revolving the region R about the line $x = -1$.

   Solution: using method of shells.

$$\begin{aligned} V &= \int_0^2 2\pi(x+1)(2x - x^2)\,dx \\ &= \int_0^2 2\pi(x^2 + 2x - x^3)\,dx \\ &= 2\pi\left[\frac{1}{3}x^3 + x^2 - \frac{1}{4}x^4\right]_0^2 \\ &= \frac{16}{3}\pi \end{aligned}$$



11. A storage tank, full of water with density ρ, is in the shape of an inverted cone, with radius at the top 2 m, and with perpendicular height 6 m. How much work is done in emptying the tank by pumping all the water up to a transfer pipe 1 m above the top of the tank?

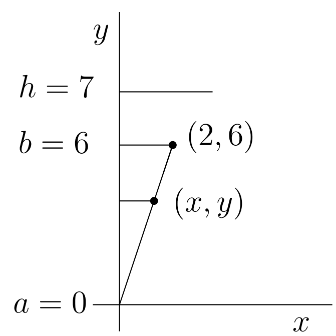
Solution:

Using similar triangles:

$$\frac{x}{y} = \frac{2}{6} \Rightarrow x = \frac{1}{3}y.$$

At height y the cross-sectional area is:

$$A(y) = \pi x^2 = \frac{\pi}{9}\,y^2.$$



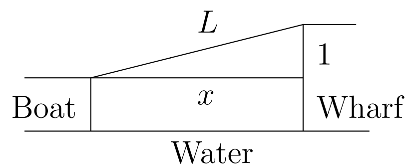
$$\begin{aligned} W &= \int_a^b \rho g A(y)(h-y)\,dy \\ &= \int_0^6 \rho g \pi \frac{1}{9}\,y^2(7-y)\,dy \\ &= \frac{\rho g \pi}{9}\int_0^6 (7y^2 - y^3)\,dy \\ &= \frac{\rho g \pi}{9}\left[\frac{7}{3}\,y^3 - \frac{1}{4}y^4\right]_0^6 \\ &= 20\pi\rho g \end{aligned}$$

12. One end of a rope is attached to the front of a boat, 1 m above sea level. The other end of the rope is pulled at a constant rate of 0.5 m/s from the top of a wharf 2 m above sea level, causing the boat to move toward the wharf. At what speed is the boat approaching the wharf when the horizontal distance between the front of the boat and the wharf is 5 m?

Solution:
Let the horizontal distance from the boat to the wharf be x, measured in metres.
Let the length of the rope be L, in metres.
Then

$$x^2 + 1 = L^2,$$

by the Pythagorean theorem.



Differentiate implicitly with respect to time, t, measured in seconds.

$$2x\frac{dx}{dt} = 2L\frac{dL}{dt} \quad \Rightarrow \quad 2x\frac{dx}{dt} = L, \text{ since } \frac{dL}{dt} = \frac{1}{2}$$
$$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{2}\frac{L}{x}$$

When $x = 5$, $L = \sqrt{26}$, and

$$\frac{dx}{dt} = \frac{1}{10}\sqrt{26}.$$

13. Let $f(x) = 3(x-2)^{1/3} - x$. Plot the graph of $y = f(x)$ on the interval $[0, 4]$, indicating absolute minimum and maximum points, all critical points, and all inflection points, if any.

Solution:

$$f'(x) = (x-2)^{-2/3} - 1; f''(x) = -\frac{2/3}{(x-2)^{5/3}}.$$

Critical Points:

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow (x-2)^2 = 1 \\ &\Leftrightarrow x - 2 = \pm 1 \\ &\Leftrightarrow x = 1 \text{ or } x = 3 \end{aligned}$$

$$\begin{aligned} f'(x) \text{ is undefined} &\Leftrightarrow x - 2 = 0 \\ &\Leftrightarrow x = 2 \end{aligned}$$

Concavity:

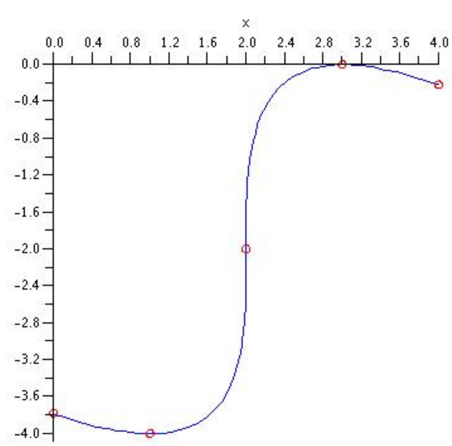
$$\begin{aligned} f''(x) > 0 &\Leftrightarrow -\frac{2/3}{(x-2)^{5/3}} > 0 \\ &\Leftrightarrow x - 2 < 0 \\ &\Leftrightarrow x < 2; \end{aligned}$$

$$\begin{aligned} f''(x) < 0 &\Leftrightarrow -\frac{2/3}{(x-2)^{5/3}} < 0 \\ &\Leftrightarrow x > 2 \end{aligned}$$

So the only inflection point is $(2, -2)$.

Graph:

Three critical points :
$(1, -4), (2, -2)$ and $(3, 0)$.
Endpoints:
$f(0) = -3 \cdot 2^{1/3} \simeq -3.77976$
$f(4) = 3 \cdot 2^{1/3} - 4 \simeq -0.220237$
Absolute maximum point: $(3, 0)$.
Absolute minimum point: $(1, -4)$.
Inflection point: $(2, -2)$.



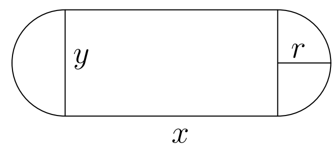
14. A school playing field is to have the shape of a rectangle with a semicircle attached at each of two opposite ends. The perimeter of the field is to be a 400 m running track. Find the dimensions of the field that

(a) maximize the total area of the field.

(b) maximize the area of the rectangular part.

Solution:

Let the dimensions of the rectangular part be $x \times y$, x and y in metres.
Let the radius of the semicircular ends be r, in metres.
Then
$$r = \frac{y}{2}.$$



The perimeter of the field is
$$P = 2x + 2\pi r = 2x + \pi y$$
and
$$P = 400 \Leftrightarrow x = \frac{400 - \pi y}{2}.$$
Since $x \geq 0$, we must have $0 \leq y \leq 400/\pi$.

Part (a): let A be the area of the whole field.
$$A = xy + \pi r^2 = \frac{400y - \pi y^2}{2} + \frac{\pi}{4} y^2.$$
$$\frac{dA}{dy} = 200 - \pi y + \frac{\pi}{2} y = 0 \Leftrightarrow y = \frac{400}{\pi}.$$
$$\frac{d^2A}{dy^2} = -\frac{\pi}{2} < 0.$$
So A is maximized if
$$x = 0, y = \frac{400}{\pi}, r = \frac{200}{\pi}.$$

Part (b): let A be the area of the rectangle.
$$A = xy = \frac{400y - \pi y^2}{2}.$$
$$\frac{dA}{dy} = \frac{400 - 2\pi y}{2} = 0 \Leftrightarrow y = \frac{200}{\pi}.$$
$$\frac{d^2A}{dy^2} = -\pi < 0.$$
So A is maximized if
$$x = 100, y = \frac{200}{\pi}, r = \frac{100}{\pi}.$$