University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, APRIL, 2009**
First Year - CHE, CIV, IND, LME, MEC, MMS

MAT186H1S - CALCULUS I
Exam Type: A

Comments:

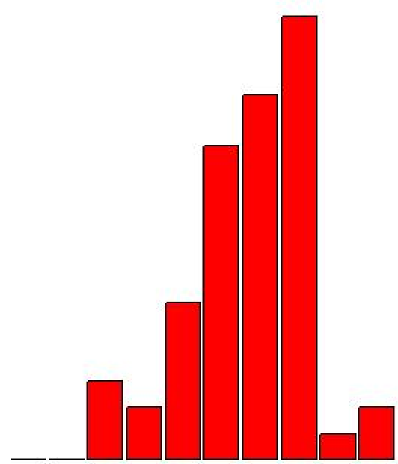
1. Very few students knew the formulas for Question 8. Most of those who did, couldn't simplify ds properly. Question 8 turned out to be the hardest question on this exam, even though it was right out of the assigned homework—#35 of Section 6.4—and should have been aced.

2. The L'Hopital's Rule questions in Question 11 are very routine, but very few students handled them correctly.

3. Unbelievably, there were many students who thought
$$\sqrt{a^2+b^2} = a+b \text{ or } \frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}.$$

Breakdown of Results: 57 students wrote this exam. The marks ranged from 21% to 91%, and the average was 61.6%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	3.5%
A	5.3%	80-89%	1.8%
B	29.8%	70-79%	29.8%
C	24.6%	60-69%	24.6%
D	21.0%	50-59%	21.0%
F	19.3%	40-49%	10.5 %
		30-39%	3.5%
		20-29%	5.3%
		10-19%	0.0%
		0-9%	0.0%



1. What is the equation of the tangent line to the graph of $f(x) = e^{2x}$ at the point $(x, y) = (0, 1)$?

(a) $y = 2x$

(b) $y = 2x + 1$

(c) $y = 2x - 1$

(d) $y = x + 1$

Solution: $f'(x) = 2e^{2x} \Rightarrow f'(0) = 2$. So the equation of the tangent line is

$$\frac{y-1}{x-0} = f'(0) \Leftrightarrow y = 2x + 1.$$

The answer is (b).

2. The volume of the solid of revolution obtained by revolving the function $f(x) = x^2$, for $0 \le x \le 1$, around the x-axis, is equal to

(a) $\frac{\pi}{2}$

(b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$

(d) $\frac{\pi}{5}$

Solution: use the method of discs.

$$\begin{aligned} V = \int_0^1 \pi (x^2)^2 \, dx &= \int_0^1 \pi x^4 \, dx \\ &= \pi \left[\frac{1}{5} x^5 \right]_0^1 \\ &= \frac{\pi}{5} \end{aligned}$$

The answer is (d).

3. How many vertical asymptotes are there to the graph of $f(x) = \dfrac{\sin(x^2-1)}{x^4 - 5x^2 + 4}$?

(a) 1

(b) 2

(c) 3

(d) 4

Solution:

$$f(x) = \frac{\sin(x^2-1)}{x^4 - 5x^2 + 4} = \frac{\sin(x^2-1)}{(x^2-1)(x^2-4)}$$

There are removable discontinuities at $x = \pm 1$; there are infinite limits as $x \to \pm 2^+$ and as $x \to \pm 2^-$. So only $x = \pm 2$ are vertical asymptotes.

The answer is (b).

4. If $\sin\theta = \dfrac{1}{4}$ and $\cos\theta < 0$, then the exact value of $\sin\left(\dfrac{\pi}{4}+\theta\right)$ is

(a) $\dfrac{1-\sqrt{15}}{4\sqrt{2}}$

(b) $\dfrac{1+\sqrt{15}}{4\sqrt{2}}$

(c) $\dfrac{-1+\sqrt{15}}{4\sqrt{2}}$

(d) $\dfrac{-1-\sqrt{15}}{4\sqrt{2}}$

Solution:

$$\begin{aligned}\sin\left(\frac{\pi}{4}+\theta\right) &= \sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta \\ &= \frac{1}{\sqrt{2}}\left(-\sqrt{1-\left(\frac{1}{4}\right)^2}\right) + \frac{1}{\sqrt{2}}\frac{1}{4} \\ &= \frac{1-\sqrt{15}}{4\sqrt{2}}\end{aligned}$$

The answer is (a).

5. $\displaystyle\int_1^2 \frac{\sqrt{4+x^6}}{x}\,dx =$

(a) $\dfrac{1}{3}\displaystyle\int_1^8 u\sqrt{4+u^2}\,du$

(b) $\dfrac{1}{3}\displaystyle\int_1^8 \frac{\sqrt{4+u^2}}{u}\,du$

(c) $\dfrac{1}{3}\displaystyle\int_1^2 u\sqrt{4+u^2}\,du$

(d) $\dfrac{1}{3}\displaystyle\int_1^2 \frac{\sqrt{4+u^2}}{u}\,du$

Solution: Let $u = x^3$; $du = 3x^2\,dx$.

$$\begin{aligned}\int_1^2 \frac{\sqrt{4+x^6}}{x}\,dx &= \frac{1}{3}\int_1^2 \frac{\sqrt{4+(x^3)^2}}{x^3}\,3x^2\,dx \\ &= \frac{1}{3}\int_1^8 \frac{\sqrt{4+u^2}}{u}\,du\end{aligned}$$

The answer is (b).

6. $\displaystyle\lim_{x\to 0}\frac{\tan x - x}{x^3} =$

(a) $\dfrac{1}{3}$

(b) $\dfrac{1}{6}$

(c) $-\dfrac{1}{3}$

(d) $-\dfrac{1}{6}$

Solution: use L'Hopital's rule three times.

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x - x}{x^3} &= \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} \\ &= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x} \\ &= \frac{1}{3}\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{3}\lim_{x\to 0}\frac{\cos x}{1} = \frac{1}{3}\end{aligned}$$

The answer is (a).

7. [24 marks] Let $f(x) = \dfrac{x^2 + x - 1}{x - 1}$, for which you may assume

$$f'(x) = \frac{x(x-2)}{(x-1)^2} \text{ and } f''(x) = \frac{2}{(x-1)^3}.$$

(a) [4 marks] Find the open intervals on which f is increasing; decreasing.

Solution:

$$f'(x) > 0 \Leftrightarrow x(x-2) > 0, x \neq 1 \Leftrightarrow x < 0 \text{ or } x > 2;$$

so f is increasing on $(-\infty, 0)$ and on $(2, \infty)$.

$$f'(x) < 0 \Leftrightarrow x(x-2) < 0, x \neq 1 \Leftrightarrow 0 < x < 2, x \neq 1;$$

so f is decreasing on $(0, 1)$ and on $(1, 2)$.

(b) [4 marks] Find the critical points of f and determine if each critical point is a relative maximum or a relative minimum point.

Solution:
$f'(x) = 0 \Leftrightarrow x = 0$ or $x = 2$.
By the first derivative test, and part (a), f has a maximum at $(x, y) = (0, 1)$, and f has a minimum at $(x, y) = (2, 5)$.
Or, use the second derivative test:
$f''(0) = -2 < 0$, so f has a maximum at $(x, y) = (0, 1)$;
and $f''(2) = 2 > 0$, so f has a maximum at $(x, y) = (2, 5)$.

(c) [4 marks] Find the open intervals on which f is concave up; concave down.

Solution:

$$f''(x) > 0 \Leftrightarrow x - 1 > 0 \Leftrightarrow x > 1;$$

so f is concave up on $(1, \infty)$.

$$f''(x) < 0 \Leftrightarrow x - 1 < 0 \Leftrightarrow x < 1;$$

so f is concave down on $(-\infty, 1)$.

(d) [6 marks] Find all asymptotes to the graph of f, if any.

Solution:

$$f(x) = x + 2 + \frac{1}{x-1};$$

so $y = x + 2$ is a slant asymptote to the graph as $x \to \pm\infty$. Consequently, there are no horizontal asymptotes.

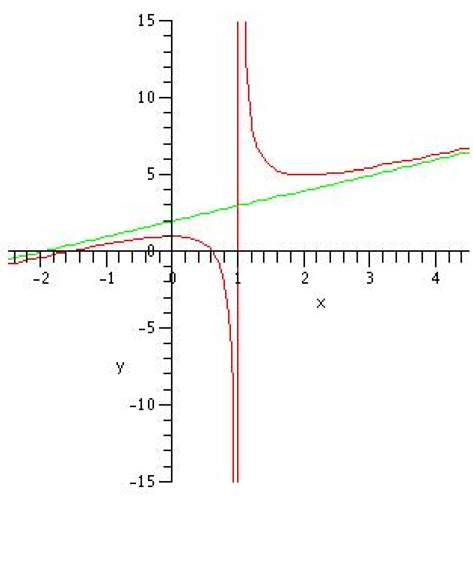
Since

$$\lim_{x \to 1^-} f(x) = -\infty \text{ and } \lim_{x \to 1^+} f(x) = \infty,$$

the graph of f has a vertical asymptote at $x = 1$.

(e) [6 marks] Sketch the graph of f labeling all critical points, inflection points and asymptotes, if any.

Solution:



You should label:

1. the maximum point at $(0, 1)$
2. the minimum point at $(2, 5)$
3. the vertical asymptote at $x = 1$
4. the slant asymptote at $y = x + 2$

There are no inflection points.

8. [12 marks; 6 for each part] Consider the curve $y = x^2 - \frac{1}{8}\ln x$, for $1 \le x \le 2$. Find the following:

(a) The length of the curve.

Solution: compute ds carefully; you need it for both parts.

$$\begin{aligned} ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx &= \sqrt{1 + \left(2x - \frac{1}{8x}\right)^2}\,dx \\ &= \sqrt{1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}}\,dx \\ &= \sqrt{4x^2 + \frac{1}{2} + \frac{1}{64x^2}}\,dx \\ &= \sqrt{\left(2x + \frac{1}{8x}\right)^2}\,dx \\ &= \left(2x + \frac{1}{8x}\right)dx, \text{ since } x > 0 \end{aligned}$$

So the length of the curve is

$$L = \int_1^2 ds = \int_1^2 \left(2x + \frac{1}{8x}\right)dx = \left[x^2 + \frac{1}{8}\ln x\right]_1^2 = 3 + \frac{1}{8}\ln 2.$$

(b) The surface area of the solid of revolution obtained by revolving the curve about the y-axis.

Solution: using ds as above.

$$\begin{aligned} SA = \int_1^2 2\pi x\,ds &= \int_1^2 2\pi x\left(2x + \frac{1}{8x}\right)dx \\ &= 2\pi\int_1^2 \left(2x^2 + \frac{1}{8}\right)dx \\ &= 2\pi\left[\frac{2}{3}x^3 + \frac{x}{8}\right]_1^2 \\ &= \frac{115\pi}{12} \end{aligned}$$

9. [10 marks] Suppose the velocity of a particle at time t is given by $v = 4t - t^2$, for $0 \le t \le 6$. Find the following:

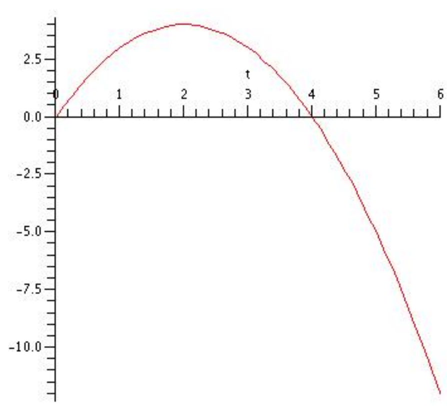
(a) [5 marks] the average velocity of the particle for $0 \le t \le 6$.

Solution:

$$\begin{aligned}
\frac{1}{6}\int_0^6 v\,dt &= \frac{1}{6}\int_0^6 (4t - t^2)\,dt \\
&= \frac{1}{6}\left[2t^2 - \frac{1}{3}t^3\right]_0^6 \\
&= \frac{1}{6}\left(72 - \frac{216}{3}\right) \\
&= 0
\end{aligned}$$

(b)[5 marks] the average speed of the particle for $0 \le t \le 6$.

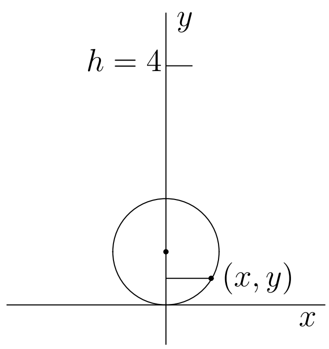
Solution: $v \le 0 \Leftrightarrow 4 \le t \le 6$.



$$\begin{aligned}
&\frac{1}{6}\int_0^6 |v|\,dt \\
=\ & \frac{1}{6}\int_0^4 v\,dt + \frac{1}{6}\int_4^6 (-v)\,dt \\
=\ & \frac{1}{6}\left[2t^2 - \frac{t^3}{3}\right]_0^4 + \frac{1}{6}\left[\frac{t^3}{3} - 2t^2\right]_4^6 \\
=\ & \frac{32}{18} + \frac{32}{18} \\
=\ & \frac{32}{9}
\end{aligned}$$

10. [10 marks] A spherical storage tank with radius 1 m is full of water with density ρ. How much work is done in pumping the water from the top half of the tank up to a transfer pipe 2 m above the top of the tank?

Solution:



$$\begin{aligned} A(y) &= \pi x^2 \\ &= \pi\left(1-(y-1)^2\right) \\ &= \pi(1-y^2+2y-1) \\ &= \pi(2y-y^2) \end{aligned}$$

$x^2 + (y-1)^2 = 1$

For the top half of the tank, $1 \le y \le 2$, so

$$\begin{aligned} W &= \int_a^b \rho g A(y)(h-y)\,dy \\ &= \int_1^2 \rho g \pi (2y-y^2)(4-y)\,dy \\ &= \rho g \pi \int_1^2 \left(8y - 6y^2 - +y^3\right)\,dy \\ &= \rho g \pi \left[4y^2 - 2y^3 + \frac{y^4}{4}\right]_1^2 \\ &= \frac{7}{4}\rho g \pi \end{aligned}$$

Alternate Solution: put the origin at the centre of the circle. Then

$$\begin{aligned} W &= \int_0^1 \rho g \pi (1-y^2)(3-y)\,dy \\ &= \rho g \pi \int_0^1 \left(3 - 3y^2 - y + y^3\right)\,dy \\ &= \rho g \pi \left[3y - y^3 - \frac{y^2}{2} + \frac{y^4}{4}\right]_0^1 \\ &= \frac{7}{4}\rho g \pi, \text{ as before.} \end{aligned}$$

11. [10 marks] Find the following limits:

(a) [5 marks] $\lim_{x\to\infty} x\left(e^{3/x}-1\right)$.

Solution: The limit is in the $\infty\cdot 0$ form; rewrite it as a fraction in the $\frac{0}{0}$ form and use L'Hopital's rule:

$$\begin{aligned}
\lim_{x\to\infty} x\left(e^{3/x}-1\right) &= \lim_{x\to\infty}\frac{e^{3/x}-1}{1/x}, \text{ which is in } \frac{0}{0} \text{ form}\\
&= \lim_{x\to\infty}\frac{-\frac{3}{x^2}e^{3/x}}{-\frac{1}{x^2}}, \text{ by L'Hopital's rule}\\
&= 3\lim_{x\to\infty} e^{3/x}\\
&= 3\,e^0\\
&= 3
\end{aligned}$$

(b) [5 marks] $\lim_{x\to 0}\left(1+\sin^{-1}(x^2)\right)^{3/x^2}$.

Solution: Since the given limit is in the 1^∞ form, let the limit be L, and evaluate the limit of $\ln L$ by using L'Hopital's Rule:

$$\begin{aligned}
\ln L &= \lim_{x\to 0}\ln\left(1+\sin^{-1}(x^2)\right)^{3/x^2}\\
&= 3\lim_{x\to 0}\frac{\ln\left(1+\sin^{-1}(x^2)\right)}{x^2}, \text{ which is in the } \frac{0}{0} \text{ form}\\
&= 3\lim_{x\to 0}\frac{\frac{1}{1+\sin^{-1}(x^2)}\left(\frac{2x}{\sqrt{1-(x^2)^2}}\right)}{2x}, \text{ by L'Hopital's rule}\\
&= 3\lim_{x\to 0}\frac{1}{1+\sin^{-1}(x^2)}\frac{1}{\sqrt{1-x^4}}\\
&= 3\\
\Rightarrow L &= e^3
\end{aligned}$$

12. [10 marks] Find the following:

(a) [5 marks] $F'(e)$, if $F(x) = \int_0^{\ln x} \sqrt{3 + t^5}\, dt$.

Soluton: By the Fundamental Theorem of Calculus, and the chain rule,

$$F'(x) = \sqrt{3 + \ln^5 x}\left(\frac{1}{x}\right).$$

So

$$F'(e) = \sqrt{3 + \ln^5 e}\left(\frac{1}{e}\right) = \frac{\sqrt{4}}{e} = \frac{2}{e}.$$

(b) [5 marks] $\int_1^e \frac{dx}{x[1 + (\ln x)^2]}$.

Soluton: Use the Evaluation Theorem, and a substitution. Let $u = \ln x$. Then $du = \frac{1}{x}\, dx$ and

$$\begin{aligned} \int_1^e \frac{dx}{x[1 + (\ln x)^2]} &= \int_0^1 \frac{du}{1 + u^2} \\ &= [\arctan u]_0^1 \\ &= \arctan 1 - \arctan 0 \\ &= \frac{\pi}{4} \end{aligned}$$