# University of Toronto <br> FACULTY OF APPLIED SCIENCE AND ENGINEERING <br> Solutions to FINAL EXAMINATION, APRIL, 2008 <br> First Year - CHE, CIV, IND, LME, MEC, MMS 

## MAT 186H1S - CALCULUS I

Exam Type: A
SURNAME:
GIVEN NAMES: $\qquad$ Examiner:
STUDENT NUMBER: $\qquad$ D. Burbulla

SIGNATURE:

## Comments:

1. This exam was very similar to the exam of Dec 07 . The main differences were: no related rates problem, and an area problem instead of a graphing question.
2. If you set up the area problem by integrating with respect to $x$, you have to integrate $\ln x$, which we haven't done yet. But you can find the integral by looking at the area with respect to the $y$-axis.
3. In the last problem, to confirm the critical point is a minimum point, it is easier to compare the length at the endpoints with the length at the critical point. The second derivative is quite messy.

Breakdown of Results: 99 students wrote this exam. The marks ranged from $25 \%$ to $94 \%$, and the average was $60.6 \%$. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

| Grade | $\%$ | Decade | $\%$ |
| ---: | :--- | ---: | :--- |
|  |  | $90-100 \%$ | $1.0 \%$ |
| A | $6.0 \%$ | $80-89 \%$ | $5.0 \%$ |
| B | $16.2 \%$ | $70-79 \%$ | $16.2 \%$ |
| C | $34.3 \%$ | $60-69 \%$ | $34.3 \%$ |
| D | $25.3 \%$ | $50-59 \%$ | $25.3 \%$ |
| F | $18.1 \%$ | $40-49 \%$ | $12.1 \%$ |
|  |  | $30-39 \%$ | $5.0 \%$ |
|  |  | $20-29 \%$ | $1.0 \%$ |
|  |  | $10-19 \%$ | $0.0 \%$ |
|  |  | $0-9 \%$ | $0.0 \%$ |



1. What is the equation of the normal line to the graph of $f(x)=e^{x}$ at the point $(x, y)=(0,1)$ ?
(a) $y=-x$
(b) $y=-x+1$
(c) $y=x+1$
(d) $y=-x+e$

Solution: $f^{\prime}(x)=e^{x} \Rightarrow f^{\prime}(0)=1$. So the equation of the normal line is

$$
\frac{y-1}{x-0}=-\frac{1}{f^{\prime}(0)} \Leftrightarrow y=-x+1
$$

The answer is (b).
2. The arc length of the curve $f(x)=2 \ln x$ for $1 \leq x \leq e$ is given by
(a) $\int_{1}^{e} \frac{\sqrt{x^{2}+4}}{x} d x$
(b) $\int_{1}^{e} \frac{\sqrt{x^{2}-4}}{x} d x$
(c) $\int_{1}^{e} \frac{\sqrt{2+x^{2}}}{x} d x$
(d) $\int_{1}^{e} \frac{\sqrt{x^{2}-2}}{x} d x$

$$
\begin{aligned}
& \text { Solution: } \\
& \begin{aligned}
& \int_{1}^{e} \sqrt{1+\left(f^{\prime}(x)\right)^{2}} d x=\int_{1}^{e} \sqrt{1+\left(\frac{2}{x}\right)^{2}} d x \\
&=\int_{1}^{e} \frac{\sqrt{x^{2}+4}}{x} d x \\
& \text { The answer is (a). }
\end{aligned} \text { }
\end{aligned}
$$

3. The equation of the slant (or oblique) asymptote to the graph of

$$
y=\frac{x^{3}+1}{x^{2}+1}
$$

is

Solution:

$$
\frac{x^{3}+1}{x^{2}+1}=x-\frac{x}{x^{2}+1}
$$

So the slant asymptote is the line with equation

$$
y=x
$$

The answer is (c).
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4. If $\sin \theta=\frac{1}{4}$ and $\cos \theta<0$, then the exact value of $\sin (2 \theta)$ is
(a) $\frac{\sqrt{15}}{8}$
(b) $\frac{7}{8}$
(c) $-\frac{\sqrt{15}}{8}$
(d) $-\frac{7}{8}$
5. $\int_{0}^{4} x^{3} \sqrt{x^{2}+9} d x=$
(a) $\frac{1}{2} \int_{0}^{4}(u+9) \sqrt{u} d u$
(b) $\frac{1}{2} \int_{9}^{25}(u+9) \sqrt{u} d u$
(c) $\frac{1}{2} \int_{0}^{4}(u-9) \sqrt{u} d u$
(d) $\frac{1}{2} \int_{9}^{25}(u-9) \sqrt{u} d u$
6. $\lim _{x \rightarrow 0} \frac{x-\sin x}{x^{3}}=$
(a) $\frac{1}{3}$
(b) $\frac{1}{6}$
(c) $-\frac{1}{3}$
(d) $-\frac{1}{6}$

Solution:

$$
\begin{aligned}
\sin (2 \theta) & =2 \sin \theta \cos \theta \\
& =2\left(\frac{1}{4}\right)\left(-\sqrt{1-\left(\frac{1}{4}\right)^{2}}\right) \\
& =-\frac{\sqrt{15}}{8}
\end{aligned}
$$

The answer is (c).

Solution: Let $u=x^{2}+9 ; d u=2 x d x$.

$$
\begin{aligned}
\int_{0}^{4} x^{3} \sqrt{x^{2}+9} d x & =\frac{1}{2} \int_{0}^{4} x^{2} \sqrt{x^{2}+9} 2 x d x \\
& =\frac{1}{2} \int_{9}^{25}(u-9) \sqrt{u} d u
\end{aligned}
$$

The answer is (d).

Solution: use L'Hopital's rule three times.

$$
\begin{aligned}
\lim _{x \rightarrow 0} \frac{x-\sin x}{x^{3}} & =\lim _{x \rightarrow 0} \frac{1-\cos x}{3 x^{2}} \\
& =\lim _{x \rightarrow 0} \frac{\sin x}{6 x} \\
& =\lim _{x \rightarrow 0} \frac{\cos x}{6}=\frac{1}{6}
\end{aligned}
$$

The answer is (b).

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7. [12 marks; 6 for each part] Find the following limits:
(a) $\lim _{x \rightarrow 0}\left(5 x^{2}+e^{-x^{2}}\right)^{3 / x^{2}}$.

Solution: this limit is in the $1^{\infty}$ form. Let the limit be $L$.

$$
\begin{aligned}
L & =\lim _{x \rightarrow 0}\left(5 x^{2}+e^{-x^{2}}\right)^{3 / x^{2}} \\
\Rightarrow \ln L & =\lim _{x \rightarrow 0} \ln \left(5 x^{2}+e^{-x^{2}}\right)^{3 / x^{2}} \\
& =\lim _{x \rightarrow 0} \frac{3}{x^{2}} \ln \left(5 x^{2}+e^{-x^{2}}\right) \\
& =3 \lim _{x \rightarrow 0} \frac{\ln \left(5 x^{2}+e^{-x^{2}}\right)}{x^{2}} \text {, in the } \frac{0}{0} \text { form } \\
\text { (by L'Hopital's rule) } & =3 \lim _{x \rightarrow 0} \frac{10 x-2 x e^{-x^{2}}}{\left(5 x^{2}+e^{-x^{2}}\right)} \frac{1}{2 x} \\
& =3 \lim _{x \rightarrow 0} \frac{1}{\left(5 x^{2}+e^{-x^{2}}\right)} \cdot \lim _{x \rightarrow 0} \frac{10 x-2 x e^{-x^{2}}}{2 x} \\
& =3 \cdot 1 \cdot \lim _{x \rightarrow 0} \frac{5-e^{-x^{2}}}{1} \\
& =3 \cdot 4=12 \\
\Rightarrow L & =e^{12}
\end{aligned}
$$

(b) $\lim _{x \rightarrow-\infty} x\left(\frac{\pi}{2}+\tan ^{-1} x\right)$.

Solution: this limit is in the $(-\infty) \cdot 0$ form. Rearrange the limit into the $\frac{0}{0}$ form.

$$
\begin{aligned}
L & =\lim _{x \rightarrow-\infty} \frac{\left(\frac{\pi}{2}+\tan ^{-1} x\right)}{1 / x} \\
\text { (by L'Hopital's rule) } & =\lim _{x \rightarrow-\infty} \frac{\frac{1}{1+x^{2}}}{-\frac{1}{x^{2}}} \\
& =(-1) \lim _{x \rightarrow-\infty} \frac{x^{2}}{1+x^{2}} \\
& =(-1) \cdot 1 \\
& =-1
\end{aligned}
$$

8. [12 marks; 6 for each part] Find the following:
(a) $F^{\prime}(\pi / 4)$, if $F(x)=\int_{0}^{\tan x} \sqrt{3+t^{5}} d t$.

Solution: use the Fundamental Theorem of Calculus, and the chain rule.

$$
\begin{aligned}
F^{\prime}(x) & =\sqrt{3+\tan ^{5} x} \cdot \sec ^{2} x \\
\Rightarrow F^{\prime}(\pi / 4) & =\sqrt{3+1^{5}} \cdot(\sqrt{2})^{2} \\
& =4
\end{aligned}
$$

(b) $\int_{0}^{1} \frac{1}{1+e^{-x}} d x$.

## Solution:

$$
\begin{aligned}
\int_{0}^{1} \frac{1}{1+e^{-x}} d x & =\int_{0}^{1} \frac{1}{1+1 / e^{x}} d x \\
& =\int_{0}^{1} \frac{e^{x}}{e^{x}+1} d x \\
\left(\text { let } u=e^{x}+1\right) & =\int_{2}^{1+e} \frac{1}{u} d u \\
& =[\ln u]_{2}^{1+e} \\
& =\ln (1+e)-\ln 2 \\
\text { (optionally) } & =\ln \left(\frac{1+e}{2}\right)
\end{aligned}
$$

9. [12 marks; 6 for each part] Let $R$ be the region in the plane bounded by $y=x$ and $y=\sqrt{x}$. Find the following:
(a) The volume of the solid obtained by revolving the region $R$ about the $x$-axis.

Solution: use the method of discs.


By method of shells: $V=\int_{0}^{1} 2 \pi y\left(y-y^{2}\right) d y$.
(b) The volume of the solid obtained by revolving the region $R$ about the line $x=2$.

Solution: use the method of shells.


$$
\begin{aligned}
V & =\int_{0}^{1} 2 \pi(2-x)(\sqrt{x}-x) d x \\
& =2 \pi \int_{0}^{1}\left(2 \sqrt{x}-2 x-x \sqrt{x}+x^{2}\right) d x \\
& =2 \pi\left[\frac{4}{3} x^{3 / 2}-x^{2}-\frac{2}{5} x^{5 / 2}+\frac{1}{3} x^{3}\right]_{0}^{1} \\
& =\frac{8 \pi}{15}
\end{aligned}
$$

By method of discs: $V=\int_{0}^{1} \pi\left(\left(2-y^{2}\right)^{2}-(2-y)^{2}\right) d y$.
10. [10 marks; 5 for each part] Suppose the velocity of a particle at time $t$ is given by $v=t^{2}-4 t$, for $0 \leq t \leq 5$. Find the following:
(a) the average velocity of the particle for $0 \leq t \leq 5$.

## Solution:

$$
\begin{aligned}
\frac{1}{5} \int_{0}^{5} v d t & =\frac{1}{5} \int_{0}^{5}\left(t^{2}-4 t\right) d t \\
& =\frac{1}{5}\left[\frac{t^{3}}{3}-2 t^{2}\right]_{0}^{5} \\
& =\frac{1}{5}\left(\frac{125}{3}-50\right) \\
& =-\frac{5}{3}
\end{aligned}
$$

(b) the average speed of the particle for $0 \leq t \leq 5$.

Solution: $v<0 \Leftrightarrow 0<t<4$.

$$
\begin{aligned}
& \frac{1}{5} \int_{0}^{5}|v| d t \\
& =\frac{1}{5} \int_{0}^{4}(-v) d t+\frac{1}{5} \int_{4}^{5} v d t \\
& =\frac{1}{5}\left[-\frac{t^{3}}{3}+2 t^{2}\right]_{0}^{4}+\frac{1}{5}\left[\frac{t^{3}}{3}-2 t^{2}\right]_{4}^{5} \\
& =\frac{32}{15}+\frac{7}{15} \\
& =\frac{13}{5}
\end{aligned}
$$

11. [10 marks] A spherical storage tank with radius 1 m is full of water with density $\rho$. How much work is done in emptying the tank by pumping all the water up to a transfer pipe 2 m above the top of the tank?

## Solution:



$$
\begin{aligned}
A(y) & =\pi x^{2} \\
& =\pi\left(1-(y-1)^{2}\right) \\
& =\pi\left(1-y^{2}+2 y-1\right) \\
& =\pi\left(2 y-y^{2}\right)
\end{aligned}
$$

$$
x^{2}+(y-1)^{2}=1
$$

$$
\begin{aligned}
W & =\int_{a}^{b} \rho g A(y)(h-y) d y \\
& =\int_{0}^{2} \rho g \pi\left(2 y-y^{2}\right)(4-y) d y \\
& =\rho g \pi \int_{0}^{2}\left(8 y-6 y^{2}-+y^{3}\right) d y \\
& =\rho g \pi\left[4 y^{2}-2 y^{3}+\frac{y^{4}}{4}\right]_{0}^{2} \\
& =4 \rho g \pi
\end{aligned}
$$

Alternate Solution: put the origin at the centre of the circle. Then

$$
\begin{aligned}
W & =\int_{-1}^{1} \rho g \pi\left(1-y^{2}\right)(3-y) d y \\
& =\rho g \pi \int_{-1}^{1}\left(3-3 y^{2}-y+y^{3}\right) d y \\
& =\rho g \pi\left[3 y-y^{3}-\frac{y^{2}}{2}+\frac{y^{4}}{4}\right]_{-1}^{1} \\
& =4 \rho g \pi, \text { as before. }
\end{aligned}
$$

12. [10 marks] Find the area of the region in the first quadrant bounded by the three curves

$$
y=2, y=\ln x \text { and } y=2 \sqrt{1-x}
$$

## Solution:



The region is indicated in the diagram to the left, where one boundary is

$$
y=\ln x \text { for } 1 \leq x \leq e^{2}
$$

and another boundary is

$$
y=2 \sqrt{1-x} \text { for } 0 \leq x \leq 1
$$

It is much simpler to integrate with respect to $y$. We have

$$
y=\ln x \Leftrightarrow x=e^{y} \text { and } y=2 \sqrt{1-x} \Leftrightarrow x=1-\left(\frac{y}{2}\right)^{2} .
$$

Then

$$
\begin{aligned}
A & =\int_{0}^{2}\left(e^{y}-1+\left(\frac{y}{2}\right)^{2}\right) d y \\
& =\int_{0}^{2}\left(e^{y}-1+\frac{y^{2}}{4}\right) d y \\
& =\left[e^{y}-y+\frac{y^{3}}{12}\right]_{0}^{2} \\
& =e^{2}-2+\frac{2}{3}-1 \\
& =e^{2}-\frac{7}{3}
\end{aligned}
$$

Note: if you integrate with respect to $x$ the area is given by two integrals:

$$
A=\int_{0}^{1}(2-2 \sqrt{1-x}) d x+\int_{1}^{e^{2}}(2-\ln x) d x=\frac{2}{3}+\left(e^{2}-3\right)=e^{2}-\frac{7}{3}
$$

13. [10 marks] Two vertical poles stand 10 m apart on level ground. A wire joining the tops of the two poles is to be attached to a point on the ground between the two poles, so that the wire goes from the top of one pole down to the ground and then up to the top of the second pole. What is the length of the shortest such wire, if one pole is 6 m high and the other pole is 4 m high?

Solution: let the distance from the base of the taller pole to the point on the ground be $x$. For $0 \leq x \leq 10$, the total length of the wire is

$L=\sqrt{6^{2}+x^{2}}+\sqrt{(10-x)^{2}+4^{2}}$
$=\sqrt{36+x^{2}}+\sqrt{(10-x)^{2}+16}$.

## Critical Points:

$$
\begin{aligned}
\frac{d L}{d x} & =\frac{2 x}{2 \sqrt{36+x^{2}}}+\frac{(-2(10-x))}{2 \sqrt{(10-x)^{2}+16}} \\
& =\frac{x}{\sqrt{36+x^{2}}}-\frac{10-x}{\sqrt{(10-x)^{2}+16}} \\
\frac{d L}{d x}=0 & \Rightarrow \frac{x}{\sqrt{36+x^{2}}}=\frac{10-x}{\sqrt{(10-x)^{2}+16}} \\
& \Rightarrow \frac{x^{2}}{36+x^{2}}=\frac{(10-x)^{2}}{(10-x)^{2}+16} \\
& \Rightarrow x^{2}\left(100-20 x+x^{2}+16\right)=\left(36+x^{2}\right)\left(100-20 x+x^{2}\right) \\
& \Rightarrow 116 x^{2}-20 x^{3}+x^{4}=3600+136 x^{2}-720 x-20 x^{3}+x^{4} \\
& \Rightarrow 20 x^{2}-720 x+3600=0 \\
& \Rightarrow x^{2}-36 x+180=0 \\
& \Rightarrow x=\frac{36 \pm \sqrt{576}}{2} \\
& \Rightarrow x=18 \pm 12 \\
& \Rightarrow x=6 \text { or } 30
\end{aligned}
$$

The only critical point is $x=6$ at which $L=\sqrt{72}+\sqrt{32}=10 \sqrt{2} \simeq 14.14$.

## End points:

$x=0 \Rightarrow L=6+\sqrt{116} \simeq 16.77 ; x=10 \Rightarrow L=\sqrt{136}+4 \simeq 15.66$.
So the minimum length of the wire is $10 \sqrt{2} \mathrm{~m}$, at the critical point.

