MAT301H1Y: Groups and Symmetry
Supplementary Material on the Symmetry Groups of the Platonic Solids

Dietrich Burbulla
Dept of Mathematics
University of Toronto

A Note to the Reader: these notes are an attempt to collect together in a coherent mathematical way supplementary material I devised for class content, or for student Problem Sets, while teaching the Groups and Symmetry course at U of T. Much of the material makes use of linear algebra since 3×3 orthogonal matrices are a convenient way to represent symmetries of a 3-dimensional object. Besides, linear algebra is a pre-requisite for MAT301, and much of this material can be considered as "applications" of linear algebra.

Some of the terminology used in these notes, e.g. deranged, counter-symmetric, is being used in idiosyncratic ways. This is mostly because I wished to emphasize to the students that some of the material is not to be found in other books, and to illustrate the joy of playing around with mathematical concepts.

Using *Maple* or *Mathematica* to perform calculations was encouraged in class, and accepted in solutions to Problem Sets as long as sufficient explanation and context were supplied. Similarly many calculations with *Maple* or *Mathematica* are included in these notes.

Some of the proofs in these notes are still not complete! When such material was used in class it was pointed out to students that it would not be examined. I have used this version of the notes to fill in some of the details, especially in Part 4.

- D. B.

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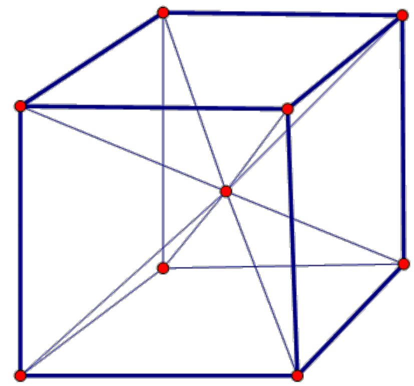
Part 1: The Symmetry Group of the Cube. Recall the Orbit-Stabilizer Theorem:

Theorem: Let G be a finite group of permutations of a set S. Then for any $i \in S$,

$$|G| = |\mathrm{orb}_G(i)|\,|\mathrm{stab}_G(i)|.$$

We can use the Orbit-Stabilizer Theorem to find out how many rotational symmetries of a cube there are. Consider the cube in the figure to the right; it has six faces, which we can number 1 through 6, and four diagonals. Let G be the group of rotational symmetries of this cube. We claim $G \cong S_4$. Each rotation can be considered as a permutation of the six faces. Relative to F_1,

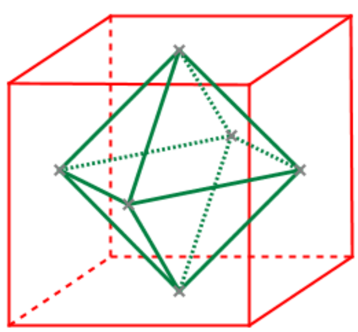


$$|\mathrm{orb}_G(F_1)| = 6.$$

For F_1 there are four rotations that fix F_1, namely rotations of 0, 90, 180 and 270 degrees around the axis through the centre of the face, so $|\mathrm{stab}_G(F_1)| = 4$. Thus $|G| = 6 \times 4 = 24$.

On the other hand, each rotation of the cube can be considered as a permutation of its four diagonals; so G can be considered as a subgroup (of order 24) of S_4, which itself has order 24. So $G \cong S_4$.

We can also use some basic linear algebra to find all the symmetries of a cube. Let the symmetry group of the cube be $S(C)$ and consider the cube in the figure to the left, with vertices $(\pm1, \pm1, \pm1)$, with respect to the usual x, y and z axes, and centre at the origin $(0, 0, 0)$. Consider the octahedron formed by joining the centres of the cube's six faces:



$$\{(\pm1, 0, 0), (0, \pm1, 0), (0, 0, \pm1)\} = \{\pm\vec{e}_1, \pm\vec{e}_2, \pm\vec{e}_3\}.$$

If A is a 3×3 orthogonal matrix that represents a symmetry of the cube, it also represents a symmetry of the octahedron. Each such matrix A must map the vertices of the cube to the vertices of the cube, and the vertices of the octahedron to the vertices of the octahedron. Recall that

$$A = \begin{bmatrix} A\vec{e}_1 & A\vec{e}_2 & A\vec{e}_3 \end{bmatrix}.$$

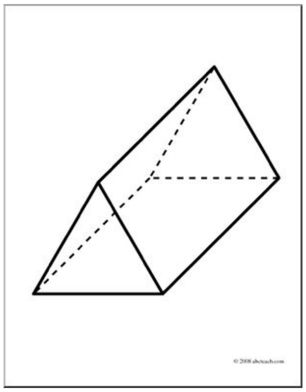
To represent a symmetry of the octahedron the three columns of A must be independent, and must be vertices of the octahedron. This gives $6 \times 4 \times 2 = 48$ possible matrices:

$$\begin{bmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{bmatrix}, \begin{bmatrix} 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \\ \pm1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \pm1 \\ \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} \pm1 & 0 & 0 \\ 0 & 0 & \pm1 \\ 0 & \pm1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \pm1 \\ 0 & \pm1 & 0 \\ \pm1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \pm1 & 0 \\ \pm1 & 0 & 0 \\ 0 & 0 & \pm1 \end{bmatrix}.$$

The 24 matrices with determinant equal to 1 represent rotations, the 24 matrices with determinant equal to -1 represent reflections or improper rotations.

Before we look at these matrices in more detail we can use the Orbit-Stabilizer Theorem to find the order of some other symmetry groups. For instance we can show that the symmetry group of a 3-prism in $\mathbb{R}^3$ must be isomorphic to D_6. A 3-prism is shown to the left. What is its symmetry group, G? The first thing you can do is use the Orbit-Stabilizer Theorem to find $|G|$. The two ends of the 3-prism consist of equilateral triangles, each with a symmetry group of D_3, which has order 6. So $|G| = 2 \times 6 = 12$. Or you can look at the 3-prism as consisting of three congruent rectangles, each of which has symmetry group $C_2 \times C_2$, which has order 4. Thus



$$|G| = 3 \times 4 = 12.$$

Either way, G contains a normal subgroup of order 6 isomorphic to D_3 and so $G \cong D_3 \times C_2 \cong D_6$.

Symmetry Groups of the Platonic Solids: the five Platonic solids are shown below in Figure 1. They are the only 3-dimensional solids with faces that are all identical, regular polygons. Using the first letter of the name for each solid we can designate their symmetry groups by

$$S(T), S(O), S(C), S(I), S(D),$$

just as we called $S(C)$ the symmetry group of the cube. How many symmetries does each platonic solid have? Making use of the facts that the tetrahedron consists of 4 equilateral triangles, the octahedron consists of 8 equilateral triangles, the cube consists of 6 squares, the icosahedron consists of 20 equilateral triangles, and the dodecahedron consists of 12 pentagons, we find

1. $|S(T)| = 4 \times |D_3| = 4 \times 6 = 24.$
2. $|S(O)| = 8 \times |D_3| = 8 \times 6 = 48.$
3. $|S(C)| = 6 \times |D_4| = 6 \times 8 = 48.$
4. $|S(I)| = 20 \times |D_3| = 20 \times 6 = 120.$
5. $|S(D)| = 12 \times |D_5| = 12 \times 10 = 120.$

It's not a consequence of the Orbit-Stabilizer Theorem, but we shall show that

$$S(T) \cong S_4, S(O) \cong S_4 \times C_2, S(C) \cong S_4 \times C_2, S(I) \cong A_5 \times C_2, S(D) \cong A_5 \times C_2.$$

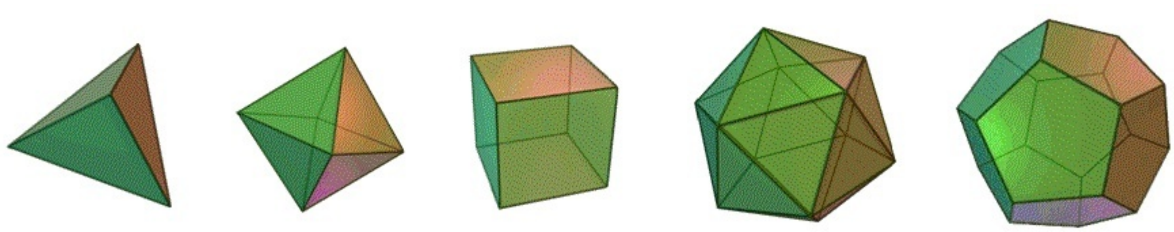


Figure 1: Tetrahedron, Octahedron, Cube, Icosahedron, Dodecahedron

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We now begin to look at the symmetry group $S(C)$ in more detail. It will be useful to classify in general the matrix of a symmetry according to its trace and determinant.

Classification of Rotations in $\mathbb{R}^3$: let A be a 3×3 matrix that represents a rotation of θ around its axis. The axis of rotation is the eigenspace corresponding to the eigenvalue $\lambda = 1$ of A. If the matrix A represents a rotation of θ then its eigenvalues are

$$\lambda_1 = 1, \lambda_2 = e^{i\theta}, \lambda_3 = e^{-i\theta},$$

so $\det(A) = \lambda_1\lambda_2\lambda_3 = 1$ and

$$\mathrm{tr}\,(A) = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2\cos\theta,$$

where we have made use of Euler's formula

$$e^{i\,\theta} = \cos\theta + i\,\sin\theta.$$

In the table to the right, the values of $\mathrm{tr}\,(A)$ and the order of A, $|A|$, are given for some simple values of θ, corresponding to low values of the order of A.

$\mathrm{tr}\,(A)$	action of A	$\lvert A\rvert$
3	identity, $A = I$	1
-1	rotation	2
0	rotation	3
1	rotation	4
$\psi = \dfrac{1-\sqrt{5}}{2}$	rotation	5
$\phi = \dfrac{1+\sqrt{5}}{2}$	rotation	5
2	rotation	6

Classification of Non-Rotations in $\mathbb{R}^3$: if the 3×3 matrix A represents a rotation then $-A$ represents a non-rotation. For exmaple, $-I$ represents inversion, a 'reflection' in the origin. (It is the composition of three separate reflections.) If $\det(A) = 1$, then $\det(-A) = -1$, and

$$\mathrm{tr}\,(-A) = -\mathrm{tr}\,(A).$$

Then A and $-A$ have the same order if $|A|$ is even, and

$$|-A| = 2|A|,$$

if $|A|$ is odd. Six simple cases of non-rotations are given to the right. A description of what reflections and improper rotations are geometrically is given below.

$\mathrm{tr}\,(A)$	action of A	$\lvert A\rvert$
-3	inversion, $A = -I$	2
1	reflection	2
0	improper rotation	6
-1	improper rotation	4
$-\psi$	improper rotation	10
$-\phi$	improper rotation	10
-2	improper rotation	6

What Is a Reflection in $\mathbb{R}^3$? The 3×3 orthogonal matrix A with $A^{-1} = A$, represents a reflection if its eigenvalues are $\lambda_1 = 1, \lambda_2 = 1$ and $\lambda_3 = -1$. Then:

- The eigenspace corresponding to the eigenvalue $\lambda = 1$ is two-dimensional; it is a plane in $\mathbb{R}^3$ which is the mirror of reflection.
- The eigenspace corresponding to the eigenvalue $\lambda = -1$ is the line normal to the plane. If the *unit* eigevector is $\vec{n}$, then $A\,\vec{n} = -\vec{n}$.

If A represents a reflection in the plane passing through the origin normal to the unit vector $\vec{n}$, then

$$A = I - 2\vec{n}\,\vec{n}^T.$$

Example 1: if

$$\vec{n} = \begin{bmatrix} 2/7 \\ 3/7 \\ 6/7 \end{bmatrix}$$

then the matrix of the reflection in the plane with equation $2x + 3y + 6z = 0$ is

$$A = I - 2\vec{n}\,\vec{n}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 2/7 \\ 3/7 \\ 6/7 \end{bmatrix} \begin{bmatrix} 2/7 & 3/7 & 6/7 \end{bmatrix}$$

$$= \frac{1}{49} \begin{bmatrix} 41 & -12 & -24 \\ -12 & 31 & -36 \\ -24 & -36 & -23 \end{bmatrix}.$$

Check that: $A^{-1} = A^T = A, \operatorname{tr}(A) = 1, \det(A) = -1, A\,\vec{n} = -\vec{n}$.

Aside: if A represents a reflection in the plane with normal vector $\vec{n}$, then $-A$ represents a rotation of order 2 around the axis parallel to the unit vector $\vec{n}$, and

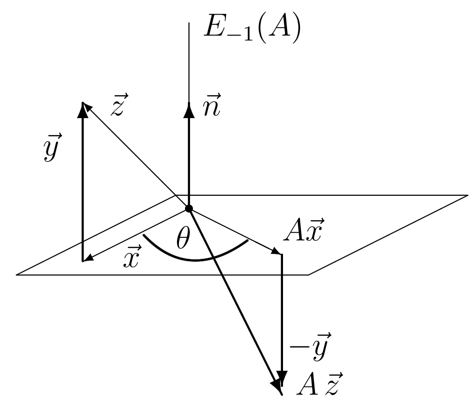
$$-A = 2\vec{n}\,\vec{n}^T - I.$$

What Is an Improper Rotation in $\mathbb{R}^3$? In terms of linear algebra, a matrix A represents an improper rotation in $\mathbb{R}^3$ if it is orthogonal, $\det(A) = -1$, and the only real eigenvalue of A is -1. If $\vec{n}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = -1$ then

$$A\,\vec{n} = -\vec{n}.$$

For every point $\vec{x}$ in the plane normal to the vector $\vec{n}$ and passing through the origin, $A\vec{x}$ is the vector obtained by rotating $\vec{x}$ by θ. The action of A on all other vectors $\vec{z} \in \mathbb{R}^3$ is a combination:

$$A\vec{z} = A(\vec{x} + \vec{y}) = A\,\vec{x} - \vec{y}.$$



(A matrix that represents a reflection is sometimes called an improper rotation, but we shall not adopt that option in these notes.)

Example 2: let

$$A = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \ \vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Check that the only real eigenvalue of A is -1, $\det(A) = -1$, and $A\,\vec{n} = -\vec{n}$. Thus A represents an improper rotation, of order 6. Restricted to the plane with equation $x + y + z = 0$, A represents a rotation of order 6, as the following calculations show:

$$\vec{v} = \begin{bmatrix} x \\ -x-z \\ z \end{bmatrix} \Rightarrow A\,\vec{v} = A \begin{bmatrix} x \\ -x-z \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -x \\ x+z \end{bmatrix}$$

and

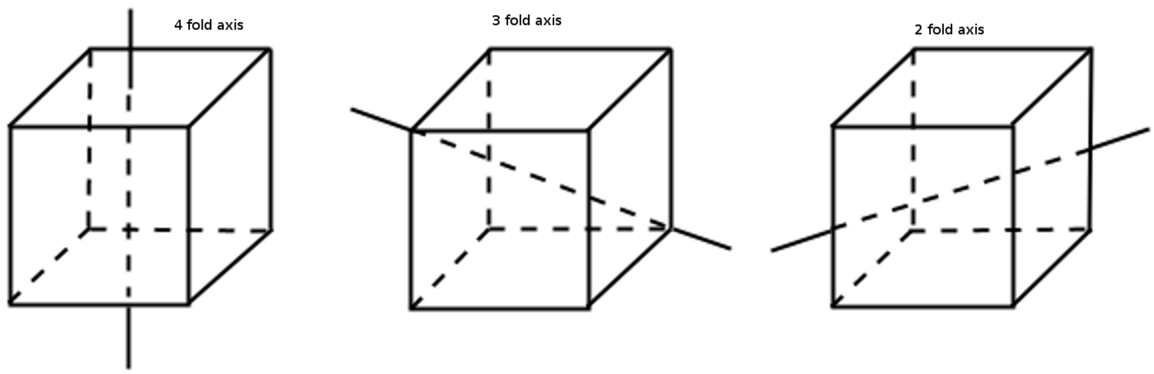
$$\cos\theta = \frac{\vec{v} \cdot A\vec{v}}{\|\vec{v}\|\,\|A\,\vec{v}\|} = \frac{x^2 + xz + z^2}{2x^2 + 2xz + 2z^2} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

The Symmetries of a Cube: we now describe in detail the action of all the symmetries of a cube. To be specific consider the cube in $\mathbb{R}^3$ with vertices $(\pm 1, \pm 1, \pm 1)$ with respect to the usual x, y and z axes, and centre at the origin $(0, 0, 0)$. As we saw above, $S(C)$ consists of the following 48 matrices, with usual matrix multiplication:

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}.$$

Rotational Symmetries of a Cube: the 24 matrices $A \in S(C)$ with $\det(A) = 1$ represent the rotational symmetries of the cube. One of these is I, the identity. What is the geometry of the other 23? The possible axes of rotation of a cube are illustrated in the figure below:



They are, respectively: the three lines joining the centres of opposite faces; the four diagonals of the cube joining opposite corners; and the six lines joining midpoints of opposite edges.

Rotations Around the Coordinate Axes:

	about x-axis	about y-axis	about z-axis	trace	order
A	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1	4
A^2	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	-1	2
A^3	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1	4

Of course, for each axis the set $\{I, A, A^2, A^3\}$ forms a subgroup of $S(C)$ isomorphic to C_4.

Rotations of Order 3 Around the Diagonals of a Cube: these matrices all have trace 0.

A	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$
A^2	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$
	rotation about diagonal parallel to the vector	rotation about diagonal parallel to the vector	rotation about diagonal parallel to the vector	rotation about diagonal parallel to the vector
axis	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

In this case, for each axis the set $\{I, A, A^2\}$ forms a subgroup of $S(C)$ isomorphic to C_3.

The Six Remaining Rotations: the remaining six rotation matrices in $S(C)$, all with trace -1, represent rotations of order 2 around lines that join the midpoints of opposite edges of the cube. They are, with the direction vectors of their axes:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \text{ about } \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}; \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ about } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix};$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \text{ about } \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ about } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix};$$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ about } \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ about } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

All in all, $S(C)$, the symmetry group of the cube has 24 rotations consisting of

- one identity element,
- nine rotations of order 2,
- eight rotations of order 3,
- six rotations of order 4.

Reflection Matrices in $S(C)$ With Their Planes of Reflection: these are the negatives of the nine rotation matrices of order 2. The table below lists all nine matrices and their fixed planes, i.e. the plane in which the reflection is being made.

reflection in plane	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $x = 0$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $y = 0$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $z = 0$
reflection in plane	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $y = z$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $x = z$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $x = y$
reflection in plane	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ $y = -z$	$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ $x = -z$	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $x = -y$

The Six Improper Rotations of Order 4 in $S(C)$: these matrices are the negatives of the six rotations of order 4 and consequently are 'rotations' about the same axes.

about x-axis	about y-axis	about z-axis
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

They all have determinant -1 and trace -1. Geometrically, an improper rotation of order 4 about an axis, rotates each point on a plane perpendicular to the axis by $\pi/2$, clockwise or counterclockwise, while also reflecting the point to the opposite plane.

Example 3: let

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos(\pi/2) & -\sin(\pi/2) \\ 0 & \sin(\pi/2) & \cos(\pi/2) \end{bmatrix},$$

and consider its action on the face of the cube in the plane $x = 1$:

$$A \begin{bmatrix} 1 \\ y \\ z \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + A \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -z \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -z \\ y \end{bmatrix},$$

which is in the plane $x = -1$. Note the presence of the 2×2 rotation matrix which rotates

$$\begin{bmatrix} 0 \\ y \\ z \end{bmatrix} \text{ by } \frac{\pi}{2} \text{ to } \begin{bmatrix} 0 \\ -z \\ y \end{bmatrix}.$$

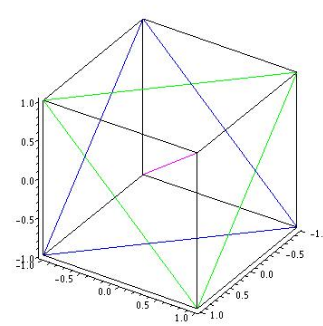
The Eight Improper Rotations of Order 6 in $S(C)$: these matrices are the negatives of the six rotations of order 3 and consequently are improper rotations about the same axes. They all have determinant -1 and trace 0.

A	$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$
A^5	$\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
	about diagonal parallel to the vector	about diagonal parallel to the vector	about diagonal parallel to the vector	about diagonal parallel to the vector
axis	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

In this case, for each axis the set $\{I, A, A^2, A^3, A^4, A^5\}$ forms a subgroup of $S(C)$ isomorphic to C_6.

Example 4: by Example 2,

$$A = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

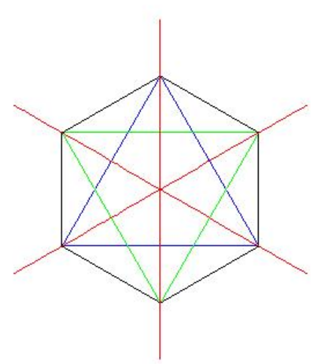


acts as a rotation of $\pi/3$ on the plane with with equation

$$x + y + x = 0.$$

What is its action on the cube? In the figures to the right, the green triangle joins the vertices of the cube in the plane

$$x + y + z = 1$$



and the blue triangle joins the vertices of the cube in the plane

$$x + y + z = -1.$$

The action of A on these triangles is to rotate a vertex on one triangle by $\pi/3$ and then to reflect it onto the other triangle. The bottom figure is the view looking down the diagonal of the cube parallel to the vector $[1\ 1\ 1]^T$.

In Summary: $S(C)$, the symmetry group of the cube has 48 elements, classified as follows:

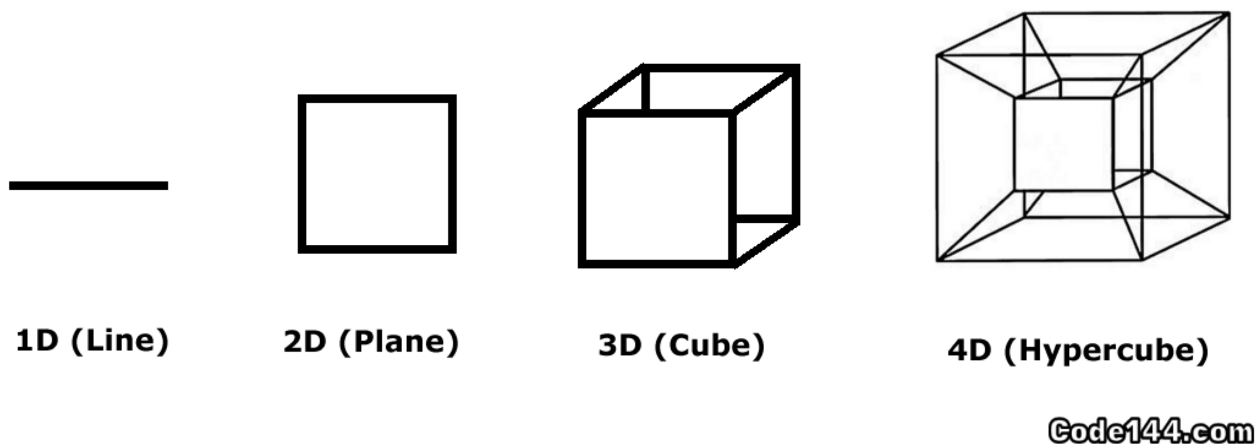
24 rotations consisting of

- one identity element, I,
- nine rotations of order 2,
- eight rotations of order 3,
- six rotations of order 4.

24 non-rotations consisting of

- one inversion element, $-I$, of order 2,
- nine reflections of order 2,
- eight improper rotations of order 6,
- six improper rotations of order 4.

The Symmetry Group of a Hypercube. Consider the figure below.



If we move a one-dimensional line perpendicular to itself through a 2-nd dimension, we obtain a square. If we now take the square and move it perpendicular to itself through a 3-rd dimension, we obtain a cube. If we take the cube and move it perpendicular to itself through a 4-th dimension we obtain a hypercube. Even though we cannot physically do this in our 3-dimensional world, we can do it mathematically. How many symmetries does the hypercube have?

1. The line with vertices ± 1 has 2 symmetries.
2. The square with vertices $(\pm 1, \pm 1)$ has four sides and has $4 \times 2 = 8 = 2^2 \times 2!$ symmetries.
3. The cube with vertices $(\pm 1, \pm 1, \pm 1)$ has six square faces and has $6 \times 8 = 48 = 2^3 \times 3!$ symmetries.
4. The hypercube with vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$ has eight cubic 'faces' and has

$$8 \times 48 = 384 = 2^4 \times 4!$$

symmetries.

(The pattern is that the number of symmetries of an n-dimensional 'cube' is $2^n \times n!$.) The symmetries of the hypercube can all be represented by the 24 possible permutations of the columns of the sixteen 4×4 matrices

$$\begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{bmatrix}.$$

While We're At It: Some Useful Observations for Future Reference.

One way to classify the order of a matrix, in terms of its trace, is to use the Cayley-Hamilton Theorem. If A is the matrix of a rotation with eigenvalues $\lambda_1 = 1, \lambda_2 = e^{i\theta}, \lambda_3 = e^{-i\theta}$, then

$$\begin{aligned} C_A(x) &= (x-\lambda_1)(x-\lambda_2)(x-\lambda_3) \\ &= x^3 - (\lambda_1+\lambda_2+\lambda_3)x^2 + (\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)x - \lambda_1\lambda_2\lambda_3 \\ &= x^3 - \mathrm{tr}\,(A)\,x^2 + \mathrm{tr}\,(A)\,x - 1 \end{aligned}$$

In particular

$$A^3 = \mathrm{tr}\,(A)\,A^2 - \mathrm{tr}\,(A)\,A + I. \tag{1}$$

Successive powers of A can be calculated in terms of I, A and A^2 recursively:

$$\begin{aligned} A^3 &= \mathrm{tr}\,(A)\,A^2 - \mathrm{tr}\,(A)\,A + I \\ \Rightarrow A^4 &= \mathrm{tr}\,(A)\,A^3 - \mathrm{tr}\,(A)\,A^2 + A \\ &= \mathrm{tr}\,(A)(\mathrm{tr}\,(A)\,A^2 - \mathrm{tr}\,(A)\,A + I) - \mathrm{tr}\,(A)\,A^2 + A \\ &= \left((\mathrm{tr}\,(A))^2 - \mathrm{tr}\,(A)\right)A^2 + \left(1 - (\mathrm{tr}\,(A))^2\right)A + \mathrm{tr}\,(A)\,I \\ \Rightarrow A^5 &= \left((\mathrm{tr}\,(A))^2 - \mathrm{tr}\,(A)\right)A^3 + \left(1 - (\mathrm{tr}\,(A))^2\right)A^2 + \mathrm{tr}\,(A)\,A \\ &= \left((\mathrm{tr}\,(A))^2 - \mathrm{tr}\,(A)\right)\left(\mathrm{tr}\,(A)\,A^2 - \mathrm{tr}\,(A)\,A + I\right) + \left(1 - (\mathrm{tr}\,(A))^2\right)A^2 + \mathrm{tr}\,(A)\,A \\ &= \left((\mathrm{tr}\,(A))^3 - 2\,(\mathrm{tr}\,(A))^2 + 1\right)A^2 - \left((\mathrm{tr}\,(A))^3 - (\mathrm{tr}\,(A))^2 - \mathrm{tr}\,(A)\right)A + \left((\mathrm{tr}\,(A))^2 - \mathrm{tr}\,(A)\right)I \end{aligned}$$

Consequently:

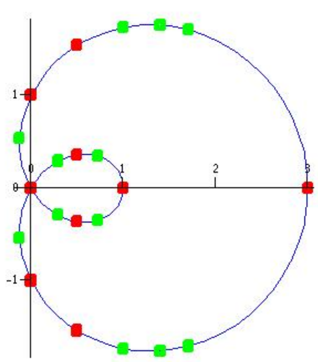
- If the order of A is 2, then $A^2 = I, A^3 = A$ and so by (1), $A = \mathrm{tr}\,(A)\,I - \mathrm{tr}\,(A)\,A + I$, implying $\mathrm{tr}\,(A) = -1$.
- If the order of A is 3, then $A^3 = I$ and so by (1), $I = \mathrm{tr}\,(A)\,A^2 - \mathrm{tr}\,(A)\,A + I$, implying $\mathrm{tr}\,(A) = 0$.
- If the order of A is 4, then $A^4 = I$ and $A^2 \neq I$, implying $\mathrm{tr}\,(A) = 1$, as you can check.
- If the order of A is 5, then $A^5 = I$, implying

  $(\mathrm{tr}\,(A))^3 - 2\,(\mathrm{tr}\,(A))^2 + 1 = 0, (\mathrm{tr}\,(A))^3 - (\mathrm{tr}\,(A))^2 - \mathrm{tr}\,(A) = 0$ and $(\mathrm{tr}\,(A))^2 - \mathrm{tr}\,(A) = 1$.

  In particular

  $$(\mathrm{tr}\,(A))^2 - \mathrm{tr}\,(A) - 1 = 0 \Rightarrow \mathrm{tr}\,(A) = \frac{1 \pm \sqrt{5}}{2} = \phi \text{ or } \psi.$$

Out of interest, you can plot the polar curve $\mathrm{tr}\,(A) = 1 + 2\cos\theta$ and see how $\mathrm{tr}\,(A)$ varies with the order of the matrix A, as long as A represents a rotation. On the limaçon to the left, the red squares represent the traces of rotation matrices of order 1, 2, 3, 4 or 5, for which



$$\theta = 0, \theta = \pi, \theta = \pm\frac{2\pi}{3}, \theta = \pm\frac{\pi}{2}, \theta = \pm\frac{2\pi}{5} \text{ or } \pm\frac{4\pi}{5},$$

respectively. They consist of the x- and y-intercepts of the limaçon, and its intersection points with the line $x = 1/2$. The green squares, for comparison, represent the traces for matrices of order 6, 7 or 8, for which there are twelve values of θ. Of course, if A represents a rotational symmetry of a Platonic solid, then the order of A can only be 1, 2, 3, 4 or 5.

The Trace of A^n: if the 3×3 matrix A represents a rotation it is possible to find a recursive formula for the trace of A^n in terms of the traces of A, A^{n-1} and A^{n-2}. First we prove a trigonometric identity:

Theorem 1 *If $n \geq 2$ then $\cos(n\theta) = 2\cos\theta\cos(n-1)\theta - \cos(n-2)\theta$.*

Proof: start with the right side:

$$\begin{aligned}
\text{RS} &= 2\cos\theta\cos(n-1)\theta - \cos(n-2)\theta \\
&= 2\cos\theta\cos(n\theta-\theta) - \cos(n\theta-2\theta) \\
&= 2\cos\theta\,(\cos(n\theta)\cos\theta + \sin(n\theta)\sin\theta) - (\cos(n\theta)\cos(2\theta) + \sin(n\theta)\sin(2\theta)) \\
&= 2\cos^2\theta\cos(n\theta) + 2\cos\theta\sin(n\theta)\sin\theta - \cos(n\theta)\cos(2\theta) - \sin(n\theta)\,2\sin\theta\cos\theta \\
&= 2\cos^2\theta\cos(n\theta) - \cos(n\theta)\,(2\cos^2\theta - 1) \\
&= 2\cos^2\theta\cos(n\theta) - 2\cos^2\theta\cos(n\theta) + \cos(n\theta) \\
&= \cos(n\theta) \\
&= \text{LS} \qquad \square
\end{aligned}$$

Now suppose A represents a rotation matrix with eigenvalues $\lambda_1 = 1, \lambda_2 = e^{i\theta}$ and $\lambda_3 = e^{-i\theta}$. Then the eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, \lambda_3^k$, and

$$\text{tr}\,(A^k) = 1 + \lambda_1^k + \lambda_2^k = 1 + 2\cos(k\theta),$$

for all $k \in \mathbb{Z}$. Thus, if $\text{tr}\,(A) = 1 + 2\cos\theta$, then

$$\text{tr}\,(A^n) = 1 + 2\cos(n\theta),\ \text{tr}\,(A^{n-1}) = 1 + 2\cos(n-1)\theta,\ \text{tr}\,(A^{n-2}) = 1 + 2\cos(n-2)\theta$$

and so

$$2\cos\theta = \text{tr}\,(A) - 1,\ \ 2\cos(n\theta) = \text{tr}\,(A^n) - 1,$$
$$2\cos(n-1)\theta = \text{tr}\,(A^{n-1}) - 1,\ 2\cos(n-2)\theta = \text{tr}\,(A^{n-2}) - 1.$$

Double both sides of the trigonometric identity proved in Theorem 1 and substitute:

$$\text{tr}\,(A^n) - 1 = (\text{tr}\,(A) - 1)\,\left(\text{tr}\,(A^{n-1}) - 1\right) - \left(\text{tr}\,(A^{n-2}) - 1\right).$$

It follows that

$$\text{tr}\,(A^n) = (\text{tr}\,(A) - 1)\,\left(\text{tr}\,(A^{n-1}) - 1\right) - \text{tr}\,(A^{n-2}) + 2;$$

and we have proved:

Theorem 2 *If $A \in O(3)$ represents a rotation, then for $n \geq 2$*

$$tr\,(A^n) = (tr\,(A) - 1)\left(tr\,(A^{n-1}) - 1\right) - tr\,(A^{n-2}) + 2.$$

Alternate proof: another way to prove this is to use the eigenvalues of A and the fact that $\lambda_2\lambda_3 = 1$:

$$\begin{aligned}
(\text{tr}\,(A) - 1)\left(\text{tr}\,(A^{n-1}) - 1\right) - \text{tr}\,(A^{n-2}) + 2 &= (\lambda_2+\lambda_3)(\lambda_2^{n-1}+\lambda_3^{n-1}) - \left(1 + \lambda_2^{n-2} + \lambda_3^{n-2}\right) + 2 \\
&= \lambda_2^n + \lambda_3^n + \lambda_2^{n-2}(\lambda_3\lambda_2 - 1) + \lambda_3^{n-2}(\lambda_2\lambda_3 - 1) + 1 \\
&= 1 + \lambda_2^n + \lambda_3^n + 0 + 0 \\
&= \text{tr}\,(A^n) \qquad \square
\end{aligned}$$

Rotation Matrices with Order 5 : let A be a matrix that represents a rotation of order 5. If $\mathrm{tr}\,(A) = \phi$, then

$$\begin{aligned} \mathrm{tr}\,(A^2) &= (\mathrm{tr}\,(A) - 1)\ (\mathrm{tr}\,(A) - 1) - \mathrm{tr}\,(I) + 2 \\ &= (\phi - 1)(\phi - 1) - 3 + 2 \\ &= \phi^2 - 2\phi + 1 - 1 \\ &= \phi + 1 - 2\phi \\ &= 1 - \phi \\ &= \psi \end{aligned}$$

This simple calculation establishes that any subgroup of $SO(3)$ that contains elements of order 5 must contain at least two conjugacy classes of elements with order 5; at least one class for elements with trace equal to ϕ and at least one class for elements with trace equal to ψ.

Rotation Matrices with Order n: if A represents a rotation of order n then

$$A^n = I,\ A^{n-1} = A^{-1} = A^T,\ \text{and}\ A^{n-2} = A^{-2}.$$

Consequently by Theorem 2, and the facts that $\mathrm{tr}\,(A^T) = \mathrm{tr}\,(A)$ and $\mathrm{tr}\,(A^{-2}) = \mathrm{tr}\,(A^2)$,

$$3 = \mathrm{tr}\,(A^n) = (\mathrm{tr}\,(A) - 1)(\mathrm{tr}\,(A^T) - 1) - \mathrm{tr}\,(A^{-2}) + 2 \Rightarrow (\mathrm{tr}\,(A) - 1)^2 = \mathrm{tr}\,(A^2) + 1.$$

Thus we have established

Theorem 3 *If the orthogonal matrix A represents a rotation of order n then*

$$(tr\,(A) - 1)^2 = tr\,(A^2) + 1.$$

(Ironically, by Theorem 2 with $n = 2$, this conclusion is true for any 3×3 rotation matrix.)

Example 5: if A represents a rotation of order 6, then A^2 has order 3, $\mathrm{tr}\,(A^2) = 0$, and

$$(\mathrm{tr}\,(A) - 1)^2 = 1 \Leftrightarrow \mathrm{tr}\,(A) = 1 \pm 1 \Leftrightarrow \mathrm{tr}\,(A) = 0 \text{ or } 2.$$

But $\mathrm{tr}\,(A) = 0$ would mean that the order of A is 3; therefore $\mathrm{tr}\,(A) = 2$. Of course this can be calculated directly by

$$\mathrm{tr}\,(A) = 1 + 2\ \cos\,(\pm 2\pi/6) = 1 + 2\ \cos(\pm\pi/3) = 1 + 2(1/2) = 2.$$

Now suppose A has order 12. Then A^2 has order 6, and so

$$(\mathrm{tr}\,(A) - 1)^2 = 2 + 1 \Leftrightarrow (\mathrm{tr}\,(A) - 1)^2 = 3 \Leftrightarrow \mathrm{tr}\,(A) = 1 \pm \sqrt{3},$$

which is equivalent to observing that

$$\cos(\pi/6) = \frac{\sqrt{3}}{2}\ \text{and}\ \cos(5\pi/6) = -\frac{\sqrt{3}}{2}.$$

Suppose A has order 24. Then A^2 has order 12, and so

$$(\mathrm{tr}\,(A) - 1)^2 = 1 \pm \sqrt{3} + 1 \Leftrightarrow (\mathrm{tr}\,(A) - 1)^2 = 2 \pm \sqrt{3} \Leftrightarrow \mathrm{tr}\,(A) = 1 \pm \sqrt{2 \pm \sqrt{3}}.$$

This is equivalent to, but easier than, calculating

$$\cos(\pi/12) = \frac{\sqrt{2 + \sqrt{3}}}{2},\ \cos(7\pi/12) = -\frac{\sqrt{2 - \sqrt{3}}}{2},$$

$$\cos(13\pi/12) = -\frac{\sqrt{2+\sqrt{3}}}{2},\ \cos(19\pi/12) = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

Now suppose A has order 48. Then A^2 has order 24, and so

$$(\mathrm{tr}\,(A)-1)^2 = 1+\mathrm{tr}\,(A^2) \Leftrightarrow (\mathrm{tr}\,(A)-1)^2 = 2\pm\sqrt{2\pm\sqrt{3}} \Leftrightarrow \mathrm{tr}\,(A) = 1\pm\sqrt{2\pm\sqrt{2\pm\sqrt{3}}}.$$

Finally suppose the order of A is 10. Then the order of A^2 is 5, $\mathrm{tr}\,(A^2) = \phi$ or ψ, and

$$(\mathrm{tr}\,(A)-1)^2 = \phi+1 \text{ or } \psi+1 \Leftrightarrow (\mathrm{tr}\,(A)-1)^2 = \phi^2 \text{ or } \psi^2 \Leftrightarrow \mathrm{tr}\,(A) = 1\pm\phi \text{ or } 1\pm\psi$$

$$\Leftrightarrow \mathrm{tr}\,(A) = \phi^2, \psi, \psi^2 \text{ or } \phi.$$

Since the order of A is *not* 5, we must have $\mathrm{tr}\,(A) = \phi^2$ or ψ^2. □

Alternate Ways to Calculate tr (A) **if** A **Has Order** n : let A be a 3×3 rotation matrix that represents a rotation of θ around its axis, $E_1(A)$. As before,

$$\mathrm{tr}\,(A) = 1 + 2\cos\theta.$$

Then A^n is a rotation of $n\,\theta$ around the same line. The order of A is the least positive integer n such that $A^n = I$, or equivalently, the least positive integer n such that $\cos(n\,\theta) = 1$. This last equation can be rephrased as $T_n(\cos\theta) = 1$, where T_n is the n-th Chebyshev polynomial of the first kind. Computationally, polynomials T_n are convenient to work with since both *Maple* and *Mathematica* can easily calculate them.

Example 6: suppose the order of A is 10. Then

$$\begin{aligned} \cos\theta &= \cos(2\pi/10),\ \cos(2\pi 3/10),\ \cos(2\pi 7/10) \text{ or } \cos(2\pi 9/10) \\ &= \cos(\pi/5),\ \cos(3\pi/5),\ \cos(7\pi/5) \text{ or } \cos(9\pi/5). \end{aligned}$$

Consider the polynomial

$$(x-\cos(\pi/5))(x-\cos(3\pi/5))(x-\cos(7\pi/5))(x-\cos(9\pi/5)).$$

If you expand and simplify, making use of *Maple* to execute the calculation, you find the polynomial is equal to

$$x^4 - x^3 - \frac{x^2}{4} + \frac{x}{4} + \frac{1}{16} = \frac{1}{16}(4x^2-2x-1)^2.$$

Note that this fourth degree polynomial factors into the square of a quadratic polynomial since

$$\cos(7\pi/5) = \cos(-3\pi/5) = \cos(3\pi/5) \text{ and } \cos(9\pi/5) = \cos(-\pi/5) = \cos(\pi/5).$$

How does this polynomial relate to $T_{10}(x)-1$? Using *Mathematica* or *Maple* we find

$$T_{10}(x) - 1 = 2(x-1)(x+1)(4x^2-2x-1)^2(4x^2+2x-1)^2,$$

which is the product of four irreducible factors. What do the other three factors represent? Using the basic composition property of Chebyshev polynomials of the first kind, namely that

$$T_{mn}(x) = T_m\left(T_n(x)\right),$$

we can say that $T_1(x)-1$, $T_2(x)-1$ and $T_5(x)-1$ must all be divisors of $T_{10}(x)-1$. For instance,

$$\begin{aligned} T_5(x) = 1 \quad &\Rightarrow \quad T_2\left(T_5(x)\right) = T_2(1) \\ &\Rightarrow \quad T_{10}(x) = 1, \text{ since } T_2(x) = 2x^2 - 1 \end{aligned}$$

Thus any root of $T_5(x) - 1 = 0$ is a root of $T_{10}(x) - 1 = 0$, so $T_5(x) - 1$ should be a divisor of $T_{10}(x) - 1$. Indeed,

$$T_1(x) - 1 = x - 1, \; T_2(x) - 1 = 2(x+1)(x-1) \text{ and } T_5(x) - 1 = (x-1)(4x^2 + 2x - 1)^2$$

are all factors of $T_{10}(x) - 1$, and together with $(4x^2 - 2x - 1)^2$ consist of all the irreducible factors of $T_{10}(x) - 1$.

Algebraically we know that if A has order 10 then A^2 has order 5. In particular, A^2 is a rotation of $2\,\theta$ with

$$\cos(2\theta) = \cos(2\pi/5) = -\frac{\psi}{2} \text{ or } \cos(2\theta) = \cos(6\pi/5) = \cos(4\pi/5) = -\frac{\phi}{2},$$

and $\operatorname{tr}(A^2) = 1 + 2\cos(2\theta) = \phi$ or ψ, respectively, confirming that the order of A^2 is 5. Now if $4x^2 + 2x - 1 = 0$ then

$$x = \frac{-2 \pm \sqrt{20}}{8} = -\frac{\phi}{2} \text{ or } -\frac{\psi}{2};$$

that is, the roots of this irreducible factor are the values of cosine for a rotation of order 5. Similarly, if A has order 10, then A^5 has order 2 and A^5 is a rotation of $5\,\theta$ with

$$\cos(5\theta) = \cos\pi = -1,$$

which is exactly the root of the irreducible factor $x + 1$. Finally, if A has order 10 then $A^{10} = I$ has order 1 and A^{10} is a rotation of $10\,\theta$ where

$$\cos(10\,\theta) = \cos(2\pi) = 1,$$

which is exactly the root of the irreducible factor $x - 1$. In general, if k is a divisor of n then $T_k(x) - 1$ is a divisor of $T_n(x) - 1$, and the number of distinct irreducible polynomial factors of $T_n(x) - 1$ is the number of divisors of n. □

Polynomials Named Ψ: these irreducible polynomials are commonly[1] denoted by Ψ_d; e.g.

$$\Psi_1(x) = x - 1, \; \Psi_2(x) = 2x + 2, \; \Psi_5(x) = 4x^2 + 2x - 1, \; \Psi_{10}(x) = 4x^2 - 2x - 1.$$

The general definition of Ψ_d, for $d > 2$ is

$$\Psi_d(x) = \prod_{k \in S_{d/2}} 2\left(x - \cos\left(2\pi\,\frac{k}{d}\right)\right),$$

with $S_{d/2} = \{k \mid (k,d) = 1, 1 \le k < d/2\}$ and $d > 2$. The degree of Ψ_d is $\phi(d)/2$, where ϕ is Euler's phi function. The first twenty are listed in Table 1.

As we have seen, the polynomial $\Psi_d(x)$ is a factor of $T_n(x) - 1$, if $d|n$. According to D. A. Wolfram, if n is even,

$$T_n(x) - 1 = \Psi_1(x)\Psi_2(x) \prod_{\substack{d\,|\,n \\ d > 2}} \Psi_d(x)^2$$

[1] D. A. Wolfram, *Factoring Variants of Chebyshev Polynomials of the First and Second Kinds with Minimal Polynomials of* $\cos(2\pi/d)$, The American Mathematical Monthly, 129:2, 172-176. Note we have defined Ψ_1 slightly differently.

d	Irreducible polynomials $\Psi_d(x)$, with degree $\phi(d)/2$ if $d > 2$	order of A
1	$x - 1$	1
2	$2x + 2$	2
3	$2x + 1$	3
4	$2x$	4
5	$4x^2 + 2x - 1$	5
6	$2x - 1$	6
7	$8x^3 + 4x^2 - 4x - 1$	7
8	$4x^2 - 2$	8
9	$8x^3 - 6x + 1$	9
10	$4x^2 - 2x - 1$	10
11	$32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1$	11
12	$4x^2 - 3$	12
13	$64x^6 + 32x^5 - 80x^4 - 32x^3 + 24x^2 + 6x - 1$	13
14	$8x^3 - 4x^2 - 4x + 1$	14
15	$16x^4 - 8x^3 - 16x^2 + 8x + 1$	15
16	$16x^4 - 16x^2 + 2$	16
17	$256x^8 + 128x^7 - 448x^6 - 192x^5 + 240x^4 + 80x^3 - 40x^2 - 8x + 1$	17
18	$8x^3 - 6x - 1$	18
19	$512x^9 + 256x^8 - 1024x^7 - 448x^6 + 672x^5 + 240x^4 - 160x^3 - 40x^2 + 10x + 1$	19
20	$16x^4 - 20x^2 + 5$	20

Table 1: Polynomials $\Psi_d(x)$ for $1 \le d \le 20$, and the order of the rotation matrix A with $\mathrm{tr}\,(A) = 1 + 2x$, where $x = \cos\theta$ and $\Psi_d(x) = 0$. If $d > 1$ and Ψ_d has degree k then the leading coefficient is 2^k; if d is a power of 2, then the coefficients of $\Psi_d(x)$ have common factor 2.

and if n is odd,

$$T_n(x) - 1 = \Psi_1(x) \prod_{\substack{d\,|\,n \\ d > 2}} \Psi_d(x)^2.$$

So one way to find the polynomials Ψ_d is to factor $T_n(x) - 1$, as we did in Example 6, using *Maple* or *Mathematica.*

For example, it's easy to calculate Ψ_3 directly:

$$\Psi_3(x) = 2(x - \cos(2\pi/3)) = 2\left(x + \frac{1}{2}\right) = 2x + 1.$$

Then using *Mathematica* we find

$$T_{15}(x) - 1 = (x - 1)(2x + 1)^2(4x^2 + 2x - 1)^2(16x^4 - 8x^3 - 16x^2 + 8x + 1)^2.$$

Since we already know

$$\Psi_1(x) = x - 1,\ \Psi_3(x) = 2x + 1 \text{ and } \Psi_5(x) = 4x^2 + 2x - 1,$$

the remaining factor must be $\Psi_{15}(x)$, namely

$$\Psi_{15}(x) = 16x^4 - 8x^3 - 16x^2 + 8x + 1.$$

n, order of A	$\Psi_n((t-1)/2)=0$ with $t=\mathrm{tr}\,(A)$	values of $\mathrm{tr}\,(A)$
1	$\frac{t}{2}-\frac{3}{2}=0$	3
2	$t+1=0$	-1
3	$t=0$	0
4	$t-1=0$	1
5	$t^2-t-1=0$	$\phi,\ \psi$
6	$t-2=0$	2
8	$t^2-2t-1=0$	$1\pm\sqrt{2}$
10	$t^2-3t+1=0$	$\phi^2,\ \psi^2$
12	$t^2-2t-2=0$	$1\pm\sqrt{3}$
16	$t^4-4t^3+2t^2+4t-1=0$	$1\pm\sqrt{2\pm\sqrt{2}}$
20	$t^4-4t^3+t^2+6t+1=0$	$1\pm\frac{\sqrt{10\pm2\sqrt{5}}}{2}$
24	$t^4-4t^3+2t^2+4t-2=0$	$1\pm\frac{\sqrt{6}}{2}\pm\frac{\sqrt{2}}{2}$
32	$t^8-8t^7+20t^6-8t^5-30t^4+24t^3+12t^2-8t-1=0$	$1\pm\sqrt{2\pm\sqrt{2\pm\sqrt{2}}}$

Table 2: Values of $\mathrm{tr}\,(A)$ if A is a 3×3 rotation matrix of order n, for some values of n.

Similarly, *Mathematica* tells us

$$T_{20}(x)-1=8(x-1)x^2(x+1)(4x^2-2x-1)^2(4x^2+2x-1)^2(16x^4-20x^2+5)^2.$$

Since we already know

$$\Psi_1(x)=x-1,\ \Psi_2(x)=2x+2,\ \Psi_4(x)=2x,\ \Psi_5(x)=4x^2+2x-1,\ \Psi_{10}(x)=4x^2-2x-1,$$

we can conclude that

$$\Psi_{20}(x)=16x^4-20x^2+5.$$

This is how we filled out Table 1.

We now use the irreducible factors Ψ_d to calculate $\mathrm{tr}\,(A)$ if A is a 3×3 rotation matrix of order n. If $\mathrm{tr}\,(A)=1+2\cos\theta$ then

$$\cos\theta=\frac{\mathrm{tr}\,(A)-1}{2}.$$

So if the order of A is n then

$$\Psi_n\left(\frac{\mathrm{tr}\,(A)-1}{2}\right)=0,$$

which we can solve for $\mathrm{tr}\,(A)$. See Table 2 in which we have only included some values of n for which the algebra works out reasonably well.

Of course a matrix A that represents a rotation of angle θ has finite order if and only if θ is a rational multiple of π, if and only if $\cos\theta$ is a root of $\Psi_n(x)$, for some n. Looking at Table 1 you can see that a quick way to demonstrate that a rotation matrix does *not* have finite order is to find an irreducible polynomial $f(x)$ with leading coefficient which is not a power of 2 such that $f(\cos\theta) = 0$. As a simple example consider a 3×3 rotation matrix A with $\text{tr}\,(A) = 5/3$. Then

$$\cos\theta = \frac{5/3 - 1}{2} = \frac{1}{3}$$

and $\cos\theta$ is a root of the irreducible polynomial $3x - 1$. Since its leading coefficient is not a power of 2, θ can't be a rational multiple of π and A can't have finite order.

Symmetric Orthogonal Matrices: if A is symmetric and orthogonal, then

$$A^{-1} = A^T = A,$$

and $A^2 = I$. Except for $\pm I$, a symmetric orthogonal matrix represents a rotation of order 2 or a reflection. We claim that:

Theorem 4 *If $Q \in O(3)$ represents a reflection in the plane passing through the origin normal to the unit vector $\vec{v}$, then*

$$Q = I - 2\vec{v}\,\vec{v}^T;$$

and if $R \in O(3)$ represents a rotation of order 2 around the axis parallel to the unit vector $\vec{v}$, then

$$R = 2\vec{v}\,\vec{v}^T - I.$$

Note that $tr\,(Q) = 3 - 2\,\vec{v}\cdot\vec{v} = 3 - 2 = 1$ and $tr\,(R) = 2\,\vec{v}\cdot\vec{v} - 3 = 2 - 3 = -1$.

Proof: pick another unit vector $\vec{u}$ which is orthogonal to $\vec{v}$. Then $\{\vec{u}, \vec{v}, \vec{u}\times\vec{v}\}$ is an orthonormal basis of $\mathbb{R}^3$ and the matrix

$$A = [\ \vec{u}\ |\ \vec{v}\ |\ \vec{u}\times\vec{v}\ ]$$

must be an orthogonal matrix, implying that $AA^T = I$. Consequently

$$[\ \vec{u}\ |\ \vec{v}\ |\ \vec{u}\times\vec{v}\ ]\begin{bmatrix} \vec{u}^T \\ \vec{v}^T \\ (\vec{u}\times\vec{v})^T \end{bmatrix} = I \Leftrightarrow \vec{u}\vec{u}^T + \vec{v}\vec{v}^T + (\vec{u}\times\vec{v})(\vec{u}\times\vec{v})^T = I.$$

Now if Q represents a reflection in the plane passing through the origin with normal vector $\vec{v}$, then

$$Q\,\vec{u} = \vec{u},\ \ Q\,\vec{v} = -\vec{v} \text{ and } Q\,(\vec{u}\times\vec{v}) = \vec{u}\times\vec{v},$$

since the plane fixed by Q is spanned by $\vec{u}$ and $\vec{u}\times\vec{v}$. In terms of matrix multiplication we have

$$QA = [\ \vec{u}\ |\ -\vec{v}\ |\ \vec{u}\times\vec{v}\ ].$$

Solve for Q:

$$\begin{aligned} Q &= [\ \vec{u}\ |\ -\vec{v}\ |\ \vec{u}\times\vec{v}\ ]\,A^T \\ &= [\ \vec{u}\ |\ -\vec{v}\ |\ \vec{u}\times\vec{v}\ ]\begin{bmatrix} \vec{u}^T \\ \vec{v}^T \\ (\vec{u}\times\vec{v})^T \end{bmatrix} \\ &= \vec{u}\vec{u}^T - \vec{v}\vec{v}^T + (\vec{u}\times\vec{v})(\vec{u}\times\vec{v})^T \\ &= \vec{u}\vec{u}^T + \vec{v}\vec{v}^T + (\vec{u}\times\vec{v})(\vec{u}\times\vec{v})^T - 2\vec{v}\vec{v}^T \\ &= I - 2\vec{v}\vec{v}^T. \end{aligned}$$

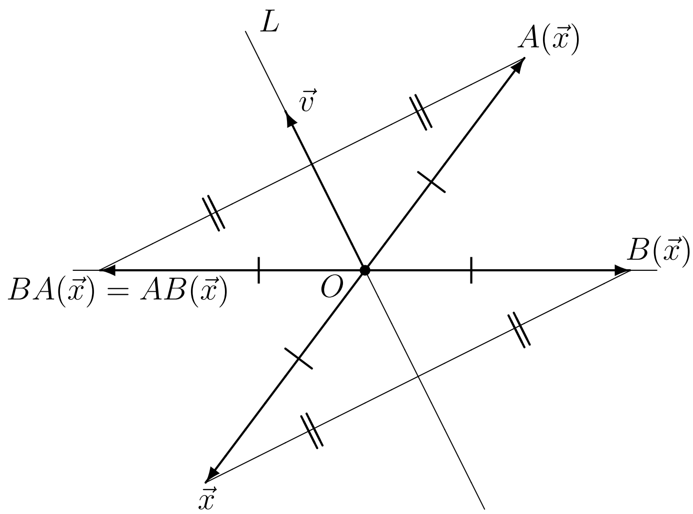


Figure 2: A is a rotation of π around O in the plane of the figure; $\vec{u}$ is orthgonal to the plane. B is a rotation of π around the line L parallel to $\vec{v}$ which is contained in the plane of the figure.

Moreover, $-Q$ represents a rotation of order 2 around an axis parallel to the vector $\vec{v}$, since

$$(-Q)\vec{v} = 2\vec{v}\,\vec{v}^T\vec{v} - \vec{v} = 2\vec{v} - \vec{v} = \vec{v},$$

where we have used the fact that $\vec{v}^T\vec{v} = \vec{v}\cdot\vec{v} = 1$. Take $R = -Q$. $\square$

In particular, it is easy to determine if symmetric orthogonal matrices in $O(3)$ commute:

Theorem 5 *Let $\vec{u}, \vec{v}$ be unit vectors in $\mathbb{R}^3$; let $A = \pm(I - 2\vec{u}\vec{u}^T), B = \pm(I - 2\vec{v}\vec{v}^T)$ be symmetric orthogonal matrices, $A, B \neq \pm I$. Then*

$$AB = BA$$

if and only if the vectors $\vec{u}$ and $\vec{v}$ are orthogonal, or parallel.

Proof:

$$\begin{aligned} AB = BA &\Leftrightarrow (I - 2\vec{u}\vec{u}^t)(I - 2\vec{v}\vec{v}^T) = (I - 2\vec{v}\vec{v}^T)(I - 2\vec{u}\vec{u}^T) \\ &\Leftrightarrow I - 2\vec{u}\vec{u}^T - 2\vec{v}\vec{v}^T + 4\vec{u}\vec{u}^T\vec{v}\vec{v}^T = I - 2\vec{v}\vec{v}^T - 2\vec{u}\vec{u}^T + 4\vec{v}\vec{v}^T\vec{u}\vec{u}^T \\ &\Leftrightarrow \vec{u}(\vec{u}\cdot\vec{v})\vec{v}^T = \vec{v}(\vec{v}\cdot\vec{u})\vec{u}^T \\ &\Leftrightarrow \vec{u}\cdot\vec{v} = 0 \text{ or } \vec{u}\vec{v}^T = \vec{v}\vec{u}^T \end{aligned}$$

Check that

$$\vec{u}\vec{v}^T = \vec{v}\vec{u}^T \Leftrightarrow u_2v_1 = u_1v_2,\ u_1v_3 = u_3v_1,\ u_3v_2 = u_2v_3 \Leftrightarrow \vec{u}\times\vec{v} = \vec{0}.$$

Thus A and B commute if and only if the vectors $\vec{u}$ and $\vec{v}$ are orthogonal, or parallel. $\square$

The theorem is illustrated geometrically in Figure 2 for A and B both rotations of order 2 and $\vec{u}\cdot\vec{v} = 0$: $\vec{u}$ is normal to the plane of the page, the line L is parallel to $\vec{v}$ and is in the plane of the page, and $\vec{x}$ is an arbitrary vector in the plane of the page. By following geometrically the calculations of both $BA(\vec{x})$ and $AB(\vec{x})$ you can see that $BA(\vec{x}) = AB(\vec{x})$. What if $\vec{z} \in \mathbb{R}^3$ is not in the plane of the page? Then $\vec{z} = \vec{x} + \vec{y}$, with $\vec{x}$ in the plane of the page and $\vec{y}$ parallel to $\vec{u}$. Consequently $A(\vec{y}) = \vec{y}$ and $B(\vec{y}) = -\vec{y}$, from which it follows that $AB(\vec{y}) = BA(\vec{y})$. So $AB(\vec{z}) = BA(\vec{z})$ as well.

By considering B alone, Figure 2 also illustrates the fact that in $\mathbb{R}^3$, every rotation of order 2, restricted to a plane containing its axis of rotation, acts as a reflection in its axis of rotation.

Products of Reflections and Rotations: As in $O(2)$, the product of two reflections in $O(3)$ is a rotation, and the product of two rotations in $O(3)$ is a rotation. This follows immediately from properties of the determinant. Unlike $O(2)$, however, the product of a reflection in $O(3)$ and a rotation in $O(3)$ need not be a reflection: the product could result in an improper rotation. For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

In this example, a matrix representing a reflection multiplied by a matrix representing a rotation of order 4 produces a matrix representing an improper rotation of order 4.

The rotation that results from the product of two reflections is actually easy to describe. Let $\vec{u}$ and $\vec{v}$ be non-parallel unit vectors and consider the two reflection matrices

$$Q_1 = I - 2\vec{u}\vec{u}^T \text{ and } Q_2 = I - 2\vec{v}\vec{v}^T.$$

Then $\det(Q_1Q_2) = 1$, and

$$Q_1Q_2 = I - 2\vec{u}\vec{u}^T - 2\vec{v}\vec{v}^T + 4\vec{u}\vec{u}^T\vec{v}\vec{v}^T = I - 2\vec{u}\vec{u}^T - 2\vec{v}\vec{v}^T + 4(\vec{u}\cdot\vec{v})\vec{u}\vec{v}^T$$

$$\Rightarrow \operatorname{tr}(Q_1Q_2) = 3 - 2 - 2 + 4(\vec{u}\cdot\vec{v})\operatorname{tr}(\vec{u}\vec{v}^T) = -1 + 4(\vec{u}\cdot\vec{v})^2,$$

since $\operatorname{tr}(\vec{u}\vec{u}^T) = \operatorname{tr}(\vec{v}\vec{v}^T) = 1, \vec{u}^T\vec{v} = \vec{u}\cdot\vec{v}$ and $\operatorname{tr}(\vec{u}\vec{v}^T) = \vec{u}\cdot\vec{v}$, as you can check. Also

$$Q_1Q_2(\vec{u}\times\vec{v}) = \vec{u}\times\vec{v} - 2\vec{u}\vec{u}^T(\vec{u}\times\vec{v}) - 2\vec{v}\vec{v}^T(\vec{u}\times\vec{v}) + 4(\vec{u}\cdot\vec{v})\,\vec{u}\vec{v}^T\,(\vec{u}\times\vec{v}) = \vec{u}\times\vec{v},$$

since $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. Thus Q_1Q_2 represents a rotation around the line passing through the origin parallel to the vector $\vec{u}\times\vec{v}$. To find the angle of rotation, θ, we can let $\operatorname{tr}(Q_1Q_2) = 1 + 2\cos\theta$. Then

$$1 + 2\cos\theta = -1 + 4(\vec{u}\cdot\vec{v})^2 \Leftrightarrow \cos^2(\theta/2) = (\vec{u}\cdot\vec{v})^2 \Leftrightarrow \cos^2(\theta/2) = \cos^2\alpha,$$

where $\alpha \in (0,\pi)$ is the angle between the vectors $\vec{u}$ and $\vec{v}$. It's just a case of picking appropriate unit vectors $\vec{u}$ and $\vec{v}$. Note that if $Q_1Q_2 = R$ then $Q_2Q_1 = R^{-1}$.

Theorem 6 *Every orthogonal matrix in $O(3)$ can be expressed as a product of reflections.*

Proof: Suppose $R \in O(3)$. If $R = I$ then pick any reflection matrix Q. Since Q has order 2, $I = Q\,Q$. Now suppose $R \neq I$ and R represents a rotation of $\theta \neq 2\pi k$ around a line parallel to the unit vector $\vec{w}$. For any two non-parallel unit vectors $\vec{u}$ and $\vec{v}$ in $(E_1(R))^\perp$ we have $\vec{u}\times\vec{v}$ parallel to $\vec{w}$. Pick $\vec{u},\vec{v}$ so that $\vec{u}\cdot\vec{v} = \cos(\theta/2)$. Then $R = Q_1Q_2$ for

$$Q_1 = I - 2\vec{u}\vec{u}^T \text{ and } Q_2 = I - 2\vec{v}\vec{v}^T.$$

Finally, if $P \in O(3)$ represents an improper rotation then $-P$ represents a rotation; let $R = -P$ and write R as a product of reflections, as above. Then $P = -R = (-I)R$, and $-I$ is also a product of reflections, namely

$$-I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

so we are done. $\square$

Example 7: let

$$R = \frac{1}{7}\begin{bmatrix} -2 & 6 & 3 \\ 3 & -2 & 6 \\ 6 & 3 & -2 \end{bmatrix}$$

which represents a rotation of θ around the axis parallel to the vector $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ such that

$$\text{tr}\,(R) = 1 + 2\cos\theta = -6/7 \Leftrightarrow \cos^2(\theta/2) = 1/28.$$

A basis of unit vectors $\{\vec{u}, \vec{v}\}$ for the plane with equation $x+y+z=0$ such that $\vec{u}\cdot\vec{v} = 1/(2\sqrt{7})$ is

$$\vec{u} = \begin{bmatrix} 2/\sqrt{14} \\ -3/\sqrt{14} \\ 1/\sqrt{14} \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix};$$

let $Q_1 = I - 2\vec{u}\vec{u}^T$, $Q_2 = 1 - 2\vec{v}\vec{v}^T$. Then

$$Q_1Q_2 = \frac{1}{7}\begin{bmatrix} 3 & 6 & -2 \\ 6 & -2 & 3 \\ -2 & 3 & 6 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -2 & 6 & 3 \\ 3 & -2 & 6 \\ 6 & 3 & -2 \end{bmatrix} = R,$$

as you can check.

Or consider the rotation matrix

$$R = \frac{1}{7}\begin{bmatrix} -3 & 2 & -6 \\ -2 & 6 & 3 \\ 6 & 3 & -2 \end{bmatrix},$$

which represents a rotation around the axis parallel to

$$\vec{w} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

with $\text{tr}\,(R) = \frac{1}{7}$. Then

$$\text{tr}\,(R) = 1 + 2\cos\theta = \frac{1}{7} \Leftrightarrow \cos^2(\theta/2) = \frac{2}{7}.$$

Thus we need to find two non-parallel vectors $\vec{u}, \vec{v}$ in the plane with equation $3y + z = 0$ such that $\vec{u}\cdot\vec{v} = \sqrt{\frac{2}{7}}$. After much trial and error we find

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v} = \frac{1}{\sqrt{14}}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

Then Q_1 is a reflection in the plane $x = 0$ but Q_2 is a bit more complicated:

$$Q_1 = I - 2\vec{u}\,\vec{u}^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ Q_2 = I - 2\vec{v}\,\vec{v}^T = \frac{1}{7}\begin{bmatrix} 3 & -2 & 6 \\ -2 & 6 & 3 \\ 6 & 3 & -2 \end{bmatrix}$$

and

$$Q_1Q_2 = \frac{1}{7}\begin{bmatrix} -3 & 2 & -6 \\ -2 & 6 & 3 \\ 6 & 3 & -2 \end{bmatrix} = R,$$

as you can check. Of course if we had interchanged our choices of $\vec{u}$ and $\vec{v}$ then, in effect, we would have calculated $Q_2Q_1 = R^{-1}$. But that is also a rotation around the axis parallel to the vector $\vec{w}$ with trace $1/7$. □

Part 2: Symmetry and Counter-Symmetry. From Part 1 we know that the symmetry group of the cube, $S(C)$, consists of the following 48 matrices, with usual matrix multiplication:

$$\begin{bmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{bmatrix}, \begin{bmatrix} 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \\ \pm1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \pm1 \\ \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} \pm1 & 0 & 0 \\ 0 & 0 & \pm1 \\ 0 & \pm1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \pm1 \\ 0 & \pm1 & 0 \\ \pm1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \pm1 & 0 \\ \pm1 & 0 & 0 \\ 0 & 0 & \pm1 \end{bmatrix}.$$

We also found that if A is an orthogonal 3×3 matrix then the geometric transformation represented by A can be classified depending on $\det(A)$ and $\text{tr}\,(A)$, as summarized below:

$\det(A)$	$\text{tr}\,(A)$	action of A	order of A	$\det(A)$	$\text{tr}\,(A)$	action of A	order of A
1	3	identity, $A = I$	1	-1	-3	inversion, $A = -I$	2
1	-1	rotation	2	-1	1	reflection	2
1	0	rotation	3	-1	0	improper rotation	6
1	1	rotation	4	-1	-1	improper rotation	4
1	ψ	rotation	5	-1	$-\psi$	improper rotation	10
1	ϕ	rotation	5	-1	$-\phi$	improper rotation	10
1	2	rotation	6	-1	-2	improper rotation	6

where

$$\phi = \frac{1+\sqrt{5}}{2} \text{ and } \psi = \frac{1-\sqrt{5}}{2}.$$

Thus we were able to conclude that the symmetry group of the cube has 48 elements, classified geometrically as follows:

24 rotations consisting of

- one identity element, I,
- nine rotations of order 2,
- eight rotations of order 3,
- six rotations of order 4.

24 non-rotations consisting of

- one inversion element, $-I$, of order 2,
- nine reflections of order 2,
- eight improper rotations of order 6,
- six improper rotations of order 4.

In Parts 2 and 3 we will look at the symmetry group of the cube less in terms of geometry and more in terms of algebra. To this end we will define various types of matrices, e.g. signed matrices, counter-diagonal matrices, counter-symmetric matrices, and deranged matrices, to name a few. Although we could state all these definitions for arbitrary matrices we shall limit ourselves to 3×3 real matrices. We shall also define various mappings and homomorphisms. But to begin with we shall formalize some of the observations we have already made in Part 1.

Structure of $S(C)$, First Pass: let $\mathcal{R}$ be the subgroup of $S(C)$ consisting of the 24 rotations of the cube. In Part I we established that $\mathcal{R} \cong S_4$. Since $\mathcal{R}$ has index 2 in $S(C)$ it is a normal subgroup of $S(C)$. From Part I we also know that every non-rotation in $S(C)$ is obtained by multiplying a rotation by $-I$. Let $\mathcal{Z} = \{I, -I\}$. It is easy to check that $\mathcal{Z}$ is also a normal subgroup of $S(C)$. In fact, $\mathcal{Z}$ is the center of $S(C)$. Thus

$$S(C) = \mathcal{R}\mathcal{Z} \cong S_4 \times C_2.$$

Signed Variations of a Matrix: if $A = [a_{ij}]$ call any one of the up to 512 possible matrices

$$\begin{bmatrix} \pm a_{11} & \pm a_{12} & \pm a_{13} \\ \pm a_{21} & \pm a_{22} & \pm a_{23} \\ \pm a_{31} & \pm a_{32} & \pm a_{33} \end{bmatrix}$$

a signed variation of A. Then $S(C)$ can be described as the group of all 48 signed variations of the six 3×3 permutation matrices,

$$P_{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; P_{(123)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \; P_{(132)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$P_{(23)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; P_{(13)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; P_{(12)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Structure of $S(C)$, Second Pass: let $\mathcal{G} = \{P_{(1)}, P_{(123)}, P_{(132)}, P_{(23)}, P_{(13)}, P_{(13)}\}$ be the subgroup of $S(C)$ consisting of the six 3×3 permutation matrices and let

$$\mathcal{D} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \middle| \, a = \pm 1, b = \pm 1, c = \pm 1 \right\}$$

be the subgroup of $S(C)$ consisting of all its diagonal matrices. Define a map $\Phi : S(C) \to S(C)$ by

$$\Phi\left((a_{ij})\right) = (a_{ij}^2)$$

for any matrix $A = (a_{ij})$ in $S(C)$. Then Φ is, perhaps surprisingly, a homomorphism:

$$\begin{aligned} \Phi(AB) = \Phi(A)\Phi(B) \;&\Leftrightarrow\; \left((a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j})^2\right) = \left(a_{i1}^2 b_{1j}^2 + a_{i2}^2 b_{2j}^2 + a_{i3}^2 b_{3j}^2\right) \\ &\Leftrightarrow\; (a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j})^2 = a_{i1}^2 b_{1j}^2 + a_{i2}^2 b_{2j}^2 + a_{i3}^2 b_{3j}^2 \\ &\Leftrightarrow\; a_{i1}a_{i2}b_{1j}b_{2j} + a_{1i}a_{i3}b_{1j}b_{3j} + a_{i2}a_{i3}b_{2j}b_{3j} = 0, \end{aligned}$$

and this last equation is true since for any matrix A or B in $S(C)$ every row of A has two zeros, as does every column of B. Check that:

1. $\Phi^2 = \Phi$,
2. $\ker(\Phi) = \mathcal{D}$,
3. $\text{im}\,(\Phi) = \mathcal{G}$.

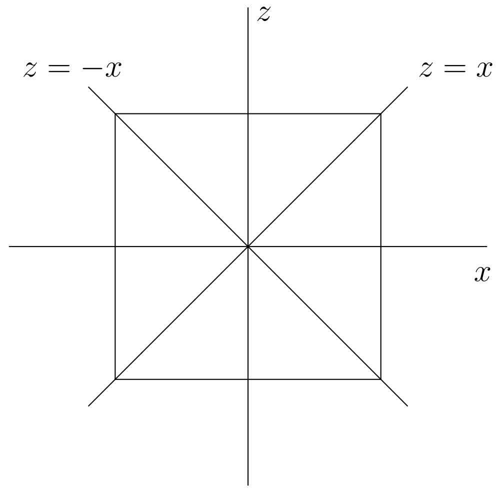


Figure 3: An arbitrary square, centered at the origin, in the plane with equation $y = 0$. Table 3 contains four copies of D_4 which each act as the symmetry group of this square.

By the First Isomorphism Theorem we can say

$$S(C)/\mathcal{D} \cong \mathcal{G} \cong S_3 \cong D_3.$$

Since $\mathcal{D}$ is a normal subgroup of $S(C)$ and $\mathcal{D} \cap \mathcal{G} = \{I\}$ we can now formalize our observation of the previous page as

$$S(C) = \mathcal{G}\mathcal{D} = \{QD \mid Q \in \mathcal{G}, D \in \mathcal{D}\}.$$

Aside: since $\mathcal{G}$ is *not* normal in $S(C)$ it is *not* true that $S(C) \cong \mathcal{G} \times \mathcal{D}$.

Diagonal and Counter-Diagonal Matrices: the counter-diagonal of a 3×3 matrix is the diagonal that goes from the bottom left to the top right of the matrix. A is called a counter-diagonal matrix if all the entries off the counter-diagonal are zero. A typical 3×3 counter-diagonal matrix A looks like

$$A = \begin{bmatrix} 0 & 0 & c \\ 0 & e & 0 \\ g & 0 & 0 \end{bmatrix}.$$

It is easy to check that the product of two diagonal matrices is a diagonal matrix, the product of two counter-diagonal matrices is a diagonal matrix, and the product of a counter-diagonal matrix and a diagonal matrix is a counter-diagonal matrix. There are eight possible orthogonal diagonal matrices, and eight possible orthogonal counter-diagonal matrices. See Table 3. As a group, these 16 matrices form a group isomorphic to $D_4 \times C_2$, as we shall see.

The eight matrices in Table 3 that represent rotations form a group isomorphic to D_4. Why should this be? D_4 is canonically interpreted as the symmetry group of the square, and with respect to the square, contains a cyclic group generated by a rotation of $\pi/2$, and four reflections in the symmetry axes of the square. See Figure 3. The cyclic group of order 4 consists of the four rotation matrices in the top left corner of Table 3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

These are all rotations around the y-axis, so act on the square in Figure 3 exactly as required. The four remaining rotations, all of order 2, from the bottom left corner of Table 3, are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

rotation matrix A	$\det(A)$	$\operatorname{tr}(A)$	order of A	non-rotation matrix A	$\det(A)$	$\operatorname{tr}(A)$	order of A
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1	3	1	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	-1	-3	2
$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	1	1	4	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	-1	-1	4
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	1	-1	2	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	-1	1	2
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	1	1	4	$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	-1	-1	4
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	1	-1	2	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	-1	1	2
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	1	-1	2	$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	-1	1	2
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1	-1	2	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	-1	1	2
$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	1	-1	2	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	-1	1	2

Table 3: The group of 16 orthogonal matrices which are diagonal or counter-diagonal. The eight diagonal matrices form a copy of $C_2 \times C_2 \times C_2$. The eight rotation matrices form a copy of D_4. This group of 16 matrices is isomorphic to $D_4 \times C_2$ and is a 2-Sylow subgroup of $S(C)$.

Their axes of rotations are lines passing through the origin parallel to the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

respectively. All these vectors are in the plane $y = 0$. These four rotations of order 2 restricted to the plane with equation $y = 0$ act as reflections, respectively, in the x-axis, in the line $z = x$, in the z-axis, and in the line $z = -x$, as shown in Figure 3.

There is another copy of D_4 in Table 3. It consists of the four rotation matrices from the top left hand corner of Table 3 and the four reflection matrices from the bottom right hand corner. These last four matrices represent true reflections in the planes $x = 0, x = -z, z = 0$ and $x = z$. Restricted to the square in Figure 3, this version of D_4 acts exactly as the symmetries of a square in the plane.

There are two more copies of D_4 in Table 3; each including the cyclic group of order 4

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}.$$

Here the 'rotations' of order 4 are the two improper rotations of order 4 around the y-axis. Restricted to the plane $y = 0$ these two improper rotations act the same as proper rotations. The 'reflections' in the symmetry axes of the square, in one additional copy of D_4, are represented by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix};$$

by their negatives, in the other copy, as you can check. In each of these two copies of D_4, of the four matrices that act as reflections on the square in Figure 3, two are true reflections, and two are rotations of order 2.

The Symmetry Group of a 4-Prism: more naturally, the sixteen matrices in Table 3 form the symmetry group of a 3-dimensional 4-prism, namely a 4-prism centered around the y-axis and symmetric with respect to the origin, having coordinates $(\pm a, \pm a, 0)$ and $(0, \pm a, \pm a)$. See Figure 4, in which $a = 1$. Each end of the 4-prism in Figure 4 is a square with side length $\sqrt{2}$; the parallel lines joining the vertices of the squares each have length 2, the length of the prism.

Viewed as the symmetries of a 4-prism, the eight rotation matrices from Table 3 all act as true rotations. The four matrices in the top left hand corner of Table 3 represent the four possible rotational symmetries of the 4-prism around the y-axis; the four matrices in the bottom left hand corner of Table 3 represent four rotations of order 2,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

around the remaining symmetry axes of the 4-prism: the x-axis, the z-axis, and the orthogonal complements of the planes with equation $x \pm z = 0$. See Figure 4, in which these axes are marked in red.

The remaining matrices in Table 3 are the above eight rotations multiplied by $-I$. These eight matrices consist of: inversion; two improper rotations of order 4 around the y-axis: and five reflections, in the planes with equations

$$x = 0,\ y = 0,\ z = 0,\ x + z = 0 \text{ and } x - z = 0,$$

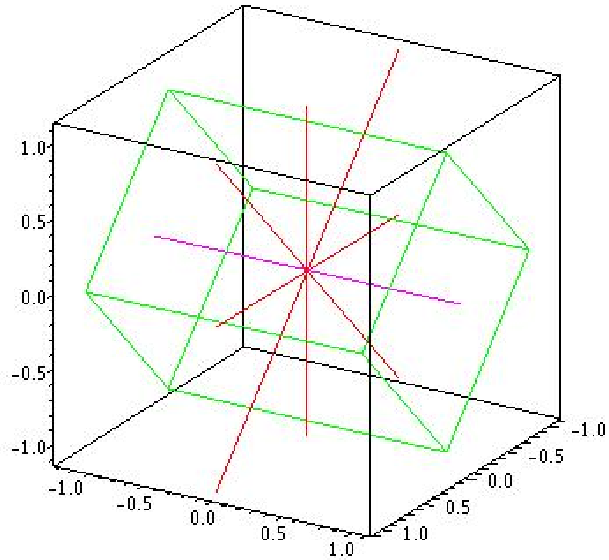


Figure 4: The 4-prism centered along the y-axis, in magenta, with vertices $(\pm1, \pm1, 0)$ and $(0, \pm1, \pm1)$. The sixteen matrices in Table 3 form the symmetry group of this prism, $D_4 \times C_2$.

as you can check. Although the two improper rotations are products of $-I$ and the two rotation matrices of order 4, the improper rotations are actually 4-fold improper rotations. That is, they can be obtained by multiplying the rotation matrices of order 4 around the y-axis, by the matrix representing a reflection in the y-axis:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Finally, the eight matrices from the top half of Table 3 form a subgroup isomorphic to $C_4 \times C_2$. In terms of the 4-prism, these eight symmetries are the full symmetry group of the 4-prism if the two pairs of opposite rectangular sides are a different colour. (CHECK.)

The sixteen matrices in Table 3 are all in $S(C)$ and the 4-prism in Figure 4 sits in the cube with vertices $(\pm1, \pm1, \pm1)$. There are two other such 4-prisms: one parallel to the x-axis, with vertices $(\pm1, 0, \pm1)$ and $(\pm1, \pm1, 0)$; and one parallel to the z-axis with vertices $(0, \pm1, \pm1)$ and $(\pm1, 0, \pm1)$. They too have their symmetry groups among the subgroups of $S(C)$. Indeed, these are the 2-Sylow subgroups of $S(C)$.

The Sylow Subgroups of $S(C)$:

$$|S(C)| = 48 = 2^4 \cdot 3.$$

Each 2-Sylow subgroup of $S(C)$ has order 16 and each 3-Sylow subgroup of $S(C)$ has order 3. Since the subgroup of 16 matrices in Table 3 is not normal in $S(C)$, the number of 2-Sylow subgroups of $S(C)$ is 3, namely the aforementioned symmetry groups of the 4-prisms parallel to the coordinate axes. As for the four 3-Sylow subgroups of $S(C)$, the eight rotations in $S(C)$ of order 3 determine four subgroups of order 3.

Geometrically, each 3-Sylow subgroup of $S(C)$ is a cyclic group of order 3 generated by a rotation of order 3 around one of the four diagonals of the cube. There is also a simple way to describe the 2-Sylow subgroups of $S(C)$. To this end consider the three matrices

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ K = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ L = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which are rotations of order 2 around the x-axis, the y-axis, and the z-axis, respectively. We show that the group of 16 matrices in Table 3 is the centralizer in $S(C)$ of the matrix K :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} -a_{11} & a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & a_{32} & -a_{33} \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ a_{21} & a_{22} & a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{bmatrix}.$$

Consequently $a_{12} = 0, a_{21} = 0, a_{23} = 0$ and $a_{32} = 0$. Thus if $A = (a_{ij}) \in C_{S(C)}(K)$ it looks like

$$A = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix}.$$

But A must itself be in $S(C)$ so that leaves only 16 possibilities,

$$A = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \text{ or } A = \begin{bmatrix} 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \end{bmatrix}.$$

That is, the centralizer in $S(C)$ of K consists of the 8 diagonal matrices in $S(C)$ and the eight counter-diagonal matrices in $S(C)$. Thus the group of matrices in Table 3 is $C_{S(C)}(K)$. Similarly, the other two 2-Sylow subgroups of $S(C)$ are the centralizers in $S(C)$ of the matrices J and L. That is, the three 2-Sylow subgroups of $S(C)$ are

$$C_{S(C)}(J),\ C_{S(C)}(K),\ C_{S(C)}(L)$$

and they are all isomorphic to $D_4 \times C_2$.

Counter-Symmetric Matrices: call a matrix counter-symmetric if its entries are symmetric in the counter-diagonal. A typical 3×3 counter-symmetric matrix A looks like

$$A = \begin{bmatrix} a & b & c \\ d & e & b \\ g & d & a \end{bmatrix}.$$

Let us call the counter-transpose of a matrix A the matrix you obtain from A by reflecting the elements of A in its counter-diagonal. Define the symbol A^t for the counter-transpose of A. That is,

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^t = \begin{bmatrix} i & f & c \\ h & e & b \\ g & d & a \end{bmatrix}.$$

Thus A is counter-symmetric if $A^t = A$. Here are some basic properties of counter-transposition, for the most part identical to those of regular transposition:

1. $(A^t)^t = A$
2. $(A+B)^t = A^t + B^t$
3. $(kA)^t = kA^t$
4. $\det(A^t) = \det(A)$
5. $(AB)^t = B^t A^t$
6. $\mathrm{tr}\,(A^t) = \mathrm{tr}\,(A)$
7. $(A^t)^{-1} = (A^{-1})^t$, if $\det(A) \neq 0$.
8. Let $A^t = A, B^t = B$. Then AB is counter-symmetric if and only if $AB = BA$.
9. For any matrix A, AA^t and A^tA are both counter-symmetric.
10. $(A^T)^t = (A^t)^T$

Every diagonal matrix is a trivial example of a symmetric matrix and every counter-diagonal matrix is a trivial example of a counter-symmetric matrix. For more examples consider the subgroup $\mathcal{G}$ of $S(C)$ consisting of the six 3×3 permutation matrices. The identity matrix is both symmetric and counter-symmetric, the two permutation matrices of order 3 are counter-symmetric, and the three permutation matrices of order 2 are symmetric.

The algebraic connection between symmetric and counter-symmetric matrices can be made explicit in terms of the matrix

$$Y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

which represents both a reflection in the plane with equation $x = z$ and the permutation matrix $P_{(13)}$. Then it is a simple computation to check that $A^t = YA^TY$ and $A^T = YA^tY$. For example:

$$\begin{aligned} YA^TY &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} g & d & a \\ h & e & b \\ i & f & c \end{bmatrix} \\ &= \begin{bmatrix} i & f & c \\ h & e & b \\ g & d & a \end{bmatrix} \\ &= A^t. \end{aligned}$$

Here are some immediate consequences of this observation:

Theorem 7 *For any 3×3 matrix A, all of A, A^T and A^t have the same determinant, the same trace, the same order, the same characteristic polynomial, and the same eigenvalues.*

Theorem 8 *A is symmetric if and only if YA and AY are counter-symmetric, and A is counter-symmetric if and only if YA and AY are symmetric.*

Proof: use the fact that Y is symmetric, counter-symmetric, and of order 2. Then

- $A^T = A \Leftrightarrow YA^tY = A \Leftrightarrow YA^t = AY \Leftrightarrow (AY)^t = AY,$
- $A^T = A \Leftrightarrow YA^tY = A \Leftrightarrow A^tY = YA \Leftrightarrow (YA)^t = YA,$
- $A^t = A \Leftrightarrow YA^TY = A \Leftrightarrow YA^T = AY \Leftrightarrow (AY)^T = AY,$
- $A^t = A \Leftrightarrow YA^TY = A \Leftrightarrow A^TY = YA \Leftrightarrow (YA)^T = YA,$

are the required calculations. $\square$

Analogous to the well-known results that for any 3×3 matrix A,

$$A + A^T \text{ is symmetric and } A - A^T \text{ is skew-symmetric,}$$

it is easy to verify that

$$A + A^t \text{ is counter-symmetric and } A - A^t \text{ is skew-counter-symmetric.}$$

Similarly, just as the set of 3×3 symmetric matrices and the set of 3×3 skew-symmetric matrices each form a subspace of the vector space of 3×3 matrices, so do the set of 3×3 counter-symmetric matrices,

$$\left\{ \begin{bmatrix} a & b & c \\ d & e & b \\ g & d & a \end{bmatrix} \middle| \, a, b, c, d, e, g \in \mathbb{R} \right\}$$

and the set of skew-counter-symmetric matrices,

$$\left\{ \begin{bmatrix} a & b & 0 \\ d & 0 & -b \\ 0 & -d & -a \end{bmatrix} \middle| \, a, b, d \in \mathbb{R} \right\},$$

as you can check.

And just as each 3×3 matrix A can be written uniquely as the sum of a symmetric matrix and a skew-symmetric matrix, namely

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2},$$

so can each 3×3 matrix A be written uniquely as the sum of counter-symmetric matrix and a skew-counter-symmetric matrix:

$$A = \frac{A + A^t}{2} + \frac{A - A^t}{2}.$$

A Linear Transformation on $M_{33}(\mathbb{R})$: Define $k : M_{33}(\mathbb{R}) \longrightarrow M_{33}(\mathbb{R})$ by

$$k(A) = A - (A^T)^t.$$

It is easy to check that for $A, B \in M_{33}(\mathbb{R})$, $c \in \mathbb{R}$,

$$k(A+B) = k(A) + k(B) \text{ and } k(cA) = c\,k(A).$$

So k is a linear transformation. Its kernel is $\{A \in M_{33}(\mathbb{R}) \mid A = (A^T)^t\}$. A typical matrix in $\ker(k)$ looks like

$$A = \begin{bmatrix} a & b & c \\ d & e & d \\ c & b & a \end{bmatrix}.$$

Such a matrix is called a centro-symmetric matrix. Note that A itself need be neither symmetric nor counter-symmetric, but that $A^T = A \Leftrightarrow A^t = A$. The image of k is the set of skew centro-symmetric matrices; that is if $B \in \text{im}\,(k)$, then

$$B = \begin{bmatrix} a & b & c \\ d & 0 & -d \\ -c & -b & -a \end{bmatrix}$$

for some values of a, b, c, d. Just as any 3×3 matrix can be written as the sum of a symmetric and a skew symmetric matrix, so any 3×3 matrix can be written as the sum of a centro-symmetric and a skew centro-symmetric matrix:

$$A = \frac{1}{2}(A + (A^T)^t) + \frac{1}{2}(A - (A^T)^t).$$

Another way to think of the kernel of k is to use the fact that $A^t = YA^TY$, where $Y = P_{(13)}$. Then

$$A = (A^T)^t \Leftrightarrow A = Y(A^T)^TY \Leftrightarrow A = YAY \Leftrightarrow YA = AY.$$

That is, A is centro-symmetric if and only if A commutes with Y. (No such simple criterion characterizes a skew centro-symmetric matrix.)

Three Isomorphisms of $GL(3,\mathbb{R})$: but the vector space $M_{33}(\mathbb{R})$ includes matrices which are not invertible. So let's move to the group of invertible 3×3 matrices, $GL(3,\mathbb{R})$. Making use of the three matrix operations—inversion, transposition and counter-transposition—we can define three isomorphisms on $GL(3,\mathbb{R})$, as follows:

1. $f(A) = (A^{-1})^T$
2. $g(A) = (A^{-1})^t$
3. $h(A) = (A^T)^t$

It is easy, but instructive, to check that each of these maps defines a homomorphism. For example,

$$\begin{aligned} f(AB) &= ((AB)^{-1})^T \\ &= (B^{-1}A^{-1})^T \\ &= (A^{-1})^T(B^{-1})^T \\ &= f(A)f(B) \end{aligned}$$

The key is that each of the three operations—inversion, transposition and counter-transposition—reverses the order of products, so that a combination of any two of them preserves order of products. Finally, each homomorphism is an isomorphism. For example:

$$h(A) = I \Rightarrow (A^T)^t = I \Rightarrow A^T = I^t = I \Rightarrow A = I^T = I.$$

Here are some properties of the isomorphisms f, g, h, which are all easy to check:

- f, g, h all have order 2; that is

$$f \circ f(A) = A; \; g \circ g(A) = A; \; h \circ h(A) = A.$$

- f, g, h all commute with each other; that is

$$f \circ g = g \circ f = h; \; g \circ h = h \circ g = f; \; h \circ g = g \circ h = f.$$

- the composition of all three is the identity; that is

$$f \circ g \circ h(A) = A,$$

- $h(A) = YAY$, so h is an inner automorphism of $GL(3, \mathbb{R})$.

- f and g are not inner automorphisms of $GL(3, \mathbb{R})$. To see why, consider f and suppose that there is an invertible matrix P such that $f(A) = PAP^{-1}$, for all A. Then

$$\begin{aligned} (A^{-1})^T = PAP^{-1} \quad &\Rightarrow \quad \mathrm{tr}\,(A^{-1})^T = \mathrm{tr}\,(PAP^{-1}) \\ &\Rightarrow \quad \mathrm{tr}\,(A^{-1}) = \mathrm{tr}\,(A), \end{aligned}$$

  which is certainly not true for all $A \in GL(3, \mathbb{R})$.

Three Subgroups of $GL(3, \mathbb{R})$: each of the above three isomorphisms determines a different group of fixed points in $GL(3, \mathbb{R})$. In particular,

$$f(A) = A \Leftrightarrow (A^{-1})^T = A \Leftrightarrow A^{-1} = A^T,$$

so

$$\{A \in GL(3, \mathbb{R}) \mid f(A) = A\} = O(3),$$

the group of 3×3 orthogonal matrices. Analogously, let us define

$$P(3) = \{A \in GL(3, \mathbb{R}) \mid g(A) = A\} \text{ and } Q(3) = \{A \in GL(3, \mathbb{R}) \mid h(A) = A\}.$$

Now

$$g(A) = A \Leftrightarrow (A^{-1})^t = A \Leftrightarrow A^{-1} = A^t \Leftrightarrow I = AA^t = A^tA$$

and

$$h(A) = A \Leftrightarrow (A^T)^t = A \Leftrightarrow A^T = A^t.$$

Then

$$P(3) = \{A \in GL(3, \mathbb{R}) \mid A^{-1} = A^t\}.$$

In terms of the more familiar transpose, $P(3)$ is the group of invertible matrices A such that $A^{-1} = YA^TY$. Such matrices satisfy $\det(A) = \pm 1$, as with matrices in $O(3)$, but otherwise are not that familiar. For comparison, let $A \in GL(3, \mathbb{R})$ such that $A = \left[\; C_1 \;\middle|\; C_2 \;\middle|\; C_3 \;\right]$. Then

$$A \in O(3) \Leftrightarrow [C_i \cdot C_j] = I;$$

the well known result that the columns of A form an orthonormal basis of $\mathbb{R}^3$. On the other hand,

$$A \in P(3) \Leftrightarrow [C_i \cdot YC_j] = Y;$$

a very similar condition algebraically to $[C_i \cdot IC_j] = I$. Perhaps we should call the fixed points of g counter orthogonal matrices? Be that as it may, the matrices in $P(3)$, according to *Maple*, look like:

$$\begin{bmatrix} a & -af & -\frac{af^2}{2} \\ 0 & 1 & f \\ 0 & 0 & \frac{1}{a} \end{bmatrix}, \begin{bmatrix} a & af & -\frac{af^2}{2} \\ 0 & -1 & f \\ 0 & 0 & \frac{1}{a} \end{bmatrix}, \begin{bmatrix} -\frac{d^2}{2g} & -\frac{d}{g} & \frac{1}{g} \\ d & 1 & 0 \\ g & 0 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{d^2}{2g} & \frac{d}{g} & \frac{1}{g} \\ d & -1 & 0 \\ g & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{g(e+1)^2}{2h^2} & -\frac{e^2-1}{2h} & \frac{(e-1)^2}{4g} \\ \frac{g(e+1)}{h} & e & -\frac{h(e-1)}{2g} \\ g & h & -\frac{h^2}{2g} \end{bmatrix} \text{ or } \begin{bmatrix} -\frac{g(e-1)^2}{2h^2} & -\frac{e^2-1}{2h} & \frac{(e+1)^2}{4g} \\ \frac{g(e-1)}{h} & e & -\frac{h(e+1)}{2g} \\ g & h & -\frac{h^2}{2g} \end{bmatrix}.$$

Finally, the fixed points of h are familiar matrices, namely centro-symmetric matrices. That is,

$$Q(3) = \{A \in GL(3, \mathbb{R}) \mid A^T = A^t\},$$

and $A^t = YA^TY$, so $A \in Q(3)$ if and only if $AY = YA$. Thus $Q(3) = C_{GL(3,\mathbb{R})}(Y)$. If $A \in Q(3)$, then A looks like

$$A = \begin{bmatrix} a & b & c \\ d & e & d \\ c & b & a \end{bmatrix},$$

and we must have $\det(A) = (a-c)(ea+ec-2bd) \neq 0$. Observe that for every $A \in Q(3)$,

$$\vec{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

is a unit eigenvector of A with corresponding eigenvalue $\lambda = a - c$.

Eigenvalues and Eigenvectors of A, A^T, A^t: suppose $A \in GL(3, \mathbb{R})$ and that λ is an eigenvalue of A with corresponding eigenvector $\vec{v}$. If $A \in O(3)$, then $A^{-1} = A^T$, so

$$A^T\vec{v} = \frac{1}{\lambda}\vec{v}.$$

And since $A^t = YA^TY$, we have

$$A^t(Y\vec{v}) = (YA^TY)(Y\vec{v}) = YA^T\vec{v} = \frac{1}{\lambda}Y\vec{v}.$$

Similarly, if $A \in P(3)$, then $A^{-1} = A^t$, and

$$A^t\vec{v} = \frac{1}{\lambda}\vec{v} \text{ and } A^T(Y\vec{v}) = \frac{1}{\lambda}Y\vec{v}.$$

If $A \in Q(3)$, then $A^t = A^T$ if and only if $AY = YA$. This means that Y must fix the eigenspaces of A, and A must fix the eigenspaces of Y. This is just a general fact about any two commuting matrices, say A and B:

$$\begin{aligned} \vec{v} \in E_\lambda(B) \;&\Rightarrow\; B\vec{v} = \lambda\vec{v} \\ &\Rightarrow\; AB\vec{v} = \lambda A\vec{v} \\ &\Rightarrow\; BA\vec{v} = \lambda A\vec{v} \\ &\Rightarrow\; A\vec{v} \in E_\lambda(B). \end{aligned}$$

Y only has two eigenvalues, $\lambda_1 = -1, \lambda_2 = 1$; the eigenspaces are

$$E_{-1}(Y) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} \text{ and } E_1(Y) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Let the eigenvalues of A be μ_1, μ_2, μ_3, with $\mu_1 = a - c$. Then $\mu_2 + \mu_3 = a + c + e$, since $\operatorname{tr}(A) = 2a + e = \mu_1 + \mu_2 + \mu_3$, and $\mu_2\mu_3 = e(a+c) - 2bd$, since $\det(A) = \mu_1\mu_2\mu_3$. Thus μ_2 and μ_3 can be considered as the eigenvalues of the 2×2 matrix

$$\begin{bmatrix} a+c & b \\ 2\,d & e \end{bmatrix},$$

whence

$$\mu_2, \mu_3 = \frac{a + c + e \pm \sqrt{(a + c - e)^2 + 8db}}{2}.$$

Let the corresponding eigenvectors be $\vec{v}_1 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$ and $\vec{v}_2$, $\vec{v}_3$. If these eigenvalues are all distinct, then each eigenspace of A is one dimensional and $Y\vec{v}_i \| \vec{v}_i$. Let $Y\vec{v}_i = k\vec{v}_i$; since $Y^2 = I$ we must have $k^2 = 1$. Consequently $\vec{v}_i$ is also an eigenvector of Y; in particular $\vec{v}_2, \vec{v}_3 \in E_1(Y)$. Can we find out what $\vec{v}_2$ and $\vec{v}_3$ are? We have

$$\begin{aligned} \vec{j} = s\,\vec{v}_2 + t\,\vec{v}_3 \;&\Rightarrow\; A\vec{j} = s\,\mu_2\vec{v}_2 + t\,\mu_3\vec{v}_3 \text{ and } \mu_2\vec{j} = s\,\mu_2\vec{v}_2 + t\,\mu_2\vec{v}_3 \\ &\Rightarrow\; A\vec{j} - \mu_2\vec{j} = t(\mu_3 - \mu_2)\vec{v}_3 \\ &\Rightarrow\; b\vec{i} + (e - \mu_2)\vec{j} + b\vec{k} = t(\mu_3 - \mu_2)\vec{v}_3 \end{aligned}$$

and

$$\begin{aligned} \vec{j} = s\,\vec{v}_2 + t\,\vec{v}_3 \;&\Rightarrow\; A\vec{j} = s\,\mu_2\vec{v}_2 + t\,\mu_3\vec{v}_3 \text{ and } \mu_3\vec{j} = s\,\mu_3\vec{v}_2 + t\,\mu_3\vec{v}_3 \\ &\Rightarrow\; A\vec{j} - \mu_3\vec{j} = s(\mu_2 - \mu_3)\vec{v}_2 \\ &\Rightarrow\; b\vec{i} + (e - \mu_3)\vec{j} + b\vec{k} = s(\mu_2 - \mu_3)\vec{v}_2. \end{aligned}$$

Dispense with the special case first: if $st = 0$, then one of s, t can be zero, but not both. If $t = 0$, then $\vec{v}_2$ is parallel to $\vec{j}$, $b = 0, \mu_2 = e$, and $\mu_3 = a + c$ and we have

$$A = \begin{bmatrix} a & 0 & c \\ d & e & d \\ c & 0 & a \end{bmatrix}, \; \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \; \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; \vec{v}_3 = \begin{bmatrix} a + c - e \\ 2d \\ a + c - e \end{bmatrix},$$

as you can check. In particular, the basis of eigenvectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is orthogonal if and only if $d = 0$, in which case A is symmetric. If $s = 0$ then things are the same accept that μ_2 and μ_3 are interchanged, as are $\vec{v}_2$ and $\vec{v}_3$. What if $st \neq 0$? Then, since a scalar multiple of an eigenvector is still an eigenvector, you can take

$$\vec{v}_2 = \begin{bmatrix} b \\ e - \mu_3 \\ b \end{bmatrix}, \; \vec{v}_3 = \begin{bmatrix} b \\ e - \mu_2 \\ b \end{bmatrix}.$$

Since $\vec{v}_2 \cdot \vec{v}_3 =$

$$2b^2+(e-\mu_3)(e-\mu_2) = 2b^2+e^2-(\mu_2+\mu_3)e+\mu_2\mu_3 = 2b^2+e^2-(a+c+e)e+e(a+c)-2bd = 2b^2-2bd,$$

we have again that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis if and only if $b = d$ and A is symmetric. (Since the eigenvalues of A are distinct, the eigenvectors are independent, so $b \neq 0$.) If instead of starting with $\vec{j}$ in the above calculations we start with the other basis vector of $E_1(Y)$, namely $\vec{i} + \vec{k}$, and set $\vec{i} + \vec{k} = u\,\vec{v}_2 + v\,\vec{v}_3$, then we would find that

$$\vec{v}_2 = \begin{bmatrix} \mu_2 - e \\ 2d \\ \mu_2 - e \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} \mu_3 - e \\ 2d \\ \mu_3 - e \end{bmatrix},$$

for $uv \neq 0$. If $uv = 0$, then

$$A = \begin{bmatrix} a & b & c \\ 0 & e & 0 \\ c & b & a \end{bmatrix},$$

with eigenvalues $a - c, a + c, e$ and eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -b \\ a+c-e \\ -b \end{bmatrix}.$$

Finally, for $A \in Q(3)$, $A^T = A^t$, and A^T is obtained from A simply by interchanging b and d. Since A and A^T have the same eigenvalues, the eigenvectors of A^T or A^t are obtained from the eigenvectors of A simply by interchanging b and d.

A Fourth Possibility: what if A is an invertible 3×3 matrix such that $A^{-1} = (A^T)^t = (A^t)^T$? This would mean

$$A\,(A^T)^t = I \Leftrightarrow A^T\,A^t = I^t = I \text{ and } (A^T)^t\,A = I \Leftrightarrow A^t\,A^T = I^t = I.$$

Consequently A^T and A^t are inverses of each other. The set of such matrices,

$$R(3) = \{A \in GL(3, \mathbb{R}) \mid A^{-1} = (A^T)^t\},$$

does not form a group, but we can still say many things about them. For example, if $A \in R(3)$:

- $\det(A^T) = \det(A^t) = \det(A) = \pm 1$.
- If λ is an eigenvalue of A with corresponding eigenvector $\vec{v}$, then

  $$(A^t)^T\vec{v} = A^{-1}\vec{v} \Rightarrow Y(A^t)^tY\vec{v} = \frac{1}{\lambda}\vec{v} \Rightarrow YAY\vec{v} = \frac{1}{\lambda}\vec{v} \Rightarrow A\,Y\vec{v} = \frac{1}{\lambda}Y\vec{v},$$

  so $Y\vec{v}$ is an eigenvector of A with eigenvalue $1/\lambda$.
- A^T and A^t swap eigenvectors corresponding to reciprocal eigenvalues:

  $$A^T\vec{v} = \lambda\vec{v} \Rightarrow A^tA^T\vec{v} = A^t(\lambda\vec{v}) \Rightarrow I\vec{v} = \lambda A^t\vec{v} \Rightarrow A^t\vec{v} = \frac{1}{\lambda}\vec{v}.$$

- A, A^t, A^T and A^{-1} all have the same characteristic polynomial and the same eigenvalues: we know $\det(A) = \det(A^T) = \det(A^t)$. So if $A^{-1} = (A^t)^T = (A^T)^t$ then

  $$\det(A^{-1}) = \det((A^T)^t) = \det(A^T) = \det(A).$$

  Thus

  $$C_A(x) = \det(xI - A) = \det((xI - A)^t) = \det(xI^t - A^t) = \det(xI - A^t) = C_{A^t}(x);$$
  $$C_A(x) = \det(xI - A^t) = \det((xI - A^t))^T) = \det(xI^T - (A^t)^T) = \det(xI - A^{-1}).$$

Example 8: let

$$A = \begin{bmatrix} -16 & -16 & -7 \\ 2 & 3 & 1 \\ 5 & 4 & 2 \end{bmatrix}$$

for which

$$A^T = \begin{bmatrix} -16 & 2 & 5 \\ -16 & 3 & 4 \\ -7 & 1 & 2 \end{bmatrix}, A^t = \begin{bmatrix} 2 & 1 & -7 \\ 4 & 3 & -16 \\ 5 & 2 & -16 \end{bmatrix} \text{ and } (A^t)^T = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 3 & 2 \\ -7 & -16 & -16 \end{bmatrix} = A^{-1}.$$

All of A, A^t, A^T, A^{-1} have determinant 1 and trace -11. Moreover $A^T\, A^t = I$ and

$$C_A(x) = C_{A^T}(x) = C_{A^t}(x) = C_{A^{-1}}(x) = x^3 + 11x^2 - 11x - 1 = (x-1)(x^2 + 12x + 1),$$

so each of A, A^T, A^t and A^{-1} has eigenvalues

$$\lambda_1 = 1,\ \ \lambda_2 = -6 + \sqrt{35},\ \lambda_3 = -6 - \sqrt{35}.$$

Note that $\lambda_2\, \lambda_3 = 1$. The eigenvectors of each matrix are given in the following table:

eigenvalue	eigenvector of A	eigenvector of A^T	eigenvector of A^t	eigenvector of A^{-1}
$\lambda_1 = 1$	$\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$
$\lambda_2 = -6 + \sqrt{35}$	$\begin{bmatrix} -3+\sqrt{35} \\ 1 \\ -3-\sqrt{35} \end{bmatrix}$	$\begin{bmatrix} 15-\sqrt{35} \\ 20 \\ 15+\sqrt{35} \end{bmatrix}$	$\begin{bmatrix} 15+\sqrt{35} \\ 20 \\ 15-\sqrt{35} \end{bmatrix}$	$\begin{bmatrix} -3-\sqrt{35} \\ 1 \\ -3+\sqrt{35} \end{bmatrix}$
$\lambda_3 = -6 - \sqrt{35}$	$\begin{bmatrix} -3-\sqrt{35} \\ 1 \\ -3+\sqrt{35} \end{bmatrix}$	$\begin{bmatrix} 15+\sqrt{35} \\ 20 \\ 15-\sqrt{35} \end{bmatrix}$	$\begin{bmatrix} 15-\sqrt{35} \\ 20 \\ 15+\sqrt{35} \end{bmatrix}$	$\begin{bmatrix} -3+\sqrt{35} \\ 1 \\ -3-\sqrt{35} \end{bmatrix}$

NB: for each matrix, corresponding elements in eigenvectors corresponding to reciprocal eigenvalues are "conjugates" of each other. □

Example 9: let

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -t \\ 0 & 1 & t \end{bmatrix}, \text{ for which } M^t = \begin{bmatrix} t & -t & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, M^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -t & t \end{bmatrix}.$$

Check that

$$M^{-1} = \begin{bmatrix} t & 1 & 0 \\ -t & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = (M^T)^t,$$

$$\det(M) = 1,\ \ \mathrm{tr}\,(M) = t,\ C_M(x) = x^3 - t\,x^2 + t\,x - 1,$$

and that the eigenvalues of M are

$$\lambda_1 = 1, \lambda_2 = \frac{-1+t+\sqrt{t^2-2t-3}}{2}, \lambda_3 = \frac{-1+t-\sqrt{t^2-2t-3}}{2}.$$

Consider the case when

$$t^2 - 2t - 3 \le 0 \Leftrightarrow -1 \le t \le 3.$$

Then λ_2 and λ_3 are complex conjugates with modulus 1. Let $\lambda_2 = e^{i\theta}$ and $\lambda_3 = e^{-i\theta}$, for some angle θ. Then $\operatorname{tr}(M) = t = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2\cos\theta$. Calculating we find:

eigenvalue	eigenvector of M	eigenvector of M^T	eigenvector of M^t	eigenvector of M^{-1}	eigenvector of $R(\theta)$
$\lambda_1 = 1$	$\begin{bmatrix} 1 \\ -\lambda_2-\lambda_3 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -\lambda_2-\lambda_3 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
$\lambda_2 = e^{i\theta}$	$\begin{bmatrix} \lambda_3 \\ -1-\lambda_3 \\ 1 \end{bmatrix}$	$\begin{bmatrix} \lambda_3 \\ 1 \\ \lambda_2 \end{bmatrix}$	$\begin{bmatrix} \lambda_2 \\ 1 \\ \lambda_3 \end{bmatrix}$	$\begin{bmatrix} \lambda_2 \\ -1-\lambda_2 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ w \\ \overline{w} \end{bmatrix}$
$\lambda_3 = e^{-i\theta}$	$\begin{bmatrix} \lambda_2 \\ -1-\lambda_2 \\ 1 \end{bmatrix}$	$\begin{bmatrix} \lambda_2 \\ 1 \\ \lambda_3 \end{bmatrix}$	$\begin{bmatrix} \lambda_3 \\ 1 \\ \lambda_2 \end{bmatrix}$	$\begin{bmatrix} \lambda_3 \\ -1-\lambda_3 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ \overline{w} \\ w \end{bmatrix}$

For comparison we have included in the last column a general rotation matrix,[2] $R(\theta)$, of θ for $\theta \neq 0, \pi$, around an axis parallel to the vector

$$\vec{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ where } w = \frac{1}{2} + \frac{\sqrt{3}}{2} i.$$

(If $\theta = 0$ or $\theta = \pi$ then the eigenvalues and eigenvectors of $R(\theta)$ are real.) All five matrices have the same trace and the same eigenvalues, but not the same eigenvectors. The eigenvectors for $R(\theta)$ form an orthogonal basis of $\mathbb{C}^3$, but this is not true for M, M^T, M^t or M^{-1}. However the second and third eigenvectors of M^T and M^t, as given in the table, do have inner product with the first eigenvector equal to $\operatorname{tr}(M)$. □

- Examples 8 and 9 illustrate the following: if $\lambda_1 = 1$ and $\lambda_2\lambda_3 = 1$ with $\lambda_2 \neq \lambda_3$, then:

$$E_{\lambda_2}(A^T) + E_{\lambda_3}(A^T) = (E_1(A))^\perp \text{ and } E_{\lambda_2}(A^t) + E_{\lambda_3}(A^t) = (E_1(A))^\perp;$$

$$E_{\lambda_2}(A) + E_{\lambda_3}(A) = (E_1(A^T))^\perp \text{ and } E_{\lambda_2}(A^{-1}) + E_{\lambda_3}(A^{-1}) = (E_1(A^T))^\perp.$$

Proof by calculation. For example, if $\vec{w}_2 \in E_{\lambda_2}(A^T)$ and $\vec{v}_1 \in E_1(A^{-1})$, then

$$\lambda_2 \vec{w}_2 \cdot \vec{v}_1 = A^T \vec{w}_2 \cdot A^{-1}\vec{v}_1 = \vec{w}_2^T A A^{-1} \overline{\vec{v}_1} = \vec{w}_2^T \overline{\vec{v}_1} = \vec{w}_2 \cdot \vec{v}_1 \Rightarrow \vec{w}_2 \cdot \vec{v}_1 = 0.$$

[2] Formula of this matrix is given in Case 2 of the section **Orthogonal Matrices in** $\mathbb{U} + \mathbb{W}$.

- According to *Maple* the general matrix $A \in R(3)$ with $\det(A) = 1$ looks like

$$A = \begin{bmatrix} -\dfrac{d(di+ef+f)}{f^2} & \dfrac{di-dei-e^2f+f}{f^2} & -\dfrac{di+ef}{f} \\ d & e & f \\ \dfrac{di+f}{f} & \dfrac{i(e-1)}{f} & i \end{bmatrix},$$

assuming $f \neq 0$. We have

$$\mathrm{tr}\,(A) = \frac{-d^2i - def - df + ef^2 + if^2}{f^2}$$

and

$$C_A(x) = \frac{1}{f^2}(x-1)(f^2x^2 - (-d^2i - def - df + ef^2 + if^2 - f^2)x + f^2),$$

which shows that one eigenvalue of A is $\lambda_1 = 1$, and the other two must be reciprocals of each other. Indeed, eigenvalues λ_2, λ_3 of A will be real if

$$(1 - \mathrm{tr}\,(A))^2 - 4 \geq 0 \Leftrightarrow (\mathrm{tr}\,(A))^2 - 2\mathrm{tr}\,(A) - 3 \geq 0 \Leftrightarrow (\mathrm{tr}\,(A) - 3)(\mathrm{tr}\,(A) + 1) \geq 0.$$

That is, eigenvalues λ_2, λ_3 will be real reciprocals if

$$\mathrm{tr}\,(A) \leq -1 \text{ or } \mathrm{tr}\,(A) \geq 3;$$

they will be complex reciprocals if

$$-1 < \mathrm{tr}\,(A) < 3.$$

- If $A \in O(3) \cap R(3)$ then $A^T = A^{-1} = (A^T)^t$, which means A is a counter-symmetric orthogonal matrix. For example,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ with } \mathrm{tr}\,(A) = 0,\ \lambda_1 = 1,\ \lambda_2 = e^{2i\pi/3},\ \lambda_3 = e^{-2i\pi/3}.$$

- If $A \in P(3) \cap R(3)$ then $A^t = A^{-1} = (A^t)^T$, which means A is a symmetric counter-orthogonal matrix. For example,

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ with } \mathrm{tr}\,(A) = -3,\ \lambda_1 = 1,\ \lambda_2 = -2+\sqrt{3},\ \lambda_3 = -2-\sqrt{3}.$$

- If $A \in Q(3) \cap R(3)$ then $A^T = A^t$ and $A^{-1} = (A^t)^T = A$, which means A is an orthogonal centro-symmetric matrix of order 2. For example,

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \text{ with } \mathrm{tr}\,(A) = -1,\ \lambda_1 = 1, \lambda_2 = \lambda_3 = -1.$$

An Application: suppose $X \in SO(3)$ represents a rotation of θ around some axis through the origin, with

$$C_X(x) = x^3 - \mathrm{tr}\,(X)\,x^2 + \mathrm{tr}\,(X)\,x - 1,$$

where $\mathrm{tr}\,(X) = \lambda_1 + \lambda_2 + \lambda_3 = 1 + e^{i\theta} + e^{-i\theta}$. For simplicity, set $\mathrm{tr}\,(X) = t$. By the Cayley-Hamilton Theorem

$$X^3 = t\,X^2 - t\,X + I.$$

Then, recursively, every power of X can be written as a linear combination of the three matrices I, X and X^2. Here are some sample calculations:

X^n	coordinates wrt I, X, X^2, where $t = \mathrm{tr}\,(X) = \lambda_1 + \lambda_2 + \lambda_3$			condition for t if $X^n = I, n \geq 3$
X^{-1}	t	$-t$	1	
I	1	0	0	
X	0	1	0	
X^2	0	0	1	
X^3	1	$-t$	t	$t = 0$
X^4	t	$1 - t^2$	$t^2 - t$	$t = 1$
X^5	$t^2 - t$	$-t^3 + t^2 + t$	$t^3 - 2\,t^2 + 1$	$t^2 - t - 1 = 0$
X^6	$t^3 - 2t^2 + 1$	$-t^4 + 2t^3 + t^2 - 2t$	$t^4 - 3t^3 + t^2 + 2t$	$t = 2$

Let $B = \{I, X, X^2\}$ and let

$$c_B(X^n) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Using $X^3 = t\,X^2 - t\,X + I$ you can calculate that

$$c_B(X^{n+1}) = \begin{bmatrix} z \\ x - t\,z \\ y + t\,z \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

where

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -t \\ 0 & 1 & t \end{bmatrix},$$

none other than the matrix in $R(3)$ from Example 9. Although M is not a rotation matrix—it isn't even orthogonal—M and X have the same determinant, the same trace, the same characteristic polynomial, the same eigenvalues, and the same order. Moreover, M^T, M^t and M^{-1} all have the same determinant, trace and eigenvalues as X too, but not the same eigenvectors. See the table in Example 9.

Part 3: Deranged Matrices and All That. A derangement of a, b, c is a permutation of a, b, c with no fixed point. There are only two derangements of a, b, c, namely

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \to \begin{bmatrix} c \\ a \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \to \begin{bmatrix} b \\ c \\ a \end{bmatrix}.$$

For arbitrary values of $a, b, c \in \mathbb{R}$, let us call the matrices

$$\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

deranged matrices[3]. It is straightforward to check that:

1. All deranged matrices have eigenvector $\vec{d} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ with eigenvalue $\lambda = a + b + c$.

2. The subspace $\mathbb{U}$ of all 3×3 counter-symmetric deranged matrices is spanned by the three even permutation matrices $I, P_{(123)}, P_{(132)}$, so that every 3×3 counter-symmetric deranged matrix can be written as
$$A = aI + cP_{(123)} + bP_{(132)},$$
for some values of a, b, c.

3. The subspace $\mathbb{W}$ of all 3×3 symmetric deranged matrices is spanned by the three odd permutation matrices $P_{(12)}, P_{(23)}, P_{(13)}$, so that every 3×3 symmetric deranged matrix can be written as
$$A = aP_{(23)} + bP_{(12)} + cP_{(13)},$$
for some values of a, b, c.

4. $\mathbb{U} \cap \mathbb{W}$ is spanned by the matrix
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$
so $\dim(\mathbb{U} \cap \mathbb{W}) = 1$ and $\dim(\mathbb{U} + \mathbb{W}) = \dim(\mathbb{U}) + \dim(\mathbb{W}) - \dim(\mathbb{U} \cap \mathbb{W}) = 3 + 3 - 1 = 5$.

5. $\dim(\mathbb{U} + \mathbb{W})^\perp = 4$ and a basis for $(\mathbb{U} + \mathbb{W})^\perp$ consists of the four matrices
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

6. The product of two counter-symmetric deranged matrices is another counter-symmetric deranged matrix and the multiplication is commutative. So $\mathbb{U}$ forms a commutative ring.

7. The inverse of an invertible counter-symmetric deranged matrix is another counter-symmetric deranged matrix.

8. The set of invertible counter-symmetric deranged matrices is an abelian subgroup of $GL(3, \mathbb{R})$.

9. The inverse of an invertible symmetric deranged matrix is a symmetric deranged matrix.

[3]More properly they are called circulants.

10. The product of two symmetric deranged matrices is a counter-symmetric deranged matrix, and $P_\sigma \mathbb{W} = \mathbb{W} P_\sigma = \mathbb{U}$ for any odd permutation σ.

11. The product of a counter-symmetric deranged matrix and a symmetric deranged matrix is a symmetric deranged matrix, and $P_\sigma \mathbb{U} = \mathbb{U} P_\sigma = \mathbb{W}$ for any odd permutation σ.

12. The set of all invertible deranged matrices is a subgroup of $GL(3, \mathbb{R})$.

Theorem 9 *Every matrix in* $(\mathbb{U} + \mathbb{W})^\perp$ *is traceless and singular.*

Proof: let $A \in (\mathbb{U} + \mathbb{W})^\perp$. Then there are scalars a, b, c, d such that

$$\begin{aligned} A &= a \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -a-b & c & -b-d \\ d & a+b+c+d & a \\ -a-c & b & -c-d \end{bmatrix} \end{aligned}$$

So tr $(A) = 0$, as is easy to see, but $\det(A)$ is a little less obvious. Making use of some properties of determinants we find

$$\begin{aligned} \det \begin{bmatrix} -a-b & c & -b-d \\ d & a+b+c+d & a \\ -a-c & b & -c-d \end{bmatrix} &= \det \begin{bmatrix} -a-b & c & -b-d \\ d+a+b & a+b+d & a+b+d \\ b-c & b-c & b-c \end{bmatrix} \\ &= (a+b+d)(b-c) \det \begin{bmatrix} -a-b & c & -b-d \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= 0, \text{ because two rows are identical.} \end{aligned}$$

Or you can just feed the matrix into *Maple* or *WolframAlpha* to confirm that $\det(A) = 0$. $\square$

When are matrices in $\mathbb{U} + \mathbb{W}$ invertible? Let $A \in \mathbb{U} + \mathbb{W}$. Then there are scalars a, b, c, d, e, f such that

$$A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} + \begin{bmatrix} f & d & e \\ d & e & f \\ e & f & d \end{bmatrix} = \begin{bmatrix} a+f & b+d & c+e \\ c+d & a+e & b+f \\ b+e & c+f & a+d \end{bmatrix}.$$

Then $\vec{d}$ is an eigenvector of A with eigenvalue $\lambda = a+b+c+d+e+f$, tr $(A) = 3a+d+e+f$, and

$$\det(A) = (a+b+c+d+e+f)(a^2+b^2+c^2-ab-ac-bc+de+ef+df-d^2-e^2-f^2),$$

where we did use *Maple* to find the determinant. So A is singular if

$$a+b+c+d+e+f = 0 \text{ or } a^2+b^2+c^2-ab-ac-bc+de+ef+df-d^2-e^2-f^2 = 0.$$

Consider the special case when $A \in \mathbb{U}$: take $d = e = f = 0$ to find

$$\det(A) = (a+b+c)(a^2+b^2+c^2-ab-ac-bc).$$

Calculating A's determinant directly you find the un-factored value, namely

$$\det \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} = a^3+b^3+c^3-3abc.$$

Thus $A \in \mathbb{U}$ is invertible if

$$a+b+c \neq 0 \text{ and } a^2+b^2+c^2-ab-ac-bc \neq 0.$$

Similarly, if $B \in \mathbb{W}$ then, taking

$$B = A\,P_{(23)} = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix},$$

we have $\det(B) = \det(A)\det(P_{(23)}) = -(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$. So in the end we have the exact same conditions for $B \in \mathbb{W}$ being invertible as for $A \in \mathbb{U}$:

$$a+b+c \neq 0 \text{ and } a^2+b^2+c^2-ab-ac-bc \neq 0.$$

Orthogonal Matrices in $\mathbb{U}+\mathbb{W}$: if A is orthogonal then $A^{-1} = A^T$, the only real eigenvalues of A are $\lambda = \pm 1$, and $\det(A) = \pm 1$. We can also say:

- If $\det(A) = 1$ then A represents a rotation. In that case

  $$\operatorname{tr}(A) = 1 + 2\cos\theta$$

  and the rotation is through an angle θ around an axis passing through the origin. The eigenvalues of A are $\lambda = 1, e^{i\theta}, e^{-i\theta}$. If $\theta = \pm\pi$ then A represents a rotation of order 2 and its eigenvalues are $\lambda = 1$ and $\lambda = -1$, repeated. Otherwise, A has a single real eigenvalue, $\lambda = 1$, and A represents a rotation of order greater than 2. If A is in $\mathbb{U}+\mathbb{W}$ one of its eigenvectors must be $\vec{d}$. So if A has order greater than 2, then the axis of rotation of A must be parallel to $\vec{d}$, that is, $E_1(A) = \operatorname{span}\{\vec{d}\}$.

- If $\det(A) = -1$ then A represents a reflection or an improper rotation.

- All orthogonal matrices of order 2 are symmetric.

To find out which orthogonal matrices are in $\mathbb{U}+\mathbb{W}$ we shall use the following formulas:

- The general rotation matrix R of order 2 is given by

  $$R = 2\vec{v}\vec{v}^T - I = \begin{bmatrix} 2v_1^2-1 & 2v_1v_2 & 2v_1v_3 \\ 2v_1v_2 & 2v_2^2-1 & 2v_2v_3 \\ 2v_1v_3 & 2v_2v_3 & 2v_3^2-1 \end{bmatrix},$$

  where $\vec{v}$ is a unit vector parallel to the axis of rotation. In this case $\dim(E_{-1}(R)) = 2$ and $E_{-1}(R)$ is the plane with equation $v_1x + v_2y + v_3z = 0$.

- The general rotation matrix R representing a rotation through θ around an axis parallel to the unit vector $\vec{u}$ is given by

$$R = \begin{bmatrix} \cos\theta + u_1^2(1-\cos\theta) & u_1u_2(1-\cos\theta) - u_3\sin\theta & u_1u_3(1-\cos\theta) + u_2\sin\theta \\ u_2u_1(1-\cos\theta) + u_3\sin\theta & \cos\theta + u_2^2(1-\cos\theta) & u_2u_3(1-\cos\theta) - u_1\sin\theta \\ u_3u_1(1-\cos\theta) - u_2\sin\theta & u_3u_2(1-\cos\theta) + u_1\sin\theta & \cos\theta + u_3^2(1-\cos\theta) \end{bmatrix}.$$

It is not clear if either of these general matrices is in $\mathbb{U}+\mathbb{W}$. To find all the orthogonal matrices in $\mathbb{U}+\mathbb{W}$ we shall consider four cases:

1. rotations of order 2.

2. rotations of order greater than 2.

3. reflections.

4. improper rotations.

Case 1. Orthogonal Matrices in $\mathbb{U}+\mathbb{W}$ That Represent Rotations of Order 2: to avoid subscripts rewrite the general rotation matrix R of order 2 as

$$R = 2\vec{v}\,\vec{v}^T - I = \begin{bmatrix} 2a^2-1 & 2ab & 2ac \\ 2ab & 2b^2-1 & 2bc \\ 2ac & 2bc & 2c^2-1 \end{bmatrix},$$

where $\vec{v} = [a\ b\ c]^T$ is a unit vector parallel to the axis of rotation. If R is to be in $\mathbb{U}+\mathbb{W}$ then $\vec{d}$ must be an eigenvector of R. If the corresponding eigenvalue is $\lambda = 1$ then

$$a = b = c = \frac{1}{\sqrt{3}},\ \ R = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \in \mathbb{U},$$

and R represents a rotation of order 2 around the line parallel to $\vec{d}$. Otherwise $\lambda = -1$ is the corresponding eigenvalue and $R\,\vec{d} = -\vec{d}$. Then

$$2\vec{v}\,\vec{v}^T\vec{d} - \vec{d} = -\vec{d} \Rightarrow 2\vec{v}(\vec{v}\cdot\vec{d}) = \vec{0}.$$

Since $\vec{v} \neq \vec{0}$ we must have

$$\vec{v}\cdot\vec{d} = 0 \Leftrightarrow a+b+c = 0.$$

If we include the fact that $\vec{v}$ is a unit vector then we obtain a system of two equations in a, b, c :

$$a^2+b^2+c^2 = 1 \text{ and } a+b+c = 0.$$

Solving these equations we find two solutions:

$$b = \frac{1}{2}\left(-a-\sqrt{2-3a^2}\right) \text{ and } c = \frac{1}{2}\left(-a+\sqrt{2-3a^2}\right)$$

or

$$b = \frac{1}{2}\left(-a+\sqrt{2-3a^2}\right) \text{ and } c = \frac{1}{2}\left(-a-\sqrt{2-3a^2}\right)$$

for $3a^2 \leq 2$. Substitute these values for b, c into R and simplify, to obtain two matrices:

$$R_1 = \begin{bmatrix} 2a^2-1 & -a^2-a\sqrt{2-3a^2} & -a^2+a\sqrt{2-3a^2} \\ -a^2-a\sqrt{2-3a^2} & -a^2+a\sqrt{2-3a^2} & 2a^2-1 \\ -a^2+a\sqrt{2-3a^2} & 2a^2-1 & -a^2-a\sqrt{2-3a^2} \end{bmatrix}$$

and

$$R_2 = \begin{bmatrix} 2a^2-1 & -a^2+a\sqrt{2-3a^2} & -a^2-a\sqrt{2-3a^2} \\ -a^2+a\sqrt{2-3a^2} & -a^2-a\sqrt{2-3a^2} & 2a^2-1 \\ -a^2-a\sqrt{2-3a^2} & 2a^2-1 & -a^2+a\sqrt{2-3a^2} \end{bmatrix}.$$

Both of these matrices[4] are in $\mathbb{W}$.

It is easy to compute an orthonormal basis of $E_{-1}(R_i)$, which is the plane normal to $\vec{v}$ with equation $ax+by+cz = 0$. That is,

$$E_{-1}(R_i) = \text{span}\left\{\frac{\vec{d}}{\|\vec{d}\|},\ \frac{\vec{v}\times\vec{d}}{\|\vec{v}\times\vec{d}\|}\right\} = \text{span}\left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix},\ \frac{1}{\sqrt{3}}\begin{bmatrix}c-b\\a-c\\b-a\end{bmatrix}\right\}.$$

NB: since $0 = (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca = 1+2(ab+bc+ca)$ it follows that

$$\|[c-b\ \ a-c\ \ b-a]^T\|^2 = c^2+b^2-2bc+a^2+c^2-2ac+b^2+a^2-2ab = 2-2(-1/2) = 3.$$

[4]In Case 3 we'll see a single way to parametrize all these matrices at once.

Case 2. Orthogonal Matrices in $\mathbb{U}+\mathbb{W}$ That Represent Other Rotations: if the general rotation matrix R is to be in $\mathbb{U}+\mathbb{W}$ then the axis of rotation must be around a line parallel to the unit vector $\vec{u}=\vec{d}/\|\vec{d}\|$. Thus we can simplify R to obtain

$$D_1(\theta)=\frac{1}{3}\begin{bmatrix} 1+2\cos\theta & 1-\cos\theta-\sqrt{3}\sin\theta & 1-\cos\theta+\sqrt{3}\sin\theta \\ 1-\cos\theta+\sqrt{3}\sin\theta & 1+2\cos\theta & 1-\cos\theta-\sqrt{3}\sin\theta \\ 1-\cos\theta-\sqrt{3}\sin\theta & 1-\cos\theta+\sqrt{3}\sin\theta & 1+2\cos\theta \end{bmatrix},$$

which is in $\mathbb{U}$! Thus the rotation matrices of order greater than 2 in $\mathbb{U}+\mathbb{W}$ are all the possible rotations of order greater than 2 around an axis parallel to the vector $\vec{d}$, and they are all counter-symmetric deranged matrices. Note that $D_1(0)=I$ and

$$D_1(\pm\pi)=\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix},$$

the specific solution in Case 1 that represents a rotation of order 2 around the line parallel to the vector $\vec{d}$. So actually $\mathbb{U}$ contains the *group* of all rotation matrices with axis parallel to $\vec{d}$.

Case 3. Orthogonal Matrices in $\mathbb{U}+\mathbb{W}$ That Represent Reflections: take $D_1(\theta)$ as in Case 2. Let $Q=P_{(23)}D_1(\theta)$; then Q will also be an orthogonal deranged matrix but Q will be in $\mathbb{W}$:

$$Q=\frac{1}{3}\begin{bmatrix} 1+2\cos\theta & 1-\cos\theta-\sqrt{3}\sin\theta & 1-\cos\theta+\sqrt{3}\sin\theta \\ 1-\cos\theta-\sqrt{3}\sin\theta & 1-\cos\theta+\sqrt{3}\sin\theta & 1+2\cos\theta \\ 1-\cos\theta+\sqrt{3}\sin\theta & 1+2\cos\theta & 1-\cos\theta-\sqrt{3}\sin\theta \end{bmatrix}.$$

Observe that

- Q is symmetric and of order 2.
- $\det(Q)=\det(P_{(23)})\det(D_1(\theta))=(-1)(1)=-1.$
- $\operatorname{tr}(Q)=(1+2\cos\theta+1-\cos\theta+\sqrt{3}\sin\theta+1-\cos\theta-\sqrt{3}\sin\theta)/3=1.$
- $Q\,\vec{d}=P_{(23)}(D_1(\theta)\,\vec{d})=P_{(23)}\,\vec{d}=\vec{d}$, so Q has eigenvector $\vec{d}$ with eigenvalue $\lambda=1$.
- The eigenvalues of Q are $\lambda_1=-1$ and $\lambda_2=1$, repeated.

Similar observations apply to $D_1(\theta)$ multiplied with any odd permutation matrix, that is, to each of $P_{(13)}D_1(\theta), P_{(12)}D_1(\theta), D_1(\theta)P_{(23)}, D_1(\theta)P_{(13)}$ and $D_1(\theta)P_{(12)}$. Thus $D_1(\theta)$ multiplied with any odd permutation matrix, in any order, results in a reflection matrix.

But how do we know if the above formula for Q generates *all* the reflection matrices in $\mathbb{U}+\mathbb{W}$? For one thing, no value of θ can give us the matrix that represents a reflection in the plane $x+y+z=0$, since then $\vec{d}$ would be in $E_{-1}(Q)$. But this matrix is

$$-R=\frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}\in\mathbb{U},$$

where R is the rotation matrix of order 2 around the axis parallel to $\vec{d}$, as given in Case 1. Indeed, as we saw in Part 1, 3×3 matrices that represent rotations of order 2 or reflections are negatives of each other. Thus $-R$, with R as in Case 1, is another way to represent reflections in $\mathbb{U}+\mathbb{W}$. That is, let

$$Q_1 = -R_1 = \begin{bmatrix} 1-2a^2 & a^2+a\sqrt{2-3a^2} & a^2-a\sqrt{2-3a^2} \\ a^2+a\sqrt{2-3a^2} & a^2-a\sqrt{2-3a^2} & 1-2a^2 \\ a^2-a\sqrt{2-3a^2} & 1-2a^2 & a^2+a\sqrt{2-3a^2} \end{bmatrix}$$

and

$$Q_2 = -R_2 = \begin{bmatrix} 1-2a^2 & a^2-a\sqrt{2-3a^2} & a^2+a\sqrt{2-3a^2} \\ a^2-a\sqrt{2-3a^2} & a^2+a\sqrt{2-3a^2} & 1-2a^2 \\ a^2+a\sqrt{2-3a^2} & 1-2a^2 & a^2-a\sqrt{2-3a^2} \end{bmatrix},$$

for $3a^2 \le 2$. Now the question is: are all the matrices Q_1, Q_2 represented by the matrix Q? The answer is yes, and this is easy to see. The obvious correspondence is to let

$$\frac{1+2\cos\theta}{3} = 1-2a^2 \Leftrightarrow \cos\theta = 1-3a^2.$$

This is well-defined since

$$-\sqrt{\frac{2}{3}} \le a \le \sqrt{\frac{2}{3}} \Rightarrow -1 \le 1-3a^2 \le 1.$$

Moreover, if $\cos\theta = 1-3a^2$ then

$$\sin^2\theta = 1-\cos^2\theta = 1-(1-3a^2)^2 = 6a^2-9a^4 = 3a^2(2-3a^2)$$

and consequently

$$\sin\theta = \sqrt{3}\,|a|\sqrt{2-3a^2}.$$

Thus

$$\frac{1}{3}(1-\cos\theta-\sqrt{3}\,\sin\theta) = \frac{1}{3}\left(1-(1-3a^2)-\sqrt{3}\,\sqrt{3}\,|a|\sqrt{2-3a^2}\right) = a^2-|a|\sqrt{2-3a^2}$$

and, similarly,

$$\frac{1}{3}(1-\cos\theta+\sqrt{3}\,\sin\theta) = a^2+|a|\sqrt{2-3a^2}.$$

So

$$Q = \begin{cases} Q_1 & \text{if} \quad a<0 \\ \\ Q_2 & \text{if} \quad a>0 \end{cases}.$$

Thus we see that every matrix in $\mathbb{U}+\mathbb{W}$ that represents a reflection, except for the reflection in the plane with equation $x+y+z=0$, is given by Q, and that all such matrices are in $\mathbb{W}$.

Let us find a basis for $E_1(Q)$. Let $\lambda = 1$ and solve the homogeneous system $(Q-I)\vec{x} = \vec{0}$:

$$\begin{bmatrix} 1+2\cos\theta-3 & 1-\cos\theta-\sqrt{3}\sin\theta & 1-\cos\theta+\sqrt{3}\sin\theta \\ 1-\cos\theta-\sqrt{3}\sin\theta & 1-\cos\theta+\sqrt{3}\sin\theta-3 & 1+2\cos\theta \\ 1-\cos\theta+\sqrt{3}\sin\theta & 1+2\cos\theta & 1-\cos\theta-\sqrt{3}\sin\theta-3 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & 0 \\ 1-\cos\theta-\sqrt{3}\sin\theta & -\cos\theta+\sqrt{3}\sin\theta-2 & 1+2\cos\theta \\ 1-\cos\theta+\sqrt{3}\sin\theta & 1+2\cos\theta & -\cos\theta-\sqrt{3}\sin\theta-2 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & 0 \\ 1-\cos\theta-\sqrt{3}\sin\theta & -\cos\theta+\sqrt{3}\sin\theta-2 & 1+2\cos\theta \\ 2\sqrt{3}\sin\theta & 3\cos\theta-\sqrt{3}\sin\theta+3 & -3\cos\theta-\sqrt{3}\sin\theta-3 \end{bmatrix}$$

Note that $\vec{d}$ is one vector in the null space of this matrix. But $\dim(E_1(Q)) = 2$ so the rank of this matrix must be 1, which means the last two rows of the matrix must be dependent. And they are, regardless of θ, since

$$\begin{bmatrix} 1-\cos\theta-\sqrt{3}\sin\theta \\ -\cos\theta+\sqrt{3}\sin\theta-2 \\ 1+2\cos\theta \end{bmatrix} \times \begin{bmatrix} 2\sqrt{3}\sin\theta \\ 3\cos\theta-\sqrt{3}\sin\theta+3 \\ -3\cos\theta-\sqrt{3}\sin\theta-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

as you can check. Of course, for certain values of θ either row could be zero, but they can't both be zero at the same time. To wit: if $\sin\theta = 0$ then $1+2\cos\theta \neq 0$, and vice versa. If $\theta \neq \pm\pi$ then the third row is not zero and

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2\sqrt{3}\sin\theta \\ 3\cos\theta-\sqrt{3}\sin\theta+3 \\ -3\cos\theta-\sqrt{3}\sin\theta-3 \end{bmatrix} = 3\begin{bmatrix} -2\cos\theta-2 \\ \cos\theta+\sqrt{3}\sin\theta+1 \\ \cos\theta-\sqrt{3}\sin\theta+1 \end{bmatrix}.$$

That is, Q represents a reflection in the plane with equation

$$(2\sqrt{3}\sin\theta)\,x + (3\cos\theta-\sqrt{3}\sin\theta+3)\,y - (3\cos\theta+\sqrt{3}\sin\theta+3)\,z = 0$$

and

$$E_1(Q) = \operatorname{span}\left\{ \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{12+12\cos\theta}}\begin{bmatrix} -2-2\cos\theta \\ \cos\theta+\sqrt{3}\sin\theta+1 \\ \cos\theta-\sqrt{3}\sin\theta+1 \end{bmatrix} \right\}.$$

If $\theta = \pm\pi$ then the third row *is* zero, but the second row is $[2\ \ -1\ \ -1]$ so we can take

$$\vec{w} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

and $\{\vec{d}, \vec{w}\}$ is a basis for $E_1(Q)$. In this case

$$Q = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix},$$

which represents a reflection in the plane with equation $2x - y - z = 0$.

Closing the circle, just as the negative of a rotation matrix of order 2 is a reflection, the negative of a reflection matrix is a rotation matrix of order 2. Therefore we can revisit Case 1 and say that every matrix in $\mathbb{U}+\mathbb{W}$ that represents a rotation of order 2 is given by

$$(-P_{(23)})D_1(\theta) = -\frac{1}{3}\begin{bmatrix} 1+2\cos\theta & 1-\cos\theta-\sqrt{3}\sin\theta & 1-\cos\theta+\sqrt{3}\sin\theta \\ 1-\cos\theta-\sqrt{3}\sin\theta & 1-\cos\theta+\sqrt{3}\sin\theta & 1+2\cos\theta \\ 1-\cos\theta+\sqrt{3}\sin\theta & 1+2\cos\theta & 1-\cos\theta-\sqrt{3}\sin\theta \end{bmatrix} \in \mathbb{W},$$

and

$$R = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \in \mathbb{U}.$$

Case 4. Orthogonal Matrices in $\mathbb{U}+\mathbb{W}$ That Represent Improper Rotations: again with $D_1(\theta)$ as in Case 2 consider the product of $-I$ and $D_1(\theta)$. The matrix $-D_1(\theta)$ will still be an orthogonal matrix in $\mathbb{U}$ but now

- $\det(-D_1(\theta)) = -1$.
- $\operatorname{tr}(-D_1(\theta)) = -1 - 2\cos\theta$.
- $\vec{d} \in E_{-1}(-D_1(\theta))$.

If $\theta \neq 0, \pm\pi$ then the only real eigenvalue of $-D_1(\theta)$ is $\lambda = -1$ so the matrix $-D_1(\theta)$ represents an improper rotation around an axis parallel to $\vec{d}$. But if $\theta = 0$ then $-D_1(\theta) = -I$, the inversion matrix; if $\theta = \pm\pi$ then

$$-D_1(\theta) = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix},$$

which represents a reflection in the plane normal to the vector $\vec{d}$.

Conversely, if X represents an improper rotation around a line parallel to the vector $\vec{d}$, then $-X$ will represent a proper rotation around a line parallel to the vector $\vec{d}$, since

$$\vec{d} \in E_{-1}(X) \Leftrightarrow \vec{d} \in E_1(-X).$$

Thus all the orthogonal matrices in $\mathbb{U} + \mathbb{W}$ that represent improper rotations are just the negatives of the rotation matrices in Case 2 with order greater than 2. □

The Group of Orthogonal Matrices in $\mathbb{U}+\mathbb{W}$: let $\mathbb{D}$ be the group of all orthogonal matrices in $\mathbb{U} + \mathbb{W}$. Let

- $\mathbb{D}_1 = \{D_1(\theta) \mid \theta \in \mathbb{R}\} \subset \mathbb{U}$.
- $\mathbb{D}_2 = (-P_{(23)})\mathbb{D}_1 \subset \mathbb{W}$.
- $\mathbb{D}_3 = -\mathbb{D}_1 \subset \mathbb{U}$.
- $\mathbb{D}_4 = P_{(23)}\,\mathbb{D}_1 \subset \mathbb{W}$.

Theorem 10 *Every matrix in $\mathbb{D}$ is a deranged matrix—it is either in $\mathbb{U}$ or it is in $\mathbb{W}$—and*

$$\mathbb{D} = \mathbb{D}_1 \cup \mathbb{D}_2 \cup \mathbb{D}_3 \cup \mathbb{D}_4.$$

Moreover, $\mathbb{D}_1$ is a normal subgroup of $\mathbb{D}$.

Proof: our analyses in Case 1 thru 4 establish the first part of the theorem. As for the second part, let P_σ be any odd 3×3 permutation matrix and let $A \in \mathbb{D}_1 \subset \mathbb{U}$. There are scalars a, b, c such that $A = a\,I + c\,P_{(123)} + b\,P_{(132)}$. then

$$P_\sigma\,A\,P_\sigma = a\,I + c\,P_\sigma P_{(123)} P_\sigma + b\,P_\sigma P_{(132)} P_\sigma = a\,I + c\,P_{(132)} + b\,P_{(123)} = A^T.$$

Since $A^{-1} = A^T$ if A is orthogonal, $P_\sigma\,A\,P_\sigma = A^{-1} \in \mathbb{D}_1$. Thus $\mathbb{D}_1 \lhd \mathbb{D}$, and $\mathbb{D}/\mathbb{D}_1 \cong C_2 \times C_2$. □

Subgroups of $\mathbb{D}$: $\mathbb{D}$ has four obvious subgroups:

1. $\mathbb{D}_1 \cong SO(2)$; that is, $\mathbb{D}_1$ consists of all possible rotations around the axis parallel to the vector $\vec{d}$. These are all counter-symmetric deranged matrices. If $A \in \mathbb{D}_1$, we know that A can be written in at least two different, but equivalent, ways:

(a)

$$A = \frac{1}{3}\begin{bmatrix} 1+2\cos\theta & 1-\sqrt{3}\sin\theta-\cos\theta & 1+\sqrt{3}\sin\theta-\cos\theta \\ 1+\sqrt{3}\sin\theta-\cos\theta & 1+2\cos\theta & 1-\sqrt{3}\sin\theta-\cos\theta \\ 1-\sqrt{3}\sin\theta-\cos\theta & 1+\sqrt{3}\sin\theta-\cos\theta & 1+2\cos\theta \end{bmatrix},$$

for some some θ.

(b) $A = a\,I + c\,P_{(123)} + b\,P_{(132)} = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$ for some a, b, c such that $a^2+b^2+c^2=1$ and $a+b+c=1$. This is because as we saw above,

$$\det(A) = (a+b+c)(a^2+b^2+c^2-ab-ac-bc).$$

So if A is orthogonal and represents a rotation, then $\det(A)=1, a^2+b^2+c^2=1$, and $ac+ba+cb=0$; consequently $a+b+c=1$.

2. $\mathbb{D}_1 \cup \mathbb{D}_2 = \{A \in \mathbb{D} \mid \det(A)=1\} \cong O(2)$. In addition to all the rotations in $\mathbb{D}_1$, this subgroup contains all the symmetric deranged matrices in $\mathbb{D}$ with determinant 1 that represent rotations of order 2.

3. $\mathbb{D}_1 \cup \mathbb{D}_3 \cong SO(2)\times C_2$. In addition to all the rotations in $\mathbb{D}_1$, this subgroup contains the inversion matrix, the matrix of order 2 that represents a reflection in the plane with equation $x+y+z=0$, and all the counter-symmetric deranged matrices with determinant -1 that represent improper rotations.

4. $\mathbb{D}_1 \cup \mathbb{D}_4 \cong O(2)$. In addition to all the rotations in $\mathbb{D}_1$, this subgroup contains all the symmetric deranged matrices in $\mathbb{D}$ with determinant -1 that represent reflections.

Calculating Orthogonal Deranged Matrices: let

$$A = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} \in \mathbb{U}$$

be a counter-symmetric deranged matrix. If $a^2+b^2+c^2=1$ and $a+b+c=\pm 1$, then automatically $ab+ac+bc=0$:

$$\begin{aligned} a+b+c=\pm 1 \;&\Rightarrow\; (a+b+c)^2=1 \\ &\Rightarrow\; a^2+b^2+c^2+2ab+2bc+2ac=1 \\ &\Rightarrow\; 2ab+2bc+2ac=0,\ \text{since } a^2+b^2+c^2=1 \\ &\Rightarrow\; ab+bc+ac=0. \end{aligned}$$

Since $\det(A)=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$, A will be orthogonal with $\det(A)=1$ if $a^2+b^2+c^2=1$ and $a+b+c=1$, and A will be orthogonal with $\det(A)=-1$ if $a^2+b^2+c^2=1$ and $a+b+c=-1$.

Now let

$$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \in \mathbb{W}$$

be a symmetric deranged matrix. In this case

$$\det(A) = -(a+b+c)(a^2+b^2+c^2-ab-ac-bc).$$

Arguing similarly we can show that A will be orthogonal with $\det(A)=1$ if $a^2+b^2+c^2=1$ and $a+b+c=-1$, and A will be orthogonal with $\det(A)=-1$ if $a^2+b^2+c^2=1$ and $a+b+c=1$.

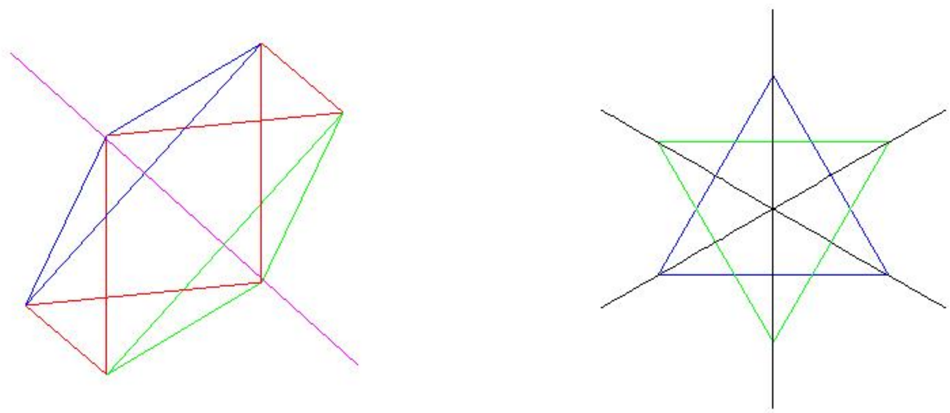


Figure 5: In the figure on the left: a 3-antiprism centered along an axis, in magenta, parallel to the vector $\vec{d}$. The symmetry group of this antiprism is D_6. In the figure on the right: the vector $\vec{d}$ is coming out of the page; the black lines represent the planes of reflection of the 3-antiprism.

Example 10: we shall now list some examples of specific sets of orthogonal deranged matrices. The 12 possible orthogonal deranged matrices with two zero's in each column are listed in Table 4. These 12 matrices form the symmetry group of a 3-antiprism. See Figure 5, which shows a 3-antiprism centered along the diagonal of a cube parallel to $\vec{d}$. The blue equilateral triangle has vertices

$$(1,1,-1),(1,-1,1),(-1,1,1)$$

and the green equilateral triangle has vertices

$$(-1,-1,1),(-1,1,-1),(1,-1,-1).$$

The symmetry group of the 3-antiprism is isomorphic to $D_3 \times C_2 \cong D_6$: the six permutation matrices in Table 4 form a copy of D_3 and the remaining matrices in Table 4 are obtained from the permutation matrices by multiplying by $-I$. The three rotations and three reflections act on *each* triangle in Figure 5 just like the symmetries of an equilateral triangle in the plane. The reflections are in the planes

$$y=z, x=z, x=y,$$

and act on the vertices of the 3-antiprism by swapping y and z coordinates, x and z coordinates, or x and y coordinates, respectively. See the right half of Figure 5, where the planes are indicated in black. Of course, the normals to these planes,

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

are the axes of rotations for the rotations of order 2 in Table 4. Finally, the improper rotations of order 6 are also called 3-fold rotoinversions; that is, they act on the 3-antiprism by rotating each triangle in Figure 5 by $2\pi/3$, followed by inversion.

Let us investigate this a little more closely. Consider the improper rotation of order 6, $A = -P_{(123)}$, which is obviously the product of a rotation of $2\pi/3$ and inversion. But how does A act on the two equilateral triangles of Figure 5? Start with a vector in the plane with

rotation matrix A	$\det(A)$	$\mathrm{tr}\,(A)$	order of A	non-rotation matrix A	$\det(A)$	$\mathrm{tr}\,(A)$	order of A
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1	3	1	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	-1	-3	2
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	1	0	3	$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$	-1	0	6
$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	1	0	3	$\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$	-1	0	6
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	1	-1	2	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	-1	1	2
$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	1	-1	2	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	-1	1	2
$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	1	-1	2	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	-1	1	2

Table 4: The group of 12 orthogonal deranged matrices with two zero's in each column. The six rotation matrices form a copy of D_3. As a group the 12 matrices form a copy of $D_3 \times C_2 \cong D_6$, the full symmetry group of a 3-antiprism. All the matrices with order 2 are symmetric and all the other matrices are counter-symmetric. The six permutation matrices form another copy of D_3. The top six counter-symmetric matrices, above the double line, form a copy of C_6. Note that the three matrices in the top left are all rotation matrices with determinant $+1$ and form a group, the three matrices in the bottom right are odd permutations of the three matrices in the top left, the three matrices in the top right are the negatives of the three matrices in the top left, and the three matrices in the bottom left are the negatives of odd permutations of the three matrices in the top left.

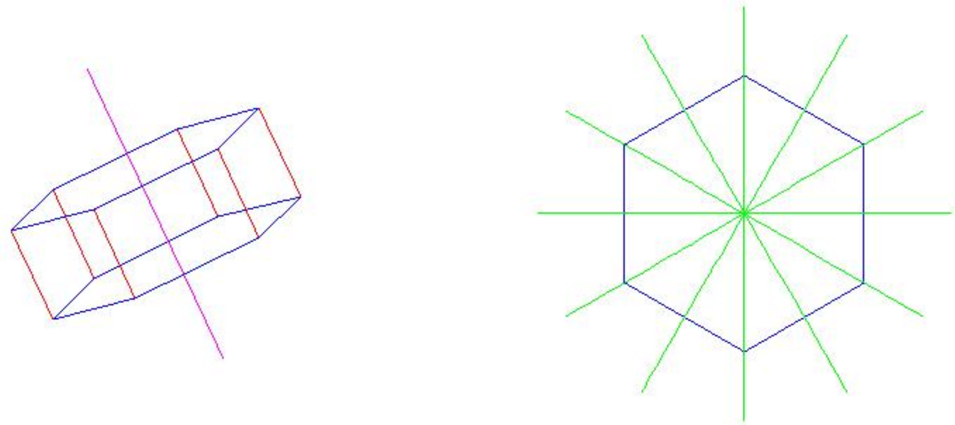


Figure 6: In the figure on the left: a 6-prism centered along the diagonal, in magenta, parallel to the vector $\vec{d}$. Its full symmetry group is $D_6 \times C_2$. The end hexagons, in blue, are in the planes $x + y + z = \pm 1$. In the figure on the right: the vector $\vec{d}$ is coming out of the page, and the six planes of reflection of the 6-prism are indicated in green.

equation $x + y + z = 0$ and compute the angle between it and its image under A:

$$\cos\theta = \frac{\begin{bmatrix} x \\ y \\ -x-y \end{bmatrix} \cdot \begin{bmatrix} -y \\ x+y \\ -x \end{bmatrix}}{\left\| \begin{bmatrix} x \\ y \\ -x-y \end{bmatrix} \right\| \left\| \begin{bmatrix} -y \\ x+y \\ -x \end{bmatrix} \right\|} = \frac{x^2 + xy + y^2}{2x^2 + 2xy + 2y^2} = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}.$$

So, projected into the plane with equation $x + y + z = 0$, the effect of A on the blue and green triangles of Figure 5 is to rotate them by $\pi/3$, moving the vertices of one "onto" the vertices of the other. Actually, the two triangles are on different planes, $x + y + z = \pm 1$, but inversion matches the vertices accordingly. □

Example 11: another triple that satisfies $a^2 + b^2 + c^2 = 1$ and $a + b + c = 1$ is

$$\{a, b, c\} = \left\{\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right\}.$$

The 12 orthogonal deranged matrices with columns taken from

$$\left\{\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right\} \text{ or } \left\{-\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\}$$

are displayed in Table 5. These 12 matrices do not form a group. But together with the 12 matrices in Table 4, they form a group of order 24 which is a copy of $D_6 \times C_2$. This group is the full symmetry group of a 6-prism; see Figure 6, in which the two end hexagons are in blue. The 12 rotations from Tables 4 and 5 form a copy of D_6; combined with the negatives of these 12 matrices you get all 24 matrices. The counter-symmetric rotations from Tables 4 and 5 form a cyclic group of order 6 about the line passing through the origin parallel to $\vec{d}$; the symmetric rotations from Tables 4 and 5 are rotations of order 2 around lines joining the centres of opposite rectangles on the sides of the 6-prism, or around lines joining the midpoints of opposite red

rotation matrix A	$\det(A)$	$\operatorname{tr}(A)$	order of A	non-rotation matrix A	$\det(A)$	$\operatorname{tr}(A)$	order of A
$\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$	1	−1	2	$\frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$	−1	1	2
$\frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$	1	2	6	$\frac{1}{3}\begin{bmatrix} -2 & -2 & 1 \\ 1 & -2 & -2 \\ -2 & 1 & -2 \end{bmatrix}$	−1	−2	6
$\frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}$	1	2	6	$\frac{1}{3}\begin{bmatrix} -2 & 1 & -2 \\ -2 & -2 & 1 \\ 1 & -2 & -2 \end{bmatrix}$	−1	−2	6
$\frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{bmatrix}$	1	−1	2	$\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$	−1	1	2
$\frac{1}{3}\begin{bmatrix} -2 & 1 & -2 \\ 1 & -2 & -2 \\ -2 & -2 & 1 \end{bmatrix}$	1	−1	2	$\frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$	−1	1	2
$\frac{1}{3}\begin{bmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{bmatrix}$	1	−1	2	$\frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$	−1	1	2

Table 5: The 12 orthogonal deranged matrices with entries in each column from $\left\{\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right\}$ or $\left\{-\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\}$. The six above the double line are counter-symmetric; the six below the double line are symmetric. These 12 matrices do not form a group. But together with the 12 matrices in Table 4, they form a group of order 24 which is a copy of $D_6 \times C_2$. This group is the full symmetry group of a 6-prism. The pattern of matrices in this table is the same as that in Table 4: the matrices in the bottom right are odd permutations of the matrices in the top left, the matrices in the top right are the negatives of the matrices in the top left, and the matrices in the bottom left are the negatives of odd permutations of the matrices in the top left.

edges. See Figure 6. These six rotations of order 2 are all around lines perpendicular to $\vec{d}$, namely lines parallel to the vectors

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

There is another copy of D_6 comprised of the six counter-symmetric rotations and six of the symmetric reflections. This copy of D_6 acts on each blue hexagon exactly as the symmetries of a hexagon in the plane. See the right half of Figure 6, in which vector $\vec{d}$ is pointing out of the page and the six planes of symmetry of the hexagon are indicated in green. The actual equations of the six planes are

$$y = z, x = z, x = y, -2x + y + z = 0, x + y - 2z = 0, x - 2y + z = 0.$$

Again, combining these 12 matrices with $-I$ results in all 24 matrices in Tables 4 and 5.

Now consider the reflection matrix

$$Q = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix},$$

which represents a reflection in the plane with equation $x + y + z = 0$. The products of Q and the 12 rotation matrices from Tables 4 and 5 produce the other 12 matrices in Tables 4 and 5. In particular, the products of Q and the two rotations of order 6 result in the two improper rotations of order 6 that are in Table 4; the products of Q and the two rotations of order 3 result in the other two improper rotations of order 6. And the product of Q and $-Q$, which represents a rotation of order 2 around $\vec{d}$, results in the inversion matrix $-I$. □

Example 12: another triple that satisfies $a^2 + b^2 + c^2 = 1$ and $a + b + c = 1$ is

$$\{a, b, c\} = \left\{ \frac{\phi}{2}, \frac{1}{2}, \frac{\psi}{2} \right\}.$$

There are 24 orthogonal deranged matrices that can be formed using this triple or the corresponding triples with $a + b + c = -1$. But the 12 counter-symmetric matrices all turn out to have infinite order. In particular, the 6 counter-symmetric matrices A with $\det(A) = 1$ satisfy

$$\operatorname{tr}(A) = \frac{3}{2}, \ \frac{3\phi}{2} \text{ or } \frac{3\psi}{2}.$$

Suppose A represents a rotation of θ around a line passing through the origin parallel to $\vec{d}$ and

$$\frac{3}{2} = 1 + 2\cos\theta \Leftrightarrow \cos\theta = \frac{1}{4}.$$

But

$$\cos\theta \neq 0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2} \text{ or } 1,$$

so θ is not[5] a rational multiple of π and so A has infinite order. Or if

$$\operatorname{tr}(A) = \frac{3\phi}{2}$$

[5]See Varona, Rational Values of the Arccosine Function, *Cent. Eur. J. Math.* **4** (2006), no. 2, 319-322

and A represents a rotation of α around a line passing through the origin parallel to $\vec{d}$ with

$$\frac{3\phi}{2} = 1 + 2\cos\alpha \Leftrightarrow \cos\alpha = \frac{3\phi}{4} - \frac{1}{2},$$

then we claim

$$\cos(\theta + \alpha) = -\frac{1}{2},$$

with $\theta = \arccos(1/4)$. We have $\sin\theta = \pm\sqrt{15}/4$ and

$$\begin{aligned}
\sin\alpha &= \pm\sqrt{1 - \left(\frac{3\phi}{4} - \frac{1}{2}\right)^2} \\
&= \pm\sqrt{1 - \left(\frac{9\phi^2}{16} - \frac{3\phi}{4} + \frac{1}{4}\right)} \\
&= \pm\sqrt{\frac{3}{4} - \frac{9\phi^2}{16} + \frac{3\phi}{4}} \\
&= \pm\sqrt{\frac{3}{4}(1 + \phi) - \frac{9\phi^2}{16}} \\
&= \pm\sqrt{\frac{3\phi^2}{4} - \frac{9\phi^2}{16}} \\
&= \pm\sqrt{\frac{3\phi^2}{16}} \\
&= \pm\frac{\sqrt{3}}{4}\phi.
\end{aligned}$$

Then we can take

$$\begin{aligned}
\cos(\theta + \alpha) &= \cos\theta\cos\alpha - \sin\theta\sin\alpha \\
&= \frac{1}{4}\left(\frac{3\phi}{4} - \frac{1}{2}\right) - \frac{\sqrt{15}}{4}\frac{\sqrt{3}}{4}\phi \\
&= \frac{3\phi}{16} - \frac{1}{8} - \frac{3\phi}{16}\sqrt{5} \\
&= \frac{3\phi}{8}\frac{1 - \sqrt{5}}{2} - \frac{1}{8} \\
&= \frac{3}{8}\phi\psi - \frac{1}{8} \\
&= -\frac{3}{8} - \frac{1}{8} \\
&= -\frac{1}{2}.
\end{aligned}$$

Thus

$$\theta + \alpha = \frac{2\pi}{3}.$$

Since θ is not a rational multiple of π, neither is α, and consequently A has infinite order. □

Example 13: the 12 symmetric deranged matrices that can be formed using the triples

$$\left\{\frac{\phi}{2}, \frac{1}{2}, \frac{\psi}{2}\right\} \text{ or } \left\{-\frac{\phi}{2}, -\frac{1}{2}, -\frac{\psi}{2}\right\}$$

are in Table 6. As we shall see, the top six matrices represent symmetries of an icosahedron with vertices

$$\begin{bmatrix} 0 \\ \pm 1 \\ \pm\phi \end{bmatrix}, \begin{bmatrix} \pm\phi \\ 0 \\ \pm 1 \end{bmatrix}, \begin{bmatrix} \pm 1 \\ \pm\phi \\ 0 \end{bmatrix},$$

and of a dodecahedron with vertices

$$\begin{bmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \pm\phi \\ \pm\psi \end{bmatrix}, \begin{bmatrix} \pm\psi \\ 0 \\ \pm\phi \end{bmatrix}, \begin{bmatrix} \pm\phi \\ \pm\psi \\ 0 \end{bmatrix};$$

while the bottom six matrices in Table 6 represent symmetries of an icosahedron with vertices

$$\begin{bmatrix} 0 \\ \pm 1 \\ \pm\psi \end{bmatrix}, \begin{bmatrix} \pm\psi \\ 0 \\ \pm 1 \end{bmatrix}, \begin{bmatrix} \pm 1 \\ \pm\psi \\ 0 \end{bmatrix},$$

and of a dodecahedron with vertices

$$\begin{bmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \pm\psi \\ \pm\phi \end{bmatrix}, \begin{bmatrix} \pm\phi \\ 0 \\ \pm\psi \end{bmatrix}, \begin{bmatrix} \pm\psi \\ \pm\phi \\ 0 \end{bmatrix}.$$

But all 12 matrices in Table 6 have order 2, so there must be many other matrices that represent symmetries of an icosahedron or a dodecahedron which are not symmetric or counter-symmetric matrices formed using the triples

$$\{a, b, c\} = \left\{\frac{\phi}{2}, \frac{1}{2}, \frac{\psi}{2}\right\}.$$

How can we generate these other matrices? □

We hope to show that the symmetry groups of all the Platonic solids are comprised of orthogonal signed variations of orthogonal deranged matrices. We already did it for the cube: consider the case $a = 1, b = c = 0$. The six orthogonal deranged matrices with $a+b+c = 1$ are none other than the permutation matrices; three of which are counter-symmetric

$$P_{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; P_{(123)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \; P_{(132)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix};$$

and three of which are symmetric

$$P_{(23)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; P_{(13)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; P_{(12)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

All eight possible signed variations of each of these six matrices are still orthogonal, resulting in a group of 48 signed 3×3 permutation matrices, $S(C)$, the symmetry group of the cube.

But before we look at the symmetry group of other Platonic solids, the rest of Part 3 will focus on the symmetry group of the cube.

rotation matrix A	basis of $E_1(A)$	$\operatorname{tr}(A)$	reflection matrix A	$\det(A)$	$\operatorname{tr}(A)$	equation of $E_1(A)$
$-\frac{1}{2}\begin{bmatrix} \psi & \phi & 1 \\ \phi & 1 & \psi \\ 1 & \psi & \phi \end{bmatrix}$	$\begin{bmatrix} \phi \\ -1 \\ \psi \end{bmatrix}$	-1	$\frac{1}{2}\begin{bmatrix} \psi & \phi & 1 \\ \phi & 1 & \psi \\ 1 & \psi & \phi \end{bmatrix}$	-1	1	$\phi x - y + \psi z = 0$
$-\frac{1}{2}\begin{bmatrix} \phi & 1 & \psi \\ 1 & \psi & \phi \\ \psi & \phi & 1 \end{bmatrix}$	$\begin{bmatrix} \psi \\ \phi \\ -1 \end{bmatrix}$	-1	$\frac{1}{2}\begin{bmatrix} \phi & 1 & \psi \\ 1 & \psi & \phi \\ \psi & \phi & 1 \end{bmatrix}$	-1	1	$\psi x + \phi y - z = 0$
$-\frac{1}{2}\begin{bmatrix} 1 & \psi & \phi \\ \psi & \phi & 1 \\ \phi & 1 & \psi \end{bmatrix}$	$\begin{bmatrix} -1 \\ \psi \\ \phi \end{bmatrix}$	-1	$\frac{1}{2}\begin{bmatrix} 1 & \psi & \phi \\ \psi & \phi & 1 \\ \phi & 1 & \psi \end{bmatrix}$	-1	1	$x - \psi y - \phi z = 0$
$-\frac{1}{2}\begin{bmatrix} \phi & \psi & 1 \\ \psi & 1 & \phi \\ 1 & \phi & \psi \end{bmatrix}$	$\begin{bmatrix} \psi \\ -1 \\ \phi \end{bmatrix}$	-1	$\frac{1}{2}\begin{bmatrix} \phi & \psi & 1 \\ \psi & 1 & \phi \\ 1 & \phi & \psi \end{bmatrix}$	-1	1	$\psi x - y + \phi z = 0$
$-\frac{1}{2}\begin{bmatrix} \psi & 1 & \phi \\ 1 & \phi & \psi \\ \phi & \psi & 1 \end{bmatrix}$	$\begin{bmatrix} \phi \\ \psi \\ -1 \end{bmatrix}$	-1	$\frac{1}{2}\begin{bmatrix} \psi & 1 & \phi \\ 1 & \phi & \psi \\ \phi & \psi & 1 \end{bmatrix}$	-1	1	$\phi x + \psi y - z = 0$
$-\frac{1}{2}\begin{bmatrix} 1 & \phi & \psi \\ \phi & \psi & 1 \\ \psi & 1 & \phi \end{bmatrix}$	$\begin{bmatrix} -1 \\ \phi \\ \psi \end{bmatrix}$	-1	$\frac{1}{2}\begin{bmatrix} 1 & \phi & \psi \\ \phi & \psi & 1 \\ \psi & 1 & \phi \end{bmatrix}$	-1	1	$x - \phi y - \psi z = 0$

Table 6: The 12 orthogonal deranged matrices of finite order with entries in each column from $\left\{\frac{\phi}{2}, \frac{1}{2}, \frac{\psi}{2}\right\}$ or $\left\{-\frac{\phi}{2}, -\frac{1}{2}, -\frac{\psi}{2}\right\}$; they are all symmetric with order 2. The rotation matrices are in the column on the left; the reflection matrices are in the column on the right. These matrices all represent symmetries of a dodecahedron or an icosahedron. The six matrices in the bottom half are obtained from the matrices in the top half by swapping ϕ and ψ.

Structure of $S(C)$, Third Pass: in terms of Φ and det we can define two homomorphisms from $S(C)$ to $\{1,-1\}$:

$$\operatorname{sgn}(A) = \det(\Phi(A)) \text{ and } \operatorname{par}(A) = \det(A\,\Phi(A)).$$

Observe that

$$\operatorname{sgn}(A) = \begin{cases} 1 & \text{if } \Phi(A) = I, P_{(123)} \text{ or } P_{(132)}, \text{ all even permutations of the columns of } I, \\ -1 & \text{if } \Phi(A) = P_{(12)}, P_{(13)} \text{ or } P_{(23)}, \text{ all odd permutations of the columns of } I. \end{cases}$$

Of course

$$\det(A) = \begin{cases} +1 & \text{if } A \text{ represents a rotation,} \\ -1 & \text{if } A \text{ does not represent a rotation.} \end{cases}$$

Now, the product of any two of $\det(A), \operatorname{par}(A), \operatorname{sgn}(A)$ equals the third one. For example,

$$\operatorname{par}(A) = \det(A\Phi(A)) = \det(A)\ \det(\Phi(A)) = \det(A)\operatorname{sgn}(A).$$

Using this formula, you can verify that

$$\operatorname{par}(A) = \begin{cases} +1 & \text{if } A \text{ has an even number of negative entries,} \\ -1 & \text{if } A \text{ has an odd number of negative entries.} \end{cases}$$

Hence we shall call $\operatorname{par}(A)$ the parity of A. Moreover, just as similar matrices have the same determinant, so do they have the same sign and parity. For example,

$$\begin{aligned} \operatorname{sgn}(BAB^{-1}) &= \operatorname{sgn}(B)\operatorname{sgn}(A)\operatorname{sgn}(B^{-1}) \\ &= \operatorname{sgn}(B)\operatorname{sgn}(B^{-1})\operatorname{sgn}(A) \\ &= \operatorname{sgn}(I)\operatorname{sgn}(A) \\ &= \operatorname{sgn}(A), \text{ since } \operatorname{sgn}(I) = 1. \end{aligned}$$

We can use these homomorphism to define three normal subgroups of $S(C)$ of order 24:

1. $\mathcal{S} = \ker(\operatorname{sgn}) = \{A \in S(C) \mid \operatorname{sgn}(A) = 1\}$. See Table 7.
2. $\mathcal{P} = \ker(\operatorname{par}) = \{A \in S(C) \mid \operatorname{par}(A) = 1\}$. See Table 8.
3. $\mathcal{R} = \ker(\det) = \{A \in S(C) \mid \det(A) = 1\}$. See Table 9.

Call the subgroup of order 12 given by their intersection $\mathcal{A}$. That is:

$$\mathcal{A} = \ker(\det) \cap \ker(\operatorname{sgn}) \cap \ker(\operatorname{par}) = \{A \in S(C) \mid \det(A) = \operatorname{sgn}(A) = \operatorname{par}(A) = 1\}.$$

Since the intersection of normal subgroups is also normal, $\mathcal{A}$ is a normal subgroup of $S(C)$. Looking at the top 12 matrices in Table 7 you can see that $\mathcal{A}$ consists of I, the three rotation matrices J, K, L of order 2, and all the rotations of order 3. So $\mathcal{A} \cong A_4$. In terms of $\mathcal{A}$, we have

1. $\ker(\operatorname{sgn}) = \{A \in S(C) \mid \operatorname{sgn}(A) = 1\} = \mathcal{A} \cup (-\mathcal{A}) \cong A_4 \times C_2$,
2. $\ker(\operatorname{par}) = \{A \in S(C) \mid \operatorname{par}(A) = 1\} = \mathcal{A} \cup (P_{(23)}\,\mathcal{A}) \cong S_4$,
3. $\ker(\det) = \{A \in S(C) \mid \det(A) = 1\} = \mathcal{A} \cup (-P_{(23)}\,\mathcal{A}) \cong S_4$,

as can easily be verified. Note that $S(C)$ contains two copies of S_4, namely $\mathcal{P}$ and $\mathcal{R}$, and that a matrix A represents an even permutation in either copy of S_4 if and only if $\operatorname{sgn}(A) = 1$, while A represents an odd permutation in either copy of S_4 if and only if $\operatorname{sgn}(A) = -1$. See Tables 8 and 9.

$A \in \ker(\text{sgn}) \Leftrightarrow \text{sgn}(A) = 1$	$\text{tr}(A)$	order of A
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	3	1
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	-1	2
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$	0	3
$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$	0	3
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	-3	2
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	1	2
$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$	0	6
$\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$	0	6

Table 7: $\mathcal{S} = \ker(\text{sgn})$. The top 12 matrices $A \in S(C)$ with $\det(A) = \text{sgn}(A) = \text{par}(A) = 1$ form a copy of A_4. The bottom 12 matrices are obtained from the top 12 by multiplying by $-I$, so $\mathcal{S} \cong A_4 \times C_2$. Moreover, for each matrix A in the bottom 12, $\det(A) = -1 = \text{par}(A)$.

$A \in \ker(\mathrm{par}) \Leftrightarrow \mathrm{par}(A) = 1$	$\mathrm{tr}(A)$	order of A
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	3	1
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	−1	2
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$	0	3
$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$	0	3
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1	2
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1	2
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	−1	4
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	−1	4

Table 8: $\mathcal{P} = \ker(\mathrm{par})$. The top 12 matrices $A \in S(C)$ with $\det(A) = \mathrm{sgn}(A) = \mathrm{par}(A) = 1$ form a copy of A_4. The bottom 12 matrices are obtained from the top 12 by multiplying by $P_{(23)}$, so $\mathcal{P} \cong S_4$. Moreover, for each matrix A in the bottom 12, $\det(A) = -1 = \mathrm{sgn}(A)$.

$A \in \ker(\det) \Leftrightarrow \det(A) = 1$	$\mathrm{tr}\,(A)$	order of A
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	3	1
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	-1	2
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$	0	3
$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$	0	3
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	-1	2
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	-1	2
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1	4
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1	4

Table 9: $\mathcal{R} = \ker(\det)$. The top 12 matrices $A \in S(C)$ with $\det(A) = \mathrm{sgn}\,(A) = \mathrm{par}\,(A) = 1$ form a copy of A_4. The bottom 12 matrices are obtained from the top 12 by multiplying by $-P_{(23)}$, so $\mathcal{R} \cong S_4$. Moreover, for each matrix A in the bottom 12, $\mathrm{sgn}\,(A) = -1 = \mathrm{par}\,(A)$.

Structure of $S(C)$, **Fourth Pass:** consider the map $\Omega : S(C) \longrightarrow S(C)$ defined by

$$\Omega(A) = \begin{bmatrix} \operatorname{sgn}(A) & 0 & 0 \\ 0 & \operatorname{par}(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}.$$

It is easy to check that Ω is a homomorphism. Moreover,

$$\ker(\Omega) = \{A \in S(C) \mid \operatorname{sgn}(A) = 1, \operatorname{par}(A) = 1, \det(A) = 1\} = \mathcal{A},$$

which is another proof that $\mathcal{A}$ is a normal subgroup of $S(C)$. The four cosets of $\mathcal{A}$, as exhibited in Tables 7, 8, 9, each containing 12 matrices, are

- $\mathcal{A} = \{A \in S(C) | \operatorname{sgn}(A) = 1, \operatorname{par}(A) = 1, \det(A) = 1\}$ and
$$\Omega(\mathcal{A}) = \{I\}.$$
- $(-I)\,\mathcal{A} = \{A \in S(C) | \operatorname{sgn}(A) = 1, \operatorname{par}(A) = -1, \det(A) = -1\}$ and
$$\Omega((-I)\,\mathcal{A}) = \{J\}.$$
- $(P_{(23)})\,\mathcal{A} = \{A \in S(C) | \operatorname{par}(A) = 1, \det(A) = -1, \operatorname{sgn}(A) = -1\}$ and
$$\Omega((P_{(23)})\,\mathcal{A}) = \{K\}.$$
- $(-P_{(23)})\,\mathcal{A} = \{A \in S(C) | \det(A) = 1, \operatorname{par}(A) = -1, \operatorname{sgn}(A) = -1\}$ and
$$\Omega((-P_{(23)})\,\mathcal{A}) = \{L\}.$$

Thus $\operatorname{im}(\Omega) = \{I, J, K, L\}$. Call this subgroup $\mathcal{V}$; it is isomorphic to the Klein 4-group. Then

$$S(C)/\mathcal{A} \cong \mathcal{V}$$

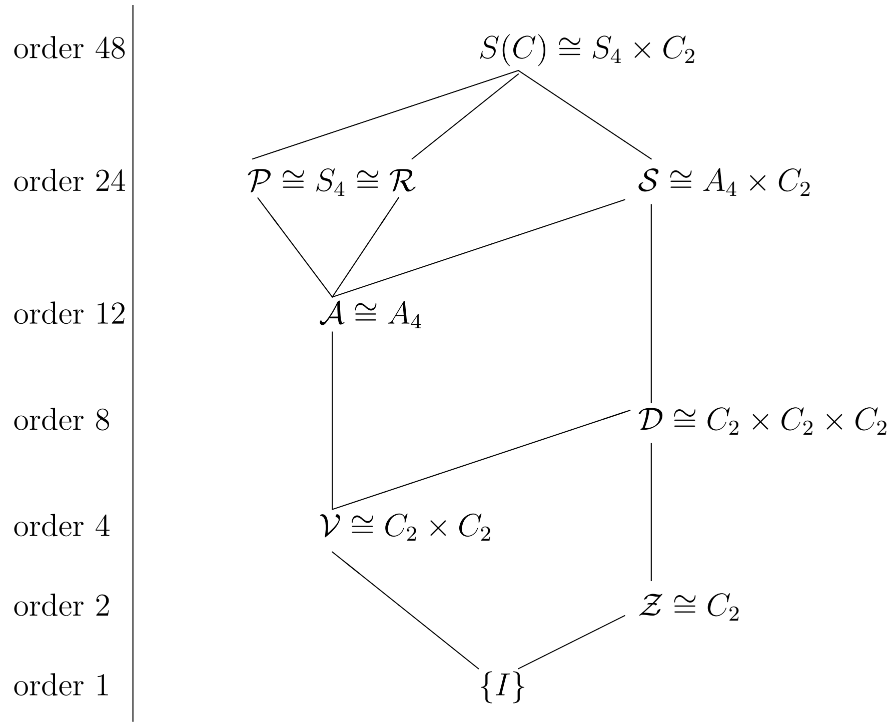
and

$$S(C) = \mathcal{A} \cup (-I)\,\mathcal{A} \cup (P_{(23)})\,\mathcal{A} \cup (-P_{(23)})\,\mathcal{A}.$$

Structure of $S(C)$, **Fifth Pass:** $S(C)$ has seven normal proper subgroups. They are

1. $\mathcal{A}$, where $\mathcal{A} = \ker(\operatorname{par}) \cap \ker(\det) \cap \ker(\operatorname{sgn})$,
2. $\mathcal{D}$, the group of diagonal matrices in $S(C)$,
3. $\mathcal{P} = \ker(\operatorname{par})$,
4. $\mathcal{R} = \ker(\det)$,
5. $\mathcal{S} = \ker(\operatorname{sgn})$,
6. $\mathcal{V} = \{I, J, K, L\}$, and
7. $\mathcal{Z} = \{I, -I\}$, the center of $S(C)$.

The lattice of normal subgroups of $S(C)$ is:



Note that of these nine normal subgroups only two of them are isomorphic to each other, namely $\mathcal{P}$ and $\mathcal{R}$. Every inner automorphism of $S(C)$ will fix its normal subgroups, but an outer automormphism need not. We proceed to find an outer automorphism of $S(C)$.

An Outer Automorphism of $S(C)$: consider the map $\rho : S(C) \to S(C)$ defined by

$$\rho(A) = (\operatorname{sgn}(A))\, A,$$

which is a homomorphism. In fact, ρ is an isomorphism since

$$\begin{aligned} \rho(A) = I \quad &\Rightarrow \quad (\operatorname{sgn}(A))\, A = I \\ &\Rightarrow \quad (\operatorname{sgn}(A))^2\, A = \operatorname{sgn}(A)\, I \\ &\Rightarrow \quad A = (\operatorname{sgn}(A))\, I, \text{ since } (\operatorname{sgn}(A))^2 = (\pm 1)^2 = 1 \end{aligned}$$

Thus $A = I$ or $A = -I$. If it is the latter then

$$\rho(A) = \rho(-I) = \operatorname{sgn}(-I)(-I) = (1)(-I) = -I \neq I,$$

so it must be the former: $A = I$.

We claim ρ is an outer automorphism of $S(C)$. If ρ were an inner automorphism of $S(C)$ then it would fix normal subgroups of $S(C)$. But we shall show that $\rho(\mathcal{P}) = \mathcal{R}$. To see this we shall use the fact that $\operatorname{sgn}(A) = \operatorname{par}(A)\det(A)$. Thus for all $A \in \mathcal{P}$,

$$\rho(A) = (\operatorname{par}(A)\ \det(A))A = (1 \cdot \det A)A = (\det(A))\, A;$$

and for all $A \in \mathcal{R}$,

$$\rho(A) = (\operatorname{par}(A)\ \det(A))A = (\operatorname{par}(A) \cdot 1)A = (\operatorname{par}(A))\, A.$$

Of course, for all $A \in \mathcal{S}$, $\rho(A) = A$. Then it is straightforward to check that

$$\rho(\mathcal{A}) = \mathcal{A},\ \rho((-P_{(23)})\, \mathcal{A}) = (P_{(23)})\, \mathcal{A},\ \rho((P_{(23)})\, \mathcal{A}) = (-P_{(23)})\, \mathcal{A} \text{ and } \rho((-I)\, \mathcal{A}) = (-I)\, \mathcal{A}.$$

In particular $\rho(\mathcal{P}) = \mathcal{R}$ and $\rho(\mathcal{R}) = \mathcal{P}$. Thus ρ maps two distinct normal subgroups of $S(C)$, namely $\mathcal{P}$ and $\mathcal{R}$, to each other, and so ρ is not an inner automorphism.

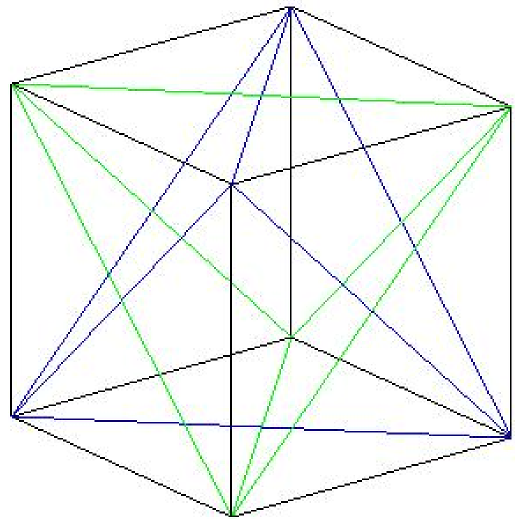


Figure 7: The cube contains two tetrahedra; they are inversions of each other.

Structure of $S(C)$, Sixth Pass: among the vertices of the cube are two tetrahedra: $\mathcal{E}$ with vertices

$$\{(1,1,1),(1,-1,-1),(-1,1,-1),(-1,-1,1)\};$$

and $\mathcal{O}$ with vertices

$$\{(-1,-1,-1),(-1,1,1),(1,-1,1),(1,1,-1)\}.$$

See Figure 7, in which $\mathcal{E}$ is the blue tetrahedron, and $\mathcal{O}$ is the green tetrahedron. Note that

1. $\mathcal{E}$ and $\mathcal{O}$ are geometric inversions of each other; that is $(-I)(\mathcal{E}) = \mathcal{O}$ and $(-I)(\mathcal{O}) = \mathcal{E}$.

2. The vertices of $\mathcal{E}$ all have an even number of negative signs.

3. The vertices of $\mathcal{O}$ all have an odd number of negative signs.

We claim every matrix $A \in \mathcal{P}$ fixes $\mathcal{E}$ and $\mathcal{O}$: every matrix $A \in \mathcal{P}$ has $\text{par}\,(A) = 1$, which means there are an even number of -1's in the entries of A. So multiplying any vertex of $\mathcal{E}$ by $A \in \mathcal{P}$ will result in another vertex with an even number of -1's, and multiplying any vertex of $\mathcal{O}$ by $A \in \mathcal{P}$ will result in another vertex with an odd number of -1's, as you may check.

On the other hand, every matrix $A \in (-I)\,\mathcal{A} \cup (-P_{(23)})\,\mathcal{A}$ will interchange the tetrahedra $\mathcal{E}$ and $\mathcal{O}$, since $\text{par}\,(A) = -1$, so the entries of A have an odd number of -1's. Additionally, every matrix $A \in \mathcal{S}$ is even, in the sense that $\text{sgn}\,(A) = 1$, but every matrix $A \in (P_{(23)})\,\mathcal{A} \cup (-P_{(23)})\,\mathcal{A}$ is odd, in the sense that $\text{sgn}\,(A) = -1$. Thus the four cosets of $\mathcal{A}$ act on $\mathcal{E}$ and $\mathcal{O}$ as follows:

- $\mathcal{A}$ contains all the even permutations of the vertices of the tetrahedron $\mathcal{E}$ (or $\mathcal{O}$.)

- $(P_{(23)})\,\mathcal{A}$ contains all the odd permutations of the vertices of the tetrahedron $\mathcal{E}$ (or $\mathcal{O}$.)

- $(-P_{(23)})\,\mathcal{A}$ contains all the odd permutations of the vertices of the tetrahedron $\mathcal{E}$ (or $\mathcal{O}$) composed with inversion. Thus each element of $(-P_{(23)})\,\mathcal{A}$ interchanges the vertices of $\mathcal{E}$ and $\mathcal{O}$.

- $(-I)\,\mathcal{A}$ contains all the even permutations of the vertices of the tetrahedron $\mathcal{E}$ (or $\mathcal{O}$) composed with inversion; so each element of $(-I)\,\mathcal{A}$ interchanges the vertices of $\mathcal{E}$ and $\mathcal{O}$.

Thus $\mathcal{P}$ is the symmetry group of the tetrahedron $\mathcal{E}$. The rotations in $S(\mathcal{E})$ are the elements $A \in \mathcal{P}$ with $\operatorname{sgn}(A) = 1$; the other symmetries of $S(\mathcal{E})$ are the elements $A \in \mathcal{P}$ with $\operatorname{sgn}(A) = -1$. That is, for either tetrahedron in Figure 7, its rotational symmetries are given by $\mathcal{A} \cong A_4$, and the rest of its symmetries are given by $P_{(23)}\mathcal{A}$. See Table 8. As we have already established that $\mathcal{P} \cong S_4$ we have proved

Theorem 11 *the symmetry group of a tetrahedron is isomorphic to S_4, That is,*

$$S(T) \cong S_4.$$

Structure of $S(C)$, Seventh Pass: we shall show that $S(C)$ is generated by the nine matrices in $S(C)$ that represent reflections. To this end recall that

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ K = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ L = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For ease of reading, let

$$X = P_{(23)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ Y = P_{(13)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ Z = P_{(12)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

and let

$$R = P_{(132)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, S = P_{(123)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then R and S are rotations of order 3 and

$$-J, -K, -L, X, XJ, Y, YK, Z, ZL$$

are the nine refections in $S(C)$. The table below gives their 81 possible products:

$\times$	$-J$	$-K$	$-L$	X	XJ	Y	YK	Z	ZL
$-J$	I	L	K	$-XJ$	$-X$	$-YL$	$-YJ$	$-ZK$	$-ZJ$
$-K$	L	I	J	$-XL$	$-XK$	$-YK$	$-Y$	$-ZJ$	$-ZK$
$-L$	K	J	I	$-XK$	$-XL$	$-YJ$	$-YL$	$-ZL$	$-Z$
X	$-XJ$	$-XK$	$-XL$	I	J	R	RK	S	SL
XJ	$-X$	$-XL$	$-XK$	J	I	RL	RJ	SK	SJ
Y	$-YJ$	$-YK$	$-YL$	S	SJ	I	K	R	RL
YK	$-YL$	$-Y$	$-YJ$	SL	SK	K	I	RJ	RK
Z	$-ZJ$	$-ZK$	$-ZL$	R	RJ	S	SK	I	L
ZL	$-ZK$	$-ZJ$	$-Z$	RK	RL	SJ	SL	L	I

The product of two reflections is always a rotation. Of the 72 products of reflections which do not equal the identity:

- the three rotations J, K, L each show up four times
- the six rotations of order 2, $-X, -XJ, -Y, -YK, -Z, -ZL$, each show up two times

- the eight rotations of order 3, $R, RJ, RK, RL, S, SJ, SK, SL$, each show up three times
- the six rotations of order 4, $-XK, -XL, -YJ, -YL, -ZJ, -ZK$, each show up four times

To obtain the full group $S(C)$ we need only to multiply all the rotations by $-I = (-J)J$. Consequently the nine reflections of $S(C)$ do in fact generate the whole group. Actually, since $Z = XYX$, $-K = Z(-J)Z$, and $-L = Y(-J)Y$, the **three** reflections $-J, X$ and Y are sufficient to generate all of $S(C)$, as you may check.

The Order of Aut $(S(C))$: if g is an automorphism of $S(C)$ then g must fix the unique (with respect to order) normal subgroups of $S(C)$. This limits the action of g on $S(C)$. That is:

1. $g : \mathcal{Z} \longrightarrow \mathcal{Z}$, so $g(-I) = -I$.
2. $g : \mathcal{V} \longrightarrow \mathcal{V}$, so $g(J), g(K), g(L)$ must be a permutation of J, K, L.
3. $g : \mathcal{D} \longrightarrow \mathcal{D}$; in light of items 1 and 2 this means g is also a permutation of $-J, -K, -L$; the same permutation as of J, K, L.
4. $g : \mathcal{A} \longrightarrow \mathcal{A}$; in light of item 2 this means g must take a rotation of order 3 to a rotation of order 3.
5. $g : \mathcal{S} \longrightarrow \mathcal{S}$; redundant, in light of items 3 and 4.

$S(C)$ is generated by the six reflections, $-J, -K, -L, X, Y, Z$. Consequently, in light of item 1 above, g is determined by what it does to J, K, L, X, Y and Z. Actually, since

$$L = JK \text{ and } Z = XYX,$$

once g is defined on the four matrices J, K, X, Y, there is no choice for $g(L)$ or $g(Z)$. Since the product of any two of X, Y and Z is a rotation of order 3, the product of any two of $g(X), g(Y)$ and $g(Z)$ must itself be a rotation of order 3. Moreover, the choices for $g(X), g(Y)$ and $g(Z)$ are limited, since

- $XJ = JX \Leftrightarrow g(X)g(J) = g(J)g(X) \Rightarrow g(X) \in C_{S(C)}(g(J))$
- $YK = KY \Leftrightarrow g(Y)g(K) = g(K)g(Y) \Rightarrow g(Y) \in C_{S(C)}(g(K))$
- $ZL = LZ \Leftrightarrow g(Z)g(L) = g(L)g(Z) \Rightarrow g(Z) \in C_{S(C)}(g(L))$

Each centralizer of $g(J), g(K)$ or $g(L)$—which equals the centralizer of either J, K or L, by item 1 above—contains four elements of order 2 which are not in $\mathcal{D}$: two reflections and two rotations. Thus there are only four choices for $g(X)$. Since the product of $g(X)$ and $g(Y)$ must be a rotation of order 3, there are only two choices for $g(Y)$: if $g(X)$ is a reflection then so is $g(Y)$; if $g(X)$ is a rotation then so is $g(Y)$. Thus there are 6 ways to define g on J, K, L, and 8 subsequent ways to define g on X, Y, Z, giving 48 automorphisms in all. Note that g is an inner automorphism if both $g(X)$ and $g(Y)$ are reflections; g is an outer automorphism if $g(X)$ and $g(Y)$ are both rotations. Note that ρ as defined above,

$$\rho(A) = \operatorname{sgn}(A)\, A,$$

is an outer automorphism since $\rho(X) = -X$ and $\rho(Y) = -Y$ are both rotations.

The Automorphism Group of $S(C)$ **Is Isomorphic to Itself:** two maps similar to ρ,

$$\iota_d(A) = (\det(A))A \text{ or } \iota_p(A) = (\operatorname{par}(A))A,$$

are *not* isomorphisms. That is,

$$(\det(A))A = I \Rightarrow A = \det(A)\, I \Rightarrow \det(A) = (\det(A))^3 \Rightarrow \det(A) = \pm 1 \Rightarrow A = \pm I$$

and

$$(\text{par}\,(A))\, A = I \Rightarrow A = \text{par}\,(A) I \Rightarrow \text{par}\,(A) = (\text{par}\,(A))^3 \Rightarrow \text{par}\,(A) = \pm 1 \Rightarrow A = \pm I.$$

The kernel of both these maps is the center of $S(C)$, $\mathcal{Z} = \{I, -I\}$. The image of ι_d is $\mathcal{R}$; the image of ι_p is $\mathcal{P}$, both of which are isomorphic to S_4. Thus

$$S(C)/\mathcal{Z} \cong S_4.$$

Since it is also true that the inner automorphisms of $S(C)$ form a normal subgroup of Aut $(S(C))$ and that

$$\text{Inn}\,((S(C)) \cong S(C)/\mathcal{Z},$$

we conclude that Inn $(S(C)) \cong S_4$.

In addition, the isomorphism ρ as defined above is an outer automorphism of $S(C)$. We claim: ρ has order 2, and it commutes with every inner automorphism of $S(C)$. Consequently

$$\text{Inn}\,(S(C))\ \langle\rho\rangle \cong S_4 \times C_2 \cong S(C).$$

But then Inn $(S(C))\ \langle\rho\rangle$ consists of 48 automorphisms of $S(C)$ so we can conclude

$$\text{Aut}\,(S(C)) = \text{Inn}\,(S(C))\ \langle\rho\rangle \cong S(C).$$

To see that ρ has order 2, calculate:

$$\begin{aligned}
\rho(\rho(A)) &= \rho(\text{sgn}\,(A)\, A) \\
&= \text{sgn}\,(\text{sgn}\,(A)\, A)(\text{sgn}\,(A)\, A) \\
&= \text{sgn}\,((\text{sgn}\,(A)\, I)\, A)(\text{sgn}\,(A))\, A \\
&= \text{sgn}\,(\text{sgn}\,(A)\, I)\,(\text{sgn}\,(A))(\text{sgn}\,(A))\, A \\
&= (1)\,(\text{sgn}\,(A))^2\, A \\
&= (1)\,(1)\, A \\
&= A.
\end{aligned}$$

To see that ρ commutes with every inner automorphism of $S(C)$, let B be any matrix in $S(C)$ and let g_B be the inner automorphism of $S(C)$ defined by

$$g_B(A) = BAB^{-1}.$$

Then

$$\begin{aligned}
(\rho \circ g_B)(A) &= \rho(g_B(A)) \\
&= \rho(BAB^{-1}) \\
&= \rho(B)\rho(A)\rho(B^{-1}) \\
&= (\text{sgn}\,(B))\,(B)\,\rho(A)\,(\text{sgn}\,(B^{-1}))\, B^{-1} \\
&= \text{sgn}\,(B)\,\text{sgn}\,(B^{-1})\, B\,\rho(A)\, B^{-1} \\
&= \text{sgn}\,(BB^{-1})\, B\,\rho(A)\, B^{-1} \\
&= \text{sgn}\,(I)\, B\,\rho(A)\, B^{-1} \\
&= (1)\, B\,\rho(A)\, B^{-1} \\
&= B\,\rho(A)\, B^{-1} \\
&= g_B(\rho(A)) \\
&= (g_B \circ \rho)(A).
\end{aligned}$$

Structure of $S(C)$, Eighth Pass: we shall outline how to find all the subgroups of $S(C)$, all 98 of them. We have already identified the seven proper normal subgroups of $S(C)$, namely

- $\mathcal{P}, \mathcal{R}$ and $\mathcal{S}$, of order 24,
- $\mathcal{A}$ of order 12,
- $\mathcal{D}$ of order 8,
- $\mathcal{V}$ of order 4,
- $\mathcal{Z}$ of order 2.

And we have mentioned many of the other subgroups as well. As for the rest: see the complete list in Tables 10 and 11, in which the matrices J, K, L, X, Y, Z, R, S are as in the Seventh Pass. How can we find all these subgroups of $S(C)$, and know that we have found them all? The answers to both these questions are provided by Goursat's Theorem, which describes how to find all the subgroups of $G \times H$ in terms of the subgroups of G and H. In particular,

- there is a bijection between the ordered triples $(A/C, B/D, \varphi)$ and the subgroups of $G \times H$, where $C \trianglelefteq A \leq G$, $D \trianglelefteq B \leq H$, and φ is an isomorphism from A/C to B/D;
- $\{(g, h) \in A \times B \mid \varphi(gC) = hD\}$ is a subgroup of $G \times H$.

In our case, we know

$$S(C) = \mathcal{P}\mathcal{Z} \cong S_4 \times C_2.$$

So to use Goursat's Theorem we must first find all the subgroups $A \leq \mathcal{P} \cong S_4$, of which there are 30. These 30 subgroups produce 60 subgroups of $S(C) = \mathcal{P}\mathcal{Z}$, of the form

$$A \times \{I\} \cong A \text{ or } A \times \mathcal{Z} \cong A\mathcal{Z}.$$

Secondly, we need to find all the quotients A/C of $\mathcal{P}$ such that

$$C \lhd A \leq \mathcal{P} \text{ and } A/C \cong C_2.$$

That is, we have to find all the subgroups A of $\mathcal{P}$ such that C is a subgroup of index 2 in A. You can check that there are 38 such quotients of $\mathcal{P}$; hence the number of subgroups of $S(C)$ will be $60 + 38 = 98$.

As a sample calculation of a subgroup of $S(C)$, consider $\mathcal{A} \lhd \mathcal{P} \leq S(C)$, for which

$$\mathcal{P}/\mathcal{A} \cong \mathcal{Z}.$$

The isomorphism φ must satisfy

$$\varphi(\mathcal{A}) = I \text{ and } \varphi(X\mathcal{A}) = -I.$$

Consequently Goursat's method produces the subgroup $\mathcal{R}$ of $S(C)$:

$$\{(M, I) \mid M \in \mathcal{A}\} \cup \{(XM, -I) \mid M \in \mathcal{A}\} \cong \mathcal{A} \cup (-X\mathcal{A}) = \mathcal{R}.$$

Note: $\rho(\mathcal{P}) = \mathcal{R}$, so there is an alternative to Goursat's method of generating the subgroup $\mathcal{R}$.

As a second example, consider the subgroup A of $\mathcal{P}$ generated by $\{J, K, L, Y\}$. $A \cong D_4$; it is one of the four copies of D_4 among the group of 16 orthogonal diagonal and counter-diagonal

order	subgroups of $S(C)$	structure	number	notes
48	$S(C)$	$S_4 \times C_2$	1	
24	$\mathcal{P} = \{A \in S(C) \mid \text{perm}\,(A) = 1\}$	S_4	1	in $\mathcal{P}$
24	$\mathcal{R} = \{A \in S(C) \mid \det(A) = 1\}$	S_4	1	in $\mathcal{R}$
24	$\mathcal{S} = \{A \in S(C) \mid \text{sgn}\,(A) = 1\}$	$A_4 \times C_2$	1	in $\mathcal{S}$
16	$C_{S(C)}(J), C_{S(C)}(K), C_{S(C)}(L)$	$D_4 \times C_2$	3	2-sylow subgroups
12	$\mathcal{A} = \mathcal{P} \cap \mathcal{R} \cap \mathcal{S}$	A_4	1	in $\mathcal{P}, R, S$
12	$\mathcal{Q}, J\mathcal{Q}J, K\mathcal{Q}K, L\mathcal{Q}L$	D_6	4	
8	$\mathcal{D} = \{\pm I, \pm J, \pm K, \pm L\}$	$C_2 \times C_2 \times C_2$	1	in $\mathcal{S}$
8	$\langle J, K, X\rangle, \langle K, L, Y\rangle, \langle J, L, Z\rangle$	D_4	3	in $\mathcal{P}$
8	$\langle J, K, -X\rangle, \langle K, L, -Y\rangle, \langle J, L, -Z\rangle$	D_4	3	in $\mathcal{R}$
8	$\langle J, -K, X\rangle, \langle K, -L, Y\rangle, \langle -J, L, Z\rangle$	D_4	3	
8	$\langle J, -K, -X\rangle, \langle K, -L, -Y\rangle, \langle -J, L, -Z\rangle$	D_4	3	
8	$C_{S(C)}(XK), C_{S(C)}(YJ), C_{S(C)}(ZJ)$	$C_4 \times C_2$	3	
8	$C_{S(C)}(X), C_{S(C)}(Y), C_{S(C)}(Z)$	$C_2 \times C_2 \times C_2$	3	
6	$C_{S(C)}(R), C_{S(C)}(RJ), C_{S(C)}(RK), C_{S(C)}(RL)$	C_6	4	in $\mathcal{S}$
6	$\mathcal{G}, J\mathcal{G}J, K\mathcal{G}K, L\mathcal{G}L$	D_3	4	in $\mathcal{P}$
6	$\mathcal{H}, J\mathcal{H}J, K\mathcal{H}K, L\mathcal{H}L$	D_3	4	in $\mathcal{R}$

Table 10: Subgroups of $S(C)$, Part 1. Forty-three subgroups of $S(C)$ with order greater than 4; conjugates are in horizontal rows. There are two copies of S_4; one copy of $A_4 \times C_2$; three copies of $D_4 \times C_2$; one copy of A_4; four copies of D_6; four copies of $C_2 \times C_2 \times C_2$; twelve copies of D_4; three copies of $C_4 \times C_2$; four copies of C_6; and eight copies of D_3. $\mathcal{Q} = \mathcal{G} \cup (-\mathcal{G}) = \mathcal{H} \cup (-\mathcal{H})$; $\mathcal{G} = \{I, R, S, X, Y, Z\}; \mathcal{H} = \{I, R, S, -X, -Y, -Z\}$. There are three subgroups of order 24; three of order 16; five of order 12; nineteen of order 8; and twelve of order 6. All in all, $S(C)$ has 98 subgroups.

order	subgroups of $S(C)$	structure	number	notes
4	$\mathcal{V} = \{I, J, K, L\}$	$C_2 \times C_2$	1	in $\mathcal{P}, R, S$
4	$\langle \pm J \rangle, \langle \pm K \rangle, \langle \pm L \rangle$	$C_2 \times C_2$	3	in $\mathcal{S}$
4	$\langle J, -K \rangle, \langle -J, K \rangle, \langle -J, -K \rangle$	$C_2 \times C_2$	3	in $\mathcal{S}$
4	$\langle J, X \rangle, \langle K, Y \rangle, \langle L, Z \rangle$	$C_2 \times C_2$	3	in $\mathcal{P}$
4	$\langle J, -X \rangle, \langle K, -Y \rangle, \langle L, -Z \rangle$	$C_2 \times C_2$	3	in $\mathcal{R}$
4	$\langle -J, X \rangle, \langle -K, Y \rangle, \langle -L, Z \rangle$	$C_2 \times C_2$	3	
4	$\langle -J, -X \rangle, \langle -K, -Y \rangle, \langle -L, -Z \rangle$	$C_2 \times C_2$	3	
4	$\langle \pm X \rangle, \langle \pm XJ \rangle, \langle \pm Y \rangle, \langle \pm YK \rangle, \langle \pm Z \rangle, \langle \pm ZL \rangle$	$C_2 \times C_2$	6	
4	$\langle XK \rangle, \langle YJ \rangle, \langle ZJ \rangle$	C_4	3	in $\mathcal{P}$
4	$\langle -XK \rangle, \langle -YJ \rangle, \langle -ZJ \rangle$	C_4	3	in $\mathcal{R}$
3	$\langle R \rangle, \langle RJ \rangle, \langle RK \rangle, \langle RL \rangle$	C_3	4	3-sylow subgroups in $\mathcal{P}, R, S$
2	$\mathcal{Z} = \{I, -I\}$	C_2	1	in $\mathcal{S}$
2	$\langle J \rangle, \langle K \rangle, \langle L \rangle$	C_2	3	in $\mathcal{P}, R, S$
2	$\langle -J \rangle, \langle -K \rangle, \langle -L \rangle$	C_2	3	in $\mathcal{S}$
2	$\langle X \rangle, \langle XJ \rangle, \langle Y \rangle, \langle YK \rangle, \langle Z \rangle, \langle ZL \rangle$	C_2	6	in $\mathcal{P}$
2	$\langle -X \rangle, \langle -XJ \rangle, \langle -Y \rangle, \langle -YK \rangle, \langle -Z \rangle, \langle -ZL \rangle$	C_2	6	in $\mathcal{R}$
1	$\{I\}$	C_1	1	in $\mathcal{P}, R, S$

Table 11: Subgroups of $S(C)$, Part 2. Fifty-five subgroups of $S(C)$ with order 4 or less; conjugates are in horizontal rows. There are 19 copies of C_2; four copies of C_3; six copies of C_4; and twenty-five copies of $C_2 \times C_2$. There are thirty-one subgroups of order 4; four of order 3; and nineteen of order 2. All in all, $S(C)$ has 98 subgroups.

matrices listed in Table 3. A contains the subgroup $\langle YJ \rangle \cong C_4$. Then the isomorphism φ between $A/\langle YJ \rangle$ and $\mathcal{Z}$ must satisfy

$$\varphi(\langle YJ \rangle) = I \text{ and } \varphi(J\langle YJ \rangle) = -I.$$

Consequently Goursat's method produces the subgroup $\langle K, -L, -Y \rangle$ of $S(C)$:

$$\begin{aligned} & \{(M, I) \mid M \in \langle YJ \rangle\} \cup \{(JM, -I) \mid M \in \langle YJ \rangle\} \\ \cong \; & \{I, YJ, K, YL\} \cup \{-J, -YK, -L, -Y\} \\ = \; & \langle K, -L, -Y \rangle. \end{aligned}$$

This is another copy of D_4, as you may check. Note: this subgroup could not have been generated by applying ρ to some other subgroup of $\mathcal{P}$, since J, K, L are all fixed by ρ.

Extreme Cases: the deranged matrices

$$\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} \in \mathbb{U} \text{ and } \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \in \mathbb{W}$$

are orthogonal if $a^2 + b^2 + c^2 = 1$ and $a + b + c = \pm 1$. Let us find for which matrices the values a, b and c are not all distinct. That is, suppose A is an orthogonal deranged matrix of either type such that $b = c$. Then $a^2 + 2b^2 = 1$, and $a + 2b = 1$ or $a + 2b = -1$. In the first case the solutions for (a, b) are

$$(a, b) = (1, 0) \text{ or } (a, b) = \left(-\frac{1}{3}, \frac{2}{3}\right);$$

in the second case,

$$(a, b) = (-1, 0) \text{ or } (a, b) = \left(\frac{1}{3}, -\frac{2}{3}\right).$$

Similarly for $a = b$ or $a = c$. All such possible orthogonal deranged matrices are displayed in Tables 4 and 5. Of course, it is impossible for $a = b = c$ in an orthogonal deranged matrix, for that would require $3a^2 = 1$ and $3a = \pm 1$.

Note that if $a^2 + b^2 + c^2 = 1$ and $a + b + c = 1$ then

$$-1/3 \leq a, b, c \leq 1;$$

and if $a + b + c = -1$ then

$$-1 \leq a, b, c \leq 1/3.$$

This is because if you try to solve for two variables in terms of the third, say x, then the solution involves $\sqrt{-3x^2 + 2x + 1}$ in the case $a + b + c = 1$, and $\sqrt{-3x^2 - 2x + 1}$ in the case $a + b + c = -1$. This means

$$-\frac{1}{3} \leq x \leq 1$$

in the first case, and

$$-1 \leq x \leq \frac{1}{3}$$

in the second case. Thus these two special cases are extreme cases in the sense that one of a, b, c is as large or as small as possible. Actually, these special cases are the solutions to the two constrained optimization problems:

1. Find the extreme values of $F(a,b,c) = abc$ subject to the constraints $a+b+c=1$ and $a^2+b^2+c^2=1$.

   The solution is

$$-\frac{4}{27} \le abc \le 0$$

   with $F(1,0,0) = F(0,1,0) = F(0,0,1) = 0$ and

$$F(-1/3, 2/3, 2/3) = F(2/3, -1/3, 2/3) = F(2/3, 2/3, -1/3) = -4/27.$$

2. Find the extreme values of $F(a,b,c) = abc$ subject to the constraints $a+b+c=-1$ and $a^2+b^2+c^2=1$.

   The solution is

$$0 \le abc \le \frac{4}{27}$$

   with $F(-1,0,0) = F(0,-1,0) = F(0,0,-1) = 0$, and

$$F(1/3, -2/3, -2/3) = F(-2/3, 1/3, -2/3) = F(-2/3, -2/3, 1/3) = 4/27.$$

For $A = D_1(\theta)$ the extreme cases occur when

$$\cos\theta = \pm 1 \text{ or } \cos\theta = \pm\frac{1}{2}$$

$$\Leftrightarrow \cos^2\theta = 1 \text{ or } \cos^2\theta = \frac{1}{4}.$$

Structure of $S(C)$, Ninth Pass: in particular, the only orthogonal matrices in $\mathbb{D} \subset O(3)$ with any zero entries are the six permutation matrices and their negatives. $S(C)$ is then the group of all possible signed variations of the orthogonal matrices in $\mathbb{D}$ with at least one zero entry. In particular all the other matrices in $\mathbb{D}$ and all their orthogonal signed variations have no zero entries at all.

Finite Subgroups of $\mathbb{D}$: suppose $A \in \mathbb{D}_1$ represents a rotation of order $n > 2$ and P_σ is an odd 3×3 permutation matrix. Then

- $\mathbb{E}_1 = \langle A \rangle \cong C_n$ is a finite subgroup of $\mathbb{D}_1$,
- $\mathbb{E}_1 \cup (-P_\sigma\,\mathbb{E}_1) \cong D_n$ is a finite subgroup of $\mathbb{D}_1 \cup \mathbb{D}_2$;
- $\mathbb{E}_1 \cup (-\mathbb{E}_1) \cong C_n \times C_2$ is a finite subgroup of $\mathbb{D}_1 \cup \mathbb{D}_3$;
- $\mathbb{E}_1 \cup (P_\sigma\,\mathbb{E}_1) \cong D_n$ is a finite subgroup of $\mathbb{D}_1 \cup \mathbb{D}_4$;

and $\mathbb{E} = \mathbb{E}_1 \cup (-P_\sigma\,\mathbb{E}_1) \cup (-\mathbb{E}_1) \cup (P_\sigma\,\mathbb{E}_1) \cong D_n \times C_2$ is a finite subgroup of $\mathbb{D}$.

Example 14: start with the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } P_{(23)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

A is a rotation matrix of order 3 and $P_{(23)}$ is a reflection matrix of order 2. (They are also both permutation matrices; A is an even permutation and $P_{(23)}$ is odd.) The six matrices in $\mathbb{E} \cong D_6$ are displayed in Table 4. The subgroup $\mathbb{E}_1$ consists of the three matrices in the top left of Table 4, the subgroup $\mathbb{E}_1 \cup (-P_\sigma\,\mathbb{E}_1)$ consists of the six matrices in the left half of Table 4, the subgroup $\mathbb{E}_1 \cup (-\mathbb{E}_1)$ consists of the the six matrices in the top half of Table 4, and the subgroup $\mathbb{E}_1 \cup P_\sigma\,\mathbb{E}_1$ consists of the three matrices in the top left of Table 4 along with the three matrices in the bottom right of Table 4. As a group, $\mathbb{E}$ is the symmetry group of the 3-antiprism illustrated in Figure 5.

Or consider

$$B = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}$$

which has order 6. Let $\mathbb{F}_1 = \langle B \rangle \cong C_6$. Then

- $\mathbb{F}_1 \cup (-P_\sigma\,\mathbb{F}_1) \cong D_6$;
- $\mathbb{F}_1 \cup (-\mathbb{F}_1) \cong C_6 \times C_2$, and
- $\mathbb{F}_1 \cup (P_\sigma\,\mathbb{F}_1) \cong D_6$.

In this case the 24 matrices in $\mathbb{F} = \mathbb{F}_1 \cup (-P_\sigma\,\mathbb{F}_1) \cup (-\mathbb{F}_1) \cup (P_\sigma\,\mathbb{F}_1)$ are listed in Tables 4 and 5; they comprise the symmetry group of the 6-prism shown in Figure 6. The six counter-symmetric rotation matrices in $\mathbb{F}_1$ are in the top left corners of Tables 4 and 5, the 12 rotations in $\mathbb{F}_1 \cup (-P_\sigma\,\mathbb{F}_1)$ are listed in the left sides, the 12 matrices in $\mathbb{F}_1 \cup (-\mathbb{F}_1)$ are listed in the top halves of the two tables, and the six additional reflections in $\mathbb{F}_1 \cup (P_\sigma\,\mathbb{F}_1)$ are in the bottom right corners. □

Useful or Not? Pick a particular permutation matrix of order 2, say $P_{(12)}$. Then

$$-P_{(12)} \in \mathbb{D}_2,\ \ D_1(\theta)(-P_{(12)}) \in \mathbb{D}_2 \text{ and } (-P_{(12)})D_1(\theta)(-P_{(12)}) = D_1(-\theta).$$

The first two statements are obvious, but the third statement requires proof. A straightforward calculation will do:

$$\begin{aligned}
& (-P_{(12)})D_1(\theta)(-P_{(12)}) \\
= \ & P_{(12)}\, D_1(\theta)\, P_{(12)} \\
= \ & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1+2\cos\theta & 1-\sqrt{3}\sin\theta-\cos\theta & 1+\sqrt{3}\sin\theta-\cos\theta \\ 1+\sqrt{3}\sin\theta-\cos\theta & 1+2\cos\theta & 1-\sqrt{3}\sin\theta-\cos\theta \\ 1-\sqrt{3}\sin\theta-\cos\theta & 1+\sqrt{3}\sin\theta-\cos\theta & 1+2\cos\theta \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
= \ & \frac{1}{3} \begin{bmatrix} 1+\sqrt{3}\sin\theta-\cos\theta & 1+2\cos\theta & 1-\sqrt{3}\sin\theta-\cos\theta \\ 1+2\cos\theta & 1-\sqrt{3}\sin\theta-\cos\theta & 1+\sqrt{3}\sin\theta-\cos\theta \\ 1-\sqrt{3}\sin\theta-\cos\theta & 1+\sqrt{3}\sin\theta-\cos\theta & 1+2\cos\theta \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
= \ & \frac{1}{3} \begin{bmatrix} 1+2\cos\theta & 1+\sqrt{3}\sin\theta-\cos\theta & 1-\sqrt{3}\sin\theta-\cos\theta \\ 1-\sqrt{3}\sin\theta-\cos\theta & 1+2\cos\theta & 1+\sqrt{3}\sin\theta-\cos\theta \\ 1+\sqrt{3}\sin\theta-\cos\theta & 1-\sqrt{3}\sin\theta-\cos\theta & 1+2\cos\theta \end{bmatrix} \\
= \ & D_1(-\theta),\ \text{since } \cos(-\theta) = \cos\theta \text{ and } \sin(-\theta) = -\sin\theta
\end{aligned}$$

Now, if $A \in \mathbb{D}_2$ then we can write $A = D_1(\theta)(-P_{(12)})$ for some θ. Let $D_2(\theta) = D_1(\theta)(-P_{(12)})$. We claim the following multiplication formulas for matrices in $\mathbb{D}_1 \cup \mathbb{D}_2$:

Theorem 12 *Let $R = -P_{(12)}$, let $D_1(\theta)$ be as defined above, and let $D_2(\theta) = D_1(\theta)\,R$, then*

1. $D_1(\alpha)D_1(\beta) = D_1(\alpha+\beta)$,
2. $D_2(\alpha)D_2(\beta) = D_1(\alpha-\beta)$,
3. $D_1(\alpha)D_2(\beta) = D_2(\alpha+\beta)$,
4. $D_2(\beta)D_1(\alpha) = D_2(\beta-\alpha)$.

Proof: property 1 is obvious, since $\mathbb{D}_1 = \{D_1(\theta) \mid \theta \in \mathbb{R}\}$ is the group of all rotations with axis of symmetry parallel to $\vec{d}$. Properties 2, 3 and 4 depend on Property 1 and R:

2. $D_2(\alpha)D_2(\beta) = D_1(\alpha)\,R\,D_1(\beta)\,R = D_1(\alpha)\,D_1(-\beta) = D_1(\alpha-\beta)$.
3. $D_1(\alpha)D_2(\beta) = D_1(\alpha)D_1(\beta)\,R = D_1(\alpha+\beta)\,R = D_2(\alpha+\beta)$.
4. $D_2(\beta)D_1(\alpha) = D_1(\beta)\,R\,RD_1(-\alpha)\,R = D_1(\beta)\,D_1(-\alpha)\,R = D_1(\beta-\alpha)\,R = D_2(\beta-\alpha)$,

where we have used the fact that $R^2 = I$. □

Of course the other matrices in $\mathbb{D}$ are just the negatives of the matrices in $\mathbb{D}_1 \cup \mathbb{D}_2$, so the above multiplication formulas are easily extended to all of $\mathbb{D}$.

Part 4: The Symmetry Groups of the Platonic Solids, or Just 240 Matrices. An attempt at something new. We now generalize some of the previous results with the aim of generating the matrices in the symmetry groups of the other Platonic solids. Actually, since the octahedron and cube are dual we can say that

$$S(O) = S(C) \cong S_4 \times C_2.$$

And as we saw in the Sixth Pass of Part 2, the symmetry group of a tetrahedron is $\mathcal{P} = \ker(\text{par}\,)$, so

$$S(T) \cong S_4.$$

That leaves only the question of the dodecahedron and icosahedron. Since these two are also dual, we only need to describe one more symmetry group.

Finding Orthogonal Signed Variations of an Orthogonal Matrix: let $A = [C_1\ C_2\ C_3]$ be a 3×3 orthogonal matrix and suppose P, Q and R are three matrices in

$$\mathcal{D} = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \middle|\, x = \pm 1, y = \pm 1, z = \pm 1 \right\}.$$

Then

$$[P\,C_1\ Q\,C_2\ R\,C_3]$$

is one of at most 512 signed variations of A. Which of these are orthogonal? Let us assume A has no zero entries, and let

$$B = \left[P \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}\ Q \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}\ R \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \right]$$

be a signed variation of A. Then the columns of B are among

$$\left\{ \begin{bmatrix} \pm a_{11} \\ \pm a_{21} \\ \pm a_{31} \end{bmatrix}, \begin{bmatrix} \pm a_{12} \\ \pm a_{22} \\ \pm a_{32} \end{bmatrix}, \begin{bmatrix} \pm a_{13} \\ \pm a_{23} \\ \pm a_{33} \end{bmatrix} \right\},$$

which are *all* unit vectors, since C_1, C_2 and C_3 are. When are the columns of B orthogonal? Let

$$P = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}, Q = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{bmatrix}.$$

Then the dot product of the first two possible columns of B is

$$p_1 q_1 a_{11} a_{12} + p_2 q_2 a_{21} a_{22} + p_3 q_3 a_{31} a_{32} = (\pm a_{11} a_{12}) + (\pm a_{21} a_{22}) + (\pm a_{31} a_{32}).$$

Setting this dot product equal to zero gives 8 possibilities, of which there are two distinct cases:

1. The coefficients all have the same sign: for example, $a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0$, in which case $PQ = I \Leftrightarrow Q = P$ or $PQ = -I \Leftrightarrow Q = -P$.

2. The coefficients do not all have the same sign. For example, $a_{11}a_{12} - a_{21}a_{22} - a_{31}a_{32} = 0$. Since $C_1 \cdot C_2 = 0$ we know that $a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0$. Adding these two equations gives $2a_{11}a_{12} = 0$, which is impossible since A has no zero entries. The remaining five cases are similarly impossible.

Thus the first two columns of B will be orthogonal if $Q = \pm P$. Similarly, the first and third columns of B will be orthogonal if $R = \pm P$. Thus there are only $8 \times 2 \times 2 = 32$ signed variations of A which are still orthogonal.

Another way to write these 32 possible matrices is as follows:

$$PA \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, PA \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, PA \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ or } PA \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

for $P \in \mathcal{D}$, since right multiplication of PA by an elementary matrix results in *column* operations on PA.

Another approach: for any orthogonal matrix A and for any two matrices $P, Q \in \mathcal{D}$, PAQ is an orthogonal signed variation of A. Why? Since P and Q are also orthogonal we have

$$(PAQ)^T = Q^T A^T P^T = Q^{-1} A^{-1} P^{-1} = (PAQ)^{-1},$$

so PAQ is orthogonal. Of these 64 possibilities there are only 32 distinct matrices, as the following theorem shows:

Theorem 13 *Suppose A is an orthogonal matrix with no zero entries. Every matrix of the form PAQ with $P, Q \in \mathcal{D}$ is an orthogonal signed variation of A. If*

$$PAQ = P'AQ',$$

with $P', Q' \in \mathcal{D}$, then $P' = P$ and $Q' = Q$, or $P' = -P$ and $Q' = -Q$, so there are only 32 distinct signed variations of A of the form PAQ.

Proof: the orthogonality of PAQ has already been established. Now suppose

$$PAQ = P'AQ' \Leftrightarrow P'PA = AQ'Q,$$

since every element in $\mathcal{D}$ has order 2. If $P'P = I$, then $A = AQ'Q \Leftrightarrow Q'Q = I$, whence $P' = P$ and $Q' = Q$. Otherwise, left multiplying A by $P'P$ will result in changing at least one row of A to its negative. This requires, $Q'Q = -I$, since right multiplying A by $Q'Q$ changes the *columns* of A. So $Q' = -Q$ and

$$P'PA = -A \Leftrightarrow P'P = -I \Leftrightarrow P' = -P.$$

Thus of the 64 possible products PAQ, with $P, Q \in \mathcal{D}$, there will only be 32 distinct results. □

Example 15: consider the deranged matrices generated by the triple $a = -1/3, b = c = 2/3$. As in Example 11 and Table 5, there are three counter-symmetric deranged possibilities

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \; P_{(123)}A = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}, \; P_{(132)}A = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$$

and three symmetric deranged possibilities

$$P_{(23)}A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}, \; P_{(12)}A = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \; P_{(13)}A = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}.$$

$$\operatorname{tr}(X) = \begin{cases} \pm\frac{1}{3}\pm\frac{1}{3}\pm\frac{1}{3}, & \text{if } X \text{ is an orthogonal signed version of } \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \\ \pm\frac{2}{3}\pm\frac{2}{3}\pm\frac{2}{3}, & \text{if } X \text{ is an orthogonal signed version of } \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix} \\ \pm\frac{2}{3}\pm\frac{2}{3}\pm\frac{2}{3}, & \text{if } X \text{ is an orthogonal signed version of } \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix} \\ \pm\frac{1}{3}\pm\frac{2}{3}\pm\frac{2}{3}, & \text{if } X \text{ is an orthogonal signed version of } \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \\ \pm\frac{2}{3}\pm\frac{1}{3}\pm\frac{2}{3}, & \text{if } X \text{ is an orthogonal signed version of } \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix} \\ \pm\frac{2}{3}\pm\frac{2}{3}\pm\frac{1}{3}, & \text{if } X \text{ is an orthogonal signed version of } \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix} \end{cases}$$

$$= \pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm\frac{5}{3} \text{ or } \pm 2.$$

Table 12: The possible values of $\operatorname{tr}(X)$ if X is an orthogonal signed variation of the six matrices in Example 15.

By Theorem 13, each of these six matrices can be signed and remain orthogonal in 32 different ways, resulting in 192 possibilities in all. If X is one of these 192 matrices, then the ten possible values of

$$\operatorname{tr}(X) = \operatorname{tr}(PAQ) = \operatorname{tr}(QPA) = \operatorname{tr}(DA) = d_{11}a_{11} + d_{22}a_{22} + d_{33}a_{33},$$

for $D \in \mathcal{D}$, are calculated in Table 12. The only elements with finite order are the matrices X with $\operatorname{tr}(X) = \pm 1$ or ± 2. There are 48 of these, namely $\pm P_\sigma A, \pm JP_\sigma AJ, \pm KP_\sigma AK, \pm LP_\sigma AL$, where A is as above, P_σ is any one of the six 3×3 permutation matrices, and as before

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; K = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; L = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Table 5 displays the 12 matrices $\pm P_\sigma A$, which are half of the symmetries of a 6-prism, oriented along the diagonal of the cube parallel to

$$\vec{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The conjugates $\pm JP_\sigma AJ, \pm KP_\sigma AK, \pm LP_\sigma AL$ are half of the symmetries of a 6-prism oriented along the other three diagonals of the cube, parallel to the vectors

$$J\vec{d} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \; K\vec{d} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \text{ and } L\vec{d} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},$$

respectively. Why? Because, for instance,

$$\begin{aligned} A\vec{d} = \vec{d} \;&\Rightarrow\; JA\vec{d} = J\vec{d} \\ &\Rightarrow\; JAJ(J\vec{d}) = J\vec{d} \\ &\Rightarrow\; J\vec{d} \in E_1(JAJ). \end{aligned}$$

So JAJ represents a rotation around the diagonal of the cube parallel to the vector $J\vec{d}$.

Let $\mathcal{F}$ be the group of 24 matrices displayed in Tables 4 and 5, which comprise the full symmetry group, isomorphic to $D_6 \times C_2$, of the 6-prism oriented along the axis parallel to $\vec{d}$. Then

$$J\mathcal{F}J, \; K\mathcal{F}K, \; L\mathcal{F}L$$

are the symmetry groups, isomorphic to $D_6 \times C_2$, of the 6-prisms oriented along the axes parallel to the vectors

$$J\vec{d}, \; K\vec{d}, \; L\vec{d},$$

respectively. □

Example 16: now consider the case

$$a = \frac{1}{2}, b = \frac{\phi}{2}, c = \frac{\psi}{2}.$$

There are six counter-symmetric deranged matrices, all with determinant 1,

$$A_1 = \frac{1}{2}\begin{bmatrix} 1 & \psi & \phi \\ \phi & 1 & \psi \\ \psi & \phi & 1 \end{bmatrix}, \; P_{(123)}A_1 = \frac{1}{2}\begin{bmatrix} \phi & 1 & \psi \\ \psi & \phi & 1 \\ 1 & \psi & \phi \end{bmatrix}, \; P_{(132)}A_1 = \frac{1}{2}\begin{bmatrix} \psi & \phi & 1 \\ 1 & \psi & \phi \\ \phi & 1 & \psi \end{bmatrix},$$

$$A_2 = \frac{1}{2}\begin{bmatrix} 1 & \phi & \psi \\ \psi & 1 & \phi \\ \phi & \psi & 1 \end{bmatrix}, \; P_{(123)}A_2 = \frac{1}{2}\begin{bmatrix} \phi & \psi & 1 \\ 1 & \phi & \psi \\ \psi & 1 & \phi \end{bmatrix}, \; P_{(132)}A_3 = \frac{1}{2}\begin{bmatrix} \psi & 1 & \phi \\ \phi & \psi & 1 \\ 1 & \phi & \psi \end{bmatrix}.$$

There are 32 signed orthogonal variations of these six matrices—192 in all. As explained in Example 12 they all have infinite order, since for any matrix X equal to any of these 192 matrices,

$$\operatorname{tr}(X) = \pm\frac{1}{2}, \; \pm\frac{3}{2}, \; \pm\frac{\phi}{2}, \; \pm\frac{3\phi}{2}, \; \pm\frac{\psi}{2}, \; \pm\frac{3\psi}{2}.$$

There are also six symmetric deranged matrices, all with determinant -1,

$$P_{(12)}A_1 = \frac{1}{2}\begin{bmatrix} \phi & 1 & \psi \\ 1 & \psi & \phi \\ \psi & \phi & 1 \end{bmatrix}, \; P_{(23)}A_1 = \frac{1}{2}\begin{bmatrix} 1 & \psi & \phi \\ \psi & \phi & 1 \\ \phi & 1 & \psi \end{bmatrix}, \; P_{(13)}A_1 = \frac{1}{2}\begin{bmatrix} \psi & \phi & 1 \\ \phi & 1 & \psi \\ 1 & \psi & \phi \end{bmatrix},$$

$$P_{(12)}A_2 = \frac{1}{2}\begin{bmatrix} \psi & 1 & \phi \\ 1 & \phi & \psi \\ \phi & \psi & 1 \end{bmatrix}, \; P_{(23)}A_2 = \frac{1}{2}\begin{bmatrix} 1 & \phi & \psi \\ \phi & \psi & 1 \\ \psi & 1 & \phi \end{bmatrix}, \; P_{(13)}A_2 = \frac{1}{2}\begin{bmatrix} \phi & \psi & 1 \\ \psi & 1 & \phi \\ 1 & \phi & \psi \end{bmatrix},$$

	$-\frac{1}{2}\begin{bmatrix} \phi & 1 & \psi \\ 1 & \psi & \phi \\ \psi & \phi & 1 \end{bmatrix}$			$-\frac{P_{(123)}}{2}\begin{bmatrix} \phi & 1 & \psi \\ 1 & \psi & \phi \\ \psi & \phi & 1 \end{bmatrix}$			$-\frac{P_{(132)}}{2}\begin{bmatrix} \phi & 1 & \psi \\ 1 & \psi & \phi \\ \psi & \phi & 1 \end{bmatrix}$		
	$A = -\frac{1}{2}\begin{bmatrix} \phi & 1 & \psi \\ 1 & \psi & \phi \\ \psi & \phi & 1 \end{bmatrix}$			$A = -\frac{1}{2}\begin{bmatrix} 1 & \psi & \phi \\ \psi & \phi & 1 \\ \phi & 1 & \psi \end{bmatrix}$			$A = -\frac{1}{2}\begin{bmatrix} \psi & \phi & 1 \\ \phi & 1 & \psi \\ 1 & \psi & \phi \end{bmatrix}$		
X	$\mathrm{tr}\,(X)$	number $x_{ij} < 0$	order of X	$\mathrm{tr}\,(X)$	number $x_{ij} < 0$	order of X	$\mathrm{tr}\,(X)$	number $x_{ij} < 0$	order of X
A	-1	6	2	-1	6	2	-1	6	2
JA	ψ	4	5	0	4	3	ϕ	4	5
KA	ϕ	4	5	ψ	4	5	0	4	3
LA	0	4	3	ϕ	4	5	ψ	4	5
AJ	ψ	4	5	0	4	3	ϕ	4	5
JAJ	-1	6	2	-1	6	2	-1	2	2
KAJ	0	6	3	ϕ	2	5	ψ	6	5
LAJ	ϕ	2	5	ψ	6	5	0	6	3
AK	ϕ	4	5	ψ	4	5	0	4	3
JAK	0	6	3	ϕ	2	5	ψ	6	5
KAK	-1	2	2	-1	6	2	-1	6	2
LAK	ψ	6	5	0	6	3	ϕ	2	5
AL	0	4	3	ϕ	4	5	ψ	4	5
JAL	ϕ	2	5	ψ	6	5	0	6	3
KAL	ψ	6	5	0	6	3	ϕ	2	5
LAL	-1	6	2	-1	2	2	-1	6	2

Table 13: 48 orthogonal signed variations with determinant 1 of symmetric deranged matrices with entries from $\{-1/2, -\phi/2, -\psi/2\}$. These 48 matrices together with the 12 matrices in $\mathcal{A}$ comprise a copy of A_5. These 60 matrices have an even number of negative entries. The 48 matrices above represent rotational symmetries of the dodecahedron D and the icosahedron I.

	$-\frac{1}{2}\begin{bmatrix} \psi & 1 & \phi \\ 1 & \phi & \psi \\ \phi & \psi & 1 \end{bmatrix}$			$-\frac{P_{(123)}}{2}\begin{bmatrix} \psi & 1 & \phi \\ 1 & \phi & \psi \\ \phi & \psi & 1 \end{bmatrix}$			$-\frac{P_{(132)}}{2}\begin{bmatrix} \psi & 1 & \phi \\ 1 & \phi & \psi \\ \phi & \psi & 1 \end{bmatrix}$		
	$A = -\frac{1}{2}\begin{bmatrix} \psi & 1 & \phi \\ 1 & \phi & \psi \\ \phi & \psi & 1 \end{bmatrix}$			$A = -\frac{1}{2}\begin{bmatrix} 1 & \phi & \psi \\ \phi & \psi & 1 \\ \psi & 1 & \phi \end{bmatrix}$			$A = -\frac{1}{2}\begin{bmatrix} \phi & \psi & 1 \\ \psi & 1 & \phi \\ 1 & \phi & \psi \end{bmatrix}$		
X	$\mathrm{tr}\,(X)$	number $x_{ij} < 0$	order of X	$\mathrm{tr}\,(X)$	number $x_{ij} < 0$	order of X	$\mathrm{tr}\,(X)$	number $x_{ij} < 0$	order of X
A	-1	6	2	-1	6	2	-1	6	2
JA	ϕ	4	5	0	4	3	ψ	4	5
KA	ψ	4	5	ϕ	4	5	0	4	3
LA	0	4	3	ψ	4	5	ϕ	4	5
AJ	ϕ	4	5	0	4	3	ψ	4	5
JAJ	-1	2	2	-1	6	2	-1	6	2
KAJ	0	6	3	ψ	6	5	ϕ	2	5
LAJ	ψ	6	5	ϕ	2	5	0	6	3
AK	ψ	4	5	ϕ	4	5	0	4	3
JAK	0	6	3	ψ	6	5	ϕ	2	5
KAK	-1	6	2	-1	2	2	-1	6	2
LAK	ϕ	2	5	0	6	3	ψ	6	5
AL	0	4	3	ψ	4	5	ϕ	4	5
JAL	ψ	6	5	ϕ	2	5	0	6	3
KAL	ϕ	2	5	0	6	3	ψ	6	5
LAL	-1	6	2	-1	6	2	-1	2	2

Table 14: 48 orthogonal matrices with determinant 1, obtained from the 48 matrices in Table 13 by swapping ϕ and ψ. These new 48 matrices together with the 12 matrices in $\mathcal{A}$ form another copy of A_5. Again, all these 60 matrices have an even number of negative entries. These 60 matrices are all the rotational symmetries of the dodecahedron D' and the icosahedron I'.

which all have order 2 and were previously listed in Table 6. Each of these six matrices can be signed and remain orthogonal in 32 different ways, resulting in 192 possibilities. If X is one of these 192 matrices then the order of X is finite, since the possibilities for $\mathrm{tr}\,(X)$ are

$$\mathrm{tr}\,(X) = \pm\frac{1}{2} \pm \frac{\phi}{2} \pm \frac{\psi}{2} = 0,\ \pm 1,\ \pm\phi,\ \pm\psi.$$

See Tables 13 and 14, in which all 96 matrices with determinant 1 are listed. Neither the 48 matrices in Table 13 nor the 48 matrices in Table 14 form a group. But each set of 48 matrices, along with the 12 matrices in $\mathcal{A}$ from the top half of Table 7, does form a group of order 60, which is isomorphic to A_5. Assuming for the moment that the 60 matrices do form a group, then the observation that Table 13 lists 24 elements of order 5 means that the group has six 5-Sylow subgroups. This is enough to identify the group as A_5.[6] There are also 24 elements of order 5 listed in Table 14, so same result. □

The Symmetry Groups of the Icosahedron and the Dodecahedron: consider the icosahedron I with the 12 vertices

$$\begin{bmatrix} 0 \\ \pm 1 \\ \pm\phi \end{bmatrix}, \begin{bmatrix} \pm\phi \\ 0 \\ \pm 1 \end{bmatrix}, \begin{bmatrix} \pm 1 \\ \pm\phi \\ 0 \end{bmatrix}.$$

Let $S(I)$ be its symmetry group and let $A \in S(I)$. Since $A\vec{\imath}, A\vec{\jmath}, A\vec{k}$ must be the first, second and third columns of A, respectively, and A must fix the set of vertices of I, we can use

$$\vec{\imath} = \frac{1}{2}\begin{bmatrix} 1 \\ \phi \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -\phi \\ 0 \end{bmatrix},\ \vec{\jmath} = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ \phi \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ -\phi \end{bmatrix},\ \vec{k} = \frac{1}{2}\begin{bmatrix} \phi \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -\phi \\ 0 \\ 1 \end{bmatrix}$$

to calculate the columns of A. For the first column we find

$$\begin{aligned}
A(\vec{\imath}) &= \frac{1}{2}A\begin{bmatrix} 1 \\ \phi \\ 0 \end{bmatrix} + \frac{1}{2}A\begin{bmatrix} 1 \\ -\phi \\ 0 \end{bmatrix} \\
&= \frac{1}{2}\left(\begin{bmatrix} 0 \\ \pm 1 \\ \pm\phi \end{bmatrix}, \begin{bmatrix} \pm\phi \\ 0 \\ \pm 1 \end{bmatrix} \text{ or } \begin{bmatrix} \pm 1 \\ \pm\phi \\ 0 \end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix} 0 \\ \pm 1 \\ \pm\phi \end{bmatrix}, \begin{bmatrix} \pm\phi \\ 0 \\ \pm 1 \end{bmatrix} \text{ or } \begin{bmatrix} \pm 1 \\ \pm\phi \\ 0 \end{bmatrix}\right) \\
&= \frac{1}{2}\begin{bmatrix} 0, \pm 1, \pm\phi, \pm 1 \pm 1, \pm 1 \pm \phi, \pm\phi \pm \phi \\ 0, \pm 1, \pm\phi, \pm 1 \pm 1, \pm 1 \pm \phi, \pm\phi \pm \phi \\ 0, \pm 1, \pm\phi, \pm 1 \pm 1, \pm 1 \pm \phi, \pm\phi \pm \phi \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 0, \pm 1, \pm\phi, \pm 2, \pm\phi^2, \pm\psi, \pm 2\phi \\ 0, \pm 1, \pm\phi, \pm 2, \pm\phi^2, \pm\psi, \pm 2\phi \\ 0, \pm 1, \pm\phi, \pm 2, \pm\phi^2, \pm\psi, \pm 2\phi \end{bmatrix} \\
\Rightarrow A(\vec{\imath}) &= \begin{bmatrix} 0, \pm\frac{1}{2}, \pm\frac{\phi}{2}, \pm\frac{\psi}{2}, \pm 1 \\ 0, \pm\frac{1}{2}, \pm\frac{\phi}{2}, \pm\frac{\psi}{2}, \pm 1 \\ 0, \pm\frac{1}{2}, \pm\frac{\phi}{2}, \pm\frac{\psi}{2}, \pm 1 \end{bmatrix},\ \text{since } \|A(\vec{\imath})\| = 1 \text{ and both } \|\pm\phi^2/2\|, \|\pm\phi\| > 1.
\end{aligned}$$

Similarly for $A(\vec{\jmath})$ and $A(\vec{k})$. Thus there are two possibilities: A is a signed permutation matrix, i.e. $A \in S(C)$; or A is one of the 384 possible orthogonal signed matrices of Example 12. But how many of these $48 + 384 = 432$ matrices fix the set of vertices of the icosahedron? Looking at

[6] See *Classification of Groups of Order 60* by Alfonso Gracia-Saz

the vertices of I it is clear that if $A \in S(I)$ then, up to sign, A must induce an even permutation of the entries in the vertices of I. So if $A \in S(I) \cap S(C)$ then $A \in \ker(\text{sgn})$; that is, A is one of the 24 matrices in Table 7, which displays all the signed even permutation matrices.

In contrast, the signed counter-symmetric matrices of Example 12 do *not* fix the vertices of I. For example, multiplication of the vertices of I by

$$A_1 = \frac{1}{2}\begin{bmatrix} 1 & \psi & \phi \\ \phi & 1 & \psi \\ \psi & \phi & 1 \end{bmatrix}$$

results in an odd permutation of the components of each vertex of I, to wit:

$$A_1\begin{bmatrix} 0 \\ 1 \\ \phi \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \phi \end{bmatrix}, \; A_1\begin{bmatrix} 0 \\ -1 \\ \phi \end{bmatrix} = \begin{bmatrix} \phi \\ -1 \\ 0 \end{bmatrix}, \; A_1\begin{bmatrix} \phi \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \phi \\ 1 \\ 0 \end{bmatrix},$$

$$A_1\begin{bmatrix} \phi \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ \phi \\ -1 \end{bmatrix}, \; A_1\begin{bmatrix} 1 \\ \phi \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \phi \\ 1 \end{bmatrix}, \; A_1\begin{bmatrix} -1 \\ \phi \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ \phi \end{bmatrix}.$$

Thus $\pm A_1, \pm P_{(123)}A_1$ and $\pm P_{(132)}A_1$—and all their signed orthogonal variations—are not symmetries of I. But $\pm P_{(23)}A_1, \pm P_{(12)}A_1$ and $P_{(13)}A_1$—and all their signed orthogonal variations—are symmetries of I. The 48 such symmetries with determinant 1 are listed in Table 13. In total, $S(I)$ consists of 120 matrices: the 24 signed even permutation matrices displayed in Table 7, the 48 matrices displayed in Table 13, and the negatives of these 48 matrices. Consequently,

Theorem 14 $S(I) \cong A_5 \times C_2$.

Proof: of the 120 matrices calculated above, the normal subgroup of 60 matrices with determinant equal to 1, consists of the matrices listed in Example 11, where it was pointed out that this subgroup must be a copy of A_5. Thus $S(I) \cong A_5 \times C_2$. □

As for the other matrices from Example 12: consider another icosahedron, I', with vertices

$$\begin{bmatrix} 0 \\ \pm 1 \\ \pm\psi \end{bmatrix}, \begin{bmatrix} \pm\psi \\ 0 \\ \pm 1 \end{bmatrix}, \begin{bmatrix} \pm 1 \\ \pm\psi \\ 0 \end{bmatrix}$$

and let

$$A_2 = \frac{1}{2}\begin{bmatrix} 1 & \phi & \psi \\ \psi & 1 & \phi \\ \phi & \psi & 1 \end{bmatrix}.$$

Then, analogously, $\pm A_2, \pm P_{(123)}A_2$ and $\pm P_{(132)}A_2$—and all their signed orthogonal variations—are not symmetries of I', but $\pm P_{(23)}A_2, \pm P_{(12)}A_2$ and $P_{(13)}A_2$—and all their signed orthogonal variations—are symmetries of I'. The 48 such matrices with determinant 1 are listed in Table 14. $S(I')$ consists of 120 matrices: the 24 signed even permutation matrices displayed in Table 7, the 48 matrices displayed in Table 14, and the negatives of these 48 matrices.

What about the symmetry group of the dodecahedron? Consider $S(I)$, where I is the icosahedron from above. $S(I)$ contains 20 matrices with order 3, generating 10 copies of C_3—the rotational symmetry group of an equilateral triangle. The following vectors,

$$\begin{bmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \pm\phi \\ \pm\psi \end{bmatrix}, \begin{bmatrix} \pm\psi \\ 0 \\ \pm\phi \end{bmatrix}, \begin{bmatrix} \pm\phi \\ \pm\psi \\ 0 \end{bmatrix},$$

are the eigenvectors, and their negatives, for the eigenvalue 1, of all the matrices $A \in S(I)$ with order 3. These 20 vectors represent the vertices of a dodecahedron, D, and $S(D) = S(I)$. (D is essentially the dodecahedron with vertices determined by the midpoints of the 20 equilateral triangular faces of I.) If instead, you consider the elements of order 3 in $S(I')$—the matrices with order 3 in Tables 7 and 14—then their eigenvectors generate the vertices

$$\begin{bmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \pm\psi \\ \pm\phi \end{bmatrix}, \begin{bmatrix} \pm\phi \\ 0 \\ \pm\psi \end{bmatrix}, \begin{bmatrix} \pm\psi \\ \pm\phi \\ 0 \end{bmatrix}$$

of another dodecahedron, D', with $S(D') = S(I')$. Thus

Theorem 15 *If P is a Platonic solid, then there are vertices for P such that $S(P)$ is a group of orthogonal matrices consisting of signed, orthogonal deranged matrices. In particular,*

$$S(T) \cong S_4, \ \ S(C) \cong S(O) \cong S_4 \times C_2, \ \ S(D) \cong S(I) \cong A_5 \times C_2.$$

Proof: the results for $S(C), S(I)$ and $S(D)$ have been displayed above. For one tetrahedron, T, take vertices

$$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};$$

then $S(T) = \ker(\mathrm{par}\,)$, the 24 matrices listed in Table 8, which from a group isomorphic to S_4. For the octahedron, O, take the vertices

$$\begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix};$$

then $S(O) = S(C)$. □

The Parity of Elements in $S(P)$: previously we defined the parity of a matrix $A \in S(C)$ by

$$\mathrm{par}\,(A) = \det(A\,\Phi(A))$$

and it was pointed out that for $A \in S(C)$,

$$\mathrm{par}\,(A) = \begin{cases} +1 & \text{if } A \text{ has an even number of negative entries,} \\ -1 & \text{if } A \text{ has an odd number of negative entries.} \end{cases}$$

In general, for any matrix $A \in S(P)$, where P is a Platonic solid, define

$$\mathrm{par}\,(A) = \begin{cases} +1 & \text{if } A \text{ has an even number of negative entries,} \\ -1 & \text{if } A \text{ has an odd number of negative entries.} \end{cases}$$

This function is not a homomorphism, as it is on $S(C)$, but it is well-defined for any matrix $A \in S(P)$. In particular, all the matrices in Tables 13 and 14 have parity $+1$.

Symmetries of Platonic Solids as Permutations: if we choose the vertices of the Platonic solids as above, then the symmetry groups of the Platonic solids can easily be constructed in terms of the matrices in $\mathcal{A}$ (see Table 7) and four other matrices. See Table 15, in which

$$P_{(23)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ U = \frac{1}{2}\begin{bmatrix} 1 & \psi & \phi \\ -\psi & -\phi & -1 \\ \phi & 1 & \psi \end{bmatrix}, \ V = \frac{1}{2}\begin{bmatrix} 1 & \phi & \psi \\ -\phi & -\psi & -1 \\ \psi & 1 & \phi \end{bmatrix}.$$

P	Vertices of P	rotations in $S(P)$	non-rotations in $S(P)$	structure of $S(P)$
T	$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\mathcal{A} \cong A_4$	$(P_{(23)})\mathcal{A}$	$\ker(\text{par}) \cong S_4$
T'	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$	$\mathcal{A} \cong A_4$	$(P_{(23)})\mathcal{A}$	$\ker(\text{par}) \cong S_4$
C	$\begin{bmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{bmatrix}$	$\ker(\det) = \mathcal{A} \cup (-P_{(23)})\mathcal{A} \cong S_4$	$(-I)\ker(\det)$	$S_4 \times C_2$
O	$\begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}$	$\ker(\det) = \mathcal{A} \cup (-P_{(23)})\mathcal{A} \cong S_4$	$(-I)\ker(\det)$	$S_4 \times C_2$
I	$\begin{bmatrix} 0 \\ \pm 1 \\ \pm\phi \end{bmatrix}, \begin{bmatrix} \pm\phi \\ 0 \\ \pm 1 \end{bmatrix}, \begin{bmatrix} \pm 1 \\ \pm\phi \\ 0 \end{bmatrix}$	$\langle U \rangle\, \mathcal{A} \cong A_5$	$(-I)\langle U \rangle\, \mathcal{A}$	$A_5 \times C_2$
D	$\begin{bmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \pm\phi \\ \pm\psi \end{bmatrix}, \begin{bmatrix} \pm\psi \\ 0 \\ \pm\phi \end{bmatrix}, \begin{bmatrix} \pm\phi \\ \pm\psi \\ 0 \end{bmatrix},$	$\langle U \rangle\, \mathcal{A} \cong A_5$	$(-I)\langle U \rangle\, \mathcal{A}$	$A_5 \times C_2$
I'	$\begin{bmatrix} 0 \\ \pm 1 \\ \pm\psi \end{bmatrix}, \begin{bmatrix} \pm\psi \\ 0 \\ \pm 1 \end{bmatrix}, \begin{bmatrix} \pm 1 \\ \pm\psi \\ 0 \end{bmatrix}$	$\langle V \rangle\, \mathcal{A} \cong A_5$	$(-I)\langle V \rangle\, \mathcal{A}$	$A_5 \times C_2$
D'	$\begin{bmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \pm\psi \\ \pm\phi \end{bmatrix}, \begin{bmatrix} \pm\phi \\ 0 \\ \pm\psi \end{bmatrix}, \begin{bmatrix} \pm\psi \\ \pm\phi \\ 0 \end{bmatrix}$	$\langle V \rangle\, \mathcal{A} \cong A_5$	$(-I)\langle V \rangle\, \mathcal{A}$	$A_5 \times C_2$

Table 15: The 240 orthogonal signed deranged matrices in the symmetry groups of the five Platonic solids. $S(P)$ always includes $\mathcal{A} = \{A \in S(C) \mid \text{sgn}\,(A) = \det(A) = \text{par}\,(A) = 1\} \cong A_4$. $S(T) = S(T') \lhd S(C) = S(O)$. Pairs I, I' and D, D' differ only in the swapping of ϕ and ψ. Note that neither $\langle U \rangle, \langle V \rangle$ or $\mathcal{A}$ are normal subgroups of $S(I), S(D), S(I')$ or $S(D')$, but that $\langle U \rangle \mathcal{A} = \mathcal{A} \langle U \rangle$ and $\langle V \rangle \mathcal{A} = \mathcal{A} \langle V \rangle$, so $\langle U \rangle \mathcal{A}$ and $\langle V \rangle \mathcal{A}$ are well determined groups.

Both U and V represent rotations of order 5; they are arbitrarily chosen matrices of order 5 from Tables 13 & 14, respectively. Call this set of 240 signed deranged orthogonal matrices $\mathbb{P}$.

For each $A \in \mathbb{P}$ define the sign of A to be $\operatorname{sgn}(A) = \det(A)\operatorname{par}(A)$, albeit $\mathbb{P}$ is neither a group, nor sgn a homomorphism on $\mathbb{P}$. Still, we can classify each $A \in \mathbb{P}$ as either even or odd, as follows:

$$\text{call } A \in \mathbb{P} \text{ even if } \operatorname{sgn}(A) = 1; \text{ and call } A \in \mathbb{P} \text{ odd if } \operatorname{sgn}(A) = -1.$$

Thus A is even if $\det(A) = \operatorname{par}(A)$ and A is odd if $\det(A) \neq \operatorname{par}(A)$. This convention, which classifies $-I$ as even, is not inconsistent with matrices in $S(P)$ viewed as elements of A_4, S_4 or A_5. That is:

- $A \in S(T)$ or $S(T')$ is even if and only if $A \in \mathcal{A} \cong A_4$.
- $A \in S(T)$ or $S(T')$ is odd if and only if $A \in \mathcal{P}_{(23)}\mathcal{A}$, which is S_4 without A_4.
- $A \in S(C)$ or $S(O)$ is even if and only if $A \in \mathcal{A} \cup (-\mathcal{A}) \cong A_4 \times C_2$.
- $A \in S(C)$ or $S(O)$ is odd if and only if $A \in P_{(23)}\mathcal{A}$ or $A \in -P_{(23)}\mathcal{A}$, both S_4 without A_4.
- $A \in S(I), S(D)$ is even, and both $S(I), S(D) = \langle U\rangle\mathcal{A} \cup (-\langle U\rangle\mathcal{A}) \cong A_5 \times C_2$.
- $A \in S(I'), S(D')$ is even, and both $S(I'), S(D') = \langle V\rangle\mathcal{A} \cup (-\langle V\rangle\mathcal{A}) \cong A_5 \times C_2$.

So every $A \in \mathbb{P}$ can be considered as an even or an odd permutation, but a permutation of what? Certainly not the components of the vertex vectors of P, as illustrated by the calculations above, and generally not even the vertices of P. We claim:

1. Every $A \in S(T)$ induces a permutation of the vertices of T, namely

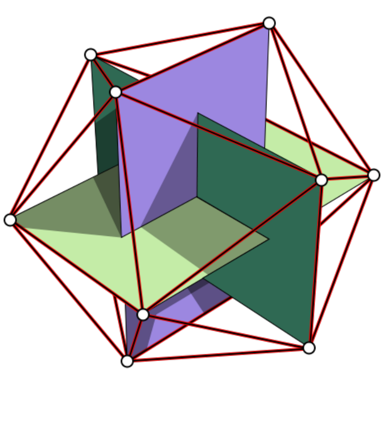
$$v_1 = (1, -1, -1),\ v_2 = (-1, 1, -1),\ v_3 = (-1, -1, 1),\ v_4 = (1, 1, 1).$$

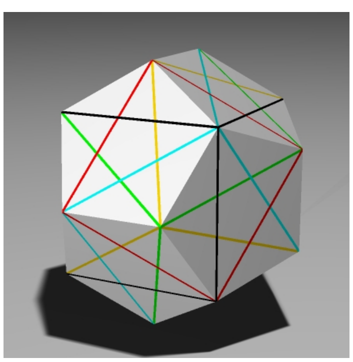
2. Every $A \in S(C)$ induces a permutation of the four diagonals of C, parallel to the vectors

$$\vec{d_1} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix},\ \vec{d_2} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix},\ \vec{d_3} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},\ \vec{d_4} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

3. Every $A \in S(O)$ induces a permutation of the four lines joining the centers of opposite faces of O; these lines are parallel to the vectors $\vec{d_1}, \vec{d_2}, \vec{d_3}, \vec{d_4}$.

4. Every $A \in S(D)$ induces a permutation of the five cubes among the 20 vertices of D. See the figure below, right, and Table 16.





cube_i	eight vertices of cube_i	stabilizer in $S(D)$ of cube_i
cube_1	$\pm\begin{bmatrix}1\\1\\1\end{bmatrix}, \pm\begin{bmatrix}1\\-1\\1\end{bmatrix}, \pm\begin{bmatrix}-1\\1\\1\end{bmatrix}, \pm\begin{bmatrix}1\\1\\-1\end{bmatrix}$	$\mathcal{S} = \mathcal{A} \cup (-\mathcal{A}) \cong A_4 \times C_2$
cube_2	$\pm\begin{bmatrix}1\\-1\\1\end{bmatrix}, \pm\begin{bmatrix}0\\-\phi\\\psi\end{bmatrix}, \pm\begin{bmatrix}-\phi\\\psi\\0\end{bmatrix}, \pm\begin{bmatrix}\psi\\0\\\phi\end{bmatrix}$	$U\,\mathcal{S}\,U^{-1}$
cube_3	$\pm\begin{bmatrix}-\phi\\\psi\\0\end{bmatrix}, \pm\begin{bmatrix}0\\\phi\\\psi\end{bmatrix}, \pm\begin{bmatrix}\psi\\0\\-\phi\end{bmatrix}, \pm\begin{bmatrix}-1\\1\\1\end{bmatrix}$	$U^2\,\mathcal{S}\,U^{-2}$
cube_4	$\pm\begin{bmatrix}\psi\\0\\-\phi\end{bmatrix}, \pm\begin{bmatrix}1\\1\\-1\end{bmatrix}, \pm\begin{bmatrix}\phi\\\psi\\0\end{bmatrix}, \pm\begin{bmatrix}0\\-\phi\\\psi\end{bmatrix}$	$U^3\,\mathcal{S}\,U^{-3}$
cube_5	$\pm\begin{bmatrix}\phi\\\psi\\0\end{bmatrix}, \pm\begin{bmatrix}\psi\\0\\\phi\end{bmatrix}, \pm\begin{bmatrix}1\\1\\1\end{bmatrix}, \pm\begin{bmatrix}0\\\phi\\\psi\end{bmatrix}$	$U^4\,\mathcal{S}\,U^{-4}$

Table 16: The five cubes among the vertices of the dodecahedron D, and their stabilizers.

5. Every $A \in S(I)$ induces a permutation of the five sets of three mutually orthogonal golden rectangles among the 12 vertices of I. See the figure above, left, and Table 17.

Note that in $S(C), S(O), S(I)$ and $S(D)$, $-I$ inverts the vertices of the diagonals, 3-antiprisms, three mutually orthogonal golden rectangles, or cubes, respectively, but does not actually transpose any of them as sets, so it is not inconsistent to call $-I$ even.

The Structure of $\mathbb{P}$: first observe that if $A \in \mathbb{P}$ and $\text{sgn}\,(A) = 1$, then $\det(A) = \text{par}\,(A)$. That is, A represents a rotation if and only if A has an even number of negative entries. Indeed it can easily be proved that

Theorem 16 *If H is a group of matrices contained in $\mathbb{P}$ such that sgn$\,(A) = 1$ for all $A \in H$, then for all $A \in H$, par$\,(A) =$ det(A) and*

$$par : H \longrightarrow \{1, -1\}$$

is a homomorphism. In particular if $A \in H$, then A represents a rotation if and only if A has an even number of negative entries.

Proof: is all in the definitions. If $\text{sgn}\,(A) = \det(A)\,\text{par}\,(A)$ for all $A \in H$, then $\text{par}\,(A) = \det(A)$, for all $A \in H$. Since $\det : H \longrightarrow \{1, -1\}$ is a homomorphism so is par , which also maps H to $\{1, -1\}$. $\square$

Set	2-sylow subgroup	Vertices of 1st Golden Rectangle	Vertices of 2nd Golden Rectangle	Vertices of 3rd Golden Rectangle
GR_1	$\mathcal{D}$	$\begin{bmatrix} 0 \\ \pm 1 \\ \pm\phi \end{bmatrix}$	$\begin{bmatrix} \pm\phi \\ 0 \\ \pm 1 \end{bmatrix}$	$\begin{bmatrix} \pm 1 \\ \pm\phi \\ 0 \end{bmatrix}$
	plane	$x = 0$	$y = 0$	$z = 0$
GR_2	$U\mathcal{D}U^{-1}$	$\pm\begin{bmatrix} -1 \\ \phi \\ 0 \end{bmatrix}, \pm\begin{bmatrix} \phi \\ 0 \\ -1 \end{bmatrix}$	$\pm\begin{bmatrix} 0 \\ 1 \\ \phi \end{bmatrix}, \pm\begin{bmatrix} \phi \\ 0 \\ 1 \end{bmatrix}$	$\pm\begin{bmatrix} 0 \\ -1 \\ \phi \end{bmatrix}, \pm\begin{bmatrix} 1 \\ \phi \\ 0 \end{bmatrix}$
	plane	$x - \psi y + \phi z = 0$	$\psi x - \phi y + z = 0$	$\phi x - y + \psi z = 0$
GR_3	$U^2\mathcal{D}U^{-2}$	$\pm\begin{bmatrix} 1 \\ \phi \\ 0 \end{bmatrix}, \pm\begin{bmatrix} 0 \\ 1 \\ \phi \end{bmatrix}$	$\pm\begin{bmatrix} \phi \\ 0 \\ 1 \end{bmatrix}, \pm\begin{bmatrix} 1 \\ -\phi \\ 0 \end{bmatrix}$	$\pm\begin{bmatrix} \phi \\ 0 \\ -1 \end{bmatrix}, \pm\begin{bmatrix} 0 \\ -1 \\ \phi \end{bmatrix}$
	plane	$\phi x - y - \psi z = 0$	$x - \psi y - \phi z = 0$	$\psi x - \phi y - z = 0$
GR_4	$U^3\mathcal{D}U^{-3}$	$\pm\begin{bmatrix} -1 \\ \phi \\ 0 \end{bmatrix}, \pm\begin{bmatrix} 0 \\ -1 \\ \phi \end{bmatrix}$	$\pm\begin{bmatrix} 1 \\ \phi \\ 0 \end{bmatrix}, \pm\begin{bmatrix} \phi \\ 0 \\ 1 \end{bmatrix}$	$\pm\begin{bmatrix} \phi \\ 0 \\ -1 \end{bmatrix}, \pm\begin{bmatrix} 0 \\ 1 \\ \phi \end{bmatrix}$
	plane	$\phi x + y - \psi z = 0$	$x + \psi y - \phi z = 0$	$\psi x + \phi y - z = 0$
GR_5	$U^4\mathcal{D}U^{-4}$	$\pm\begin{bmatrix} 1 \\ \phi \\ 0 \end{bmatrix}, \pm\begin{bmatrix} \phi \\ 0 \\ -1 \end{bmatrix}$	$\pm\begin{bmatrix} 0 \\ -1 \\ \phi \end{bmatrix}, \pm\begin{bmatrix} \phi \\ 0 \\ 1 \end{bmatrix}$	$\pm\begin{bmatrix} -1 \\ \phi \\ 0 \end{bmatrix}, \pm\begin{bmatrix} 0 \\ 1 \\ \phi \end{bmatrix}$
	plane	$x + \psi y + \phi z = 0$	$\psi x + \phi y + z = 0$	$\phi x + y + \psi z = 0$

Table 17: The five sets of three mutually orthogonal golden rectangles among the vertices of I, the planes they are in, and the conjugate of $\mathcal{D}$ in $S(I)$ that fixes each set of golden rectangles 'point'-wise. That is, not only is the set of rectangles fixed, but so too each rectangle in the set.

	I	J	K	L	P	JP	KP	LP	P^2	JP^2	KP^2	LP^2
I	I	J	K	L	P	JP	KP	LP	P^2	JP^2	KP^2	LP^2
J	J	I	L	K	JP	P	LP	KP	JP^2	P^2	LP^2	KP^2
K	K	L	I	J	KP	LP	P	JP	KP^2	LP^2	P^2	JP^2
L	L	K	J	I	LP	KP	JP	P	LP^2	KP^2	JP^2	P^2
P	P	LP	JP	KP	P^2	LP^2	JP^2	KP^2	I	L	J	K
JP	JP	KP	P	LP	JP^2	KP^2	P^2	LP^2	J	K	I	L
KP	KP	JP	LP	P	KP^2	JP^2	LP^2	P^2	K	J	L	I
LP	LP	P	KP	JP	LP^2	P^2	KP^2	JP^2	L	I	K	J
P^2	P^2	KP^2	LP^2	JP^2	I	K	L	J	P	KP	LP	JP
JP^2	JP^2	LP^2	KP^2	P^2	J	L	K	I	JP	LP	KP	P
KP^2	KP^2	P^2	JP^2	LP^2	K	I	J	L	KP	P	JP	LP
LP^2	P^2	JP^2	P^2	KP^2	L	J	I	K	LP	JP	P	KP

Table 18: The multiplication table for the group $\mathcal{A} \cong A_4$, with $P = P_{(123)}$ and $P^2 = P_{(132)}$. Also note that: $PK = JP,\ PL = KP,\ PJ = LP$ and $P^2L = JP^2,\ P^2J = KP^2,\ P^2K = LP^2$.

An Equivalence Relation: let $\mathbb{SD} = \{DAD' \mid A \in \mathbb{D}, D, D' \in \mathcal{D}\}$. By Theorem 13, $\mathbb{SD}$ is the set of all possible orthogonal signed variations of orthogonal deranged 3×3 matrices in $\mathbb{D}$. We can extend the definition of parity and sign to $M \in \mathbb{SD}$, as usual:

$$\operatorname{par}(M) = \begin{cases} +1 & \text{if } M \text{ has an even number of negative entries,} \\ -1 & \text{if } M \text{ has an odd number of negative entries.} \end{cases}$$

and $\operatorname{sgn}(M) = \det(M)\operatorname{par}(M)$. You can check that if $M = DAD'$ with $A \in \mathbb{D}, D, D' \in \mathcal{D}$ then $\operatorname{par}(M) = \operatorname{par}(D)\operatorname{par}(A)\operatorname{par}(D')$. More importantly let us define a relation on $\mathbb{SD}$ by

$$M \sim N \Leftrightarrow N = XMY, \text{ for some } X, Y \in \mathcal{A},$$

where $\mathcal{A} = \{I, J, K, L, P, JP, KP, LP, P^2, JP^2, KP^2, LP^2\}$, with J, K and L as in the Seventh Pass of Part 3, $P = P_{(123)}$ and $P^2 = P_{(132)}$. The multiplication table for $\mathcal{A}$ is listed in Table 18.

It is straightforward to check that $\sim$ is an equivalence relation on $\mathbb{SD}$. Let $[M]$ be the equivalence class of $M \in \mathbb{SD}$ with respect to this equivalence relation. To find all the matrices in $[M]$ it requires 144 calculations, in theory. Let's break these calculations down into 9 sets of 16 products and see if any simplifications or short cuts can be found.

Each set exhibits the same pattern: each set consists of the 16 elements in $[M]$ obtained by forming the 16 products XZY for

$$X, Y \in \{I, J, K, L\},$$

where Z, in the top left corner of the set, ranges over the nine possible products UMV for

$$U, V \in \{I, P, P^2\}.$$

M	JM	KM	LM
MJ	JMJ	KMJ	LMJ
MK	JMK	KMK	LMK
ML	JML	KML	LML

Set 1: $Z = M$

PM	JPM	KPM	LPM
PMJ	$JPMJ$	$KPMJ$	$LPMJ$
PMK	$JPMK$	$KPMK$	$LPMK$
PML	$JPML$	$KPML$	$LPML$

Set 2: $Z = PM$

P^2M	JP^2M	KP^2M	LP^2M
P^2MJ	JP^2MJ	KP^2MJ	LP^2MJ
P^2MK	JP^2MK	KP^2MK	LP^2MK
P^2ML	JP^2ML	KP^2ML	LP^2ML

Set 3: $Z = P^2M$

MP	JMP	KMP	LMP
MPJ	$JMPJ$	$KMPJ$	$LMPJ$
MPK	$JMPK$	$KMPK$	$LMPK$
MPL	$JMPL$	$KMLP$	$LMPL$

Set 4: $Z = MP$

PMP	$JPMP$	$KPMP$	$LPMP$
$PMPJ$	$JPMPJ$	$KPMPJ$	$LPMPJ$
$PMPK$	$JPMPK$	$KPMPK$	$LPMPK$
$PMPL$	$JPMPL$	$KPMPL$	$LPMPL$

Set 5: $Z = PMP$

P^2MP	JP^2MP	KP^2MP	LP^2MP
P^2MPJ	JP^2MPJ	KP^2MPJ	LP^2MPJ
P^2MPK	JP^2MPK	KP^2MPK	LP^2MPK
P^2MPL	JP^2MPL	KP^2MPL	LP^2MPL

Set 6: $Z = P^2MP$

MP^2	JMP^2	KMP^2	LMP^2
MP^2J	JMP^2J	KMP^2J	LMP^2J
MP^2K	JMP^2K	KMP^2K	LMP^2K
MP^2L	JMP^2L	KMP^2L	LMP^2L

Set 7: $Z = MP^2$

PMP^2	$JPMP^2$	$KPMP^2$	$LPMP^2$
PMP^2J	$JPMP^2J$	$KPMP^2J$	$LPMP^2J$
PMP^2K	$JPMP^2K$	$KPMP^2K$	$LPMP^2K$
PMP^2L	$JPMP^2L$	$KPMP^2L$	$LPMP^2L$

Set 8: $Z = PMP^2$

P^2MP^2	JP^2MP^2	KP^2MP^2	LP^2MP^2
P^2MP^2J	JP^2MP^2J	KP^2MP^2J	LP^2MP^2J
P^2MP^2K	JP^2MP^2K	KP^2MP^2K	LP^2MP^2K
P^2MP^2L	JP^2MP^2L	KP^2MP^2L	LP^2MP^2L

Set 9: $Z = P^2MP^2$

Some simplifications can be made. If $M \in \mathbb{U}$, then

$$PM = MP \text{ and } P^2M = MP^2$$

as you can check, and so

- Set 1 = Set 6 = Set 8
- Set 2 = Set 4 = Set 9
- Set 3 = Set 5 = Set 7.

If $M \in \mathbb{W}$ check that $PM = MP^2$ and $P^2M = MP$, so

- Set 1 = Set 5 = Set 9
- Set 2 = Set 6 = Set 7
- Set 3 = Set 4 = Set 8.

Thus, for example, Table 13 displays the 48 matrices in the equivalence class of

$$M = -\frac{1}{2}\begin{pmatrix} \phi & 1 & \psi \\ 1 & \psi & \phi \\ \psi & \phi & 1 \end{pmatrix} \in \mathbb{W}$$

and Table 14 displays the 48 matrices in the equivalence class of

$$M = -\frac{1}{2}\begin{pmatrix} \psi & 1 & \phi \\ 1 & \phi & \psi \\ \phi & \psi & 1 \end{pmatrix} \in \mathbb{W}.$$

In each case, the matrices corresponding to Set 1 are in the first column, those to Set 2 in the second column, those to Set 3 in the third column.

Now, if $X \in \mathcal{A}$ then $\det(X) = \operatorname{sgn}(X) = \operatorname{par}(X) = 1$, so it follows that all matrices in $[M]$ have the same determinant, the same parity and the same sign as M. But not the same order, as shown in Tables 13 and 14.

Three Sets are Enough: let $M = DAD'$ for some $A \in \mathbb{D}$, some $D, D' \in \mathcal{D} = \{\pm I, \pm J, \pm K, \pm L\}$. Half of the elements in $\mathcal{D}$ are also in $\mathcal{A}$, specifically $\mathcal{V} = \{I, J, K, L\} \subset \mathcal{A}$. So if $M = DAD'$ for $D, D' \in \mathcal{V}$, then $M \sim A$ and consequently $[M] = [A]$. This holds true if $-D, -D' \in \mathcal{V}$ as well, since $(-D)A(-D') = DAD' = M$. But if only one of D, D' is in $\mathcal{V}$, say D, then $-D' \in \mathcal{V}$, and so

$$-M = DA(-D') \Rightarrow M = D(-A)(-D') \Rightarrow M \sim -A \Rightarrow [M] = [-A].$$

Of course, $-A \in \mathbb{D}$ if $A \in \mathbb{D}$. Thus for each equivalence class $[M]$ there is an $A \in \mathbb{D}$ such that $[M] = [A]$. By Theorem 10, $A \in \mathbb{U}$ or $A \in \mathbb{W}$, thus the above simplifications apply and $[M]$ is equal to the union of Sets 1, 2 and 3. In particular, $[M]$ contains at most 48 matrices.

Special Cases of $[M]$**:** let us look at $[M]$ in more detail in the cases when $[M] = [X]$ for $X \in \mathbb{D}_1 \cup \mathbb{D}_2$. That is, suppose X is an orthogonal deranged matrix with determinant $+1$. Then either X is counter-symmetric or X is symmetric. To set things up

$$\text{let } A = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} \in \mathbb{D}_1 \subset \mathbb{U} \text{ and let } B = -P_{(12)}A = \begin{bmatrix} -b & -a & -c \\ -a & -c & -b \\ -c & -b & -a \end{bmatrix} \in \mathbb{D}_2 \subset \mathbb{W}$$

for some a, b, c such that $a^2 + b^2 + c^2 = 1$ and $a + b + c = 1$.

Three Sets for $[A]$**:** let $A = aI + cP + bP^2$, with $a^2 + b^2 + c^2 = 1$, $a + b + c = 1$, and assume a, b, c are all distinct.[7] Then the 48 matrices in $[A]$ are all distinct. This is a consequence of Theorem 13. That is, each of A, PA and P^2A have 32 distinct orthogonal, signed variations, only 16 of which have determinant $+1$. So the significance of $[A]$ is that it contains all the orthogonal, signed variations of the three orthogonal counter-symmetric matrices with a, b, c distinct and $a^2 + b^2 + c^2 = 1, a + b + c = 1$, that have the same determinant as A. We list all the elements in Sets 1, 2 and 3 of $[A]$, and their traces and axes of rotation, in Table 19. Let

$$\vec{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{r} = \begin{bmatrix} 1+3a \\ c-b \\ b-c \end{bmatrix}, \vec{s} = \begin{bmatrix} b-c \\ 1+3a \\ c-b \end{bmatrix}, \vec{t} = \begin{bmatrix} c-b \\ b-c \\ 1+3a \end{bmatrix}.$$

Check that $A\vec{d} = \vec{d}, A\vec{r} = J\vec{r}, A\vec{s} = K\vec{s}, A\vec{t} = L\vec{t}$. Then it is an easy exercise to calculate the traces and axes of rotation of the 16 matrices in $[A]$ which are in Set 1. See the second column of Table 19.

Similar calculations for $P\,A$ and $P^2\,A$ are in the third and fourth columns of Table 19, where the vectors $\vec{u}, \vec{v}, \vec{w}$ and $\vec{x}, \vec{y}, \vec{z}$ are obtained by deranging the parameters a, b, c in the two possible ways:

$$\vec{u} = \begin{bmatrix} 1+3b \\ a-c \\ c-a \end{bmatrix}, \vec{v} = \begin{bmatrix} c-a \\ 1+3b \\ a-c \end{bmatrix}, \vec{w} = \begin{bmatrix} a-c \\ c-a \\ 1+3b \end{bmatrix};$$

$$\vec{x} = \begin{bmatrix} 1+3c \\ b-a \\ a-b \end{bmatrix}, \vec{y} = \begin{bmatrix} a-b \\ 1+3c \\ b-a \end{bmatrix}, \vec{z} = \begin{bmatrix} b-a \\ a-b \\ 1+3c \end{bmatrix}.$$

The data in the third and fourth columns of Table 19 depend on the equations

$$P\,A\,\vec{d} = \vec{d},\ \ P\,A\,\vec{u} = J\,\vec{u},\ \ P\,A\,\vec{v} = K\,\vec{v}, P\,A\,\vec{w} = L\,\vec{w};$$

$$P^2\,A\,\vec{d} = \vec{d},\ \ P^2\,A\,\vec{x} = J\,\vec{x},\ \ P^2\,A\,\vec{y} = K\,\vec{y},\ \ P^2\,A\vec{z} = L\,\vec{z}.$$

Now, as pointed out in the proof of Theorem 10, $A = D_1(\theta)$, for some θ, with

$$\begin{aligned} D_1(\theta) &= \frac{1}{3}\begin{bmatrix} 1+2\cos\theta & 1-\sqrt{3}\sin\theta-\cos\theta & 1+\sqrt{3}\sin\theta-\cos\theta \\ 1+\sqrt{3}\sin\theta-\cos\theta & 1+2\cos\theta & 1-\sqrt{3}\sin\theta-\cos\theta \\ 1-\sqrt{3}\sin\theta-\cos\theta & 1+\sqrt{3}\sin\theta-\cos\theta & 1+2\cos\theta \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 1+2\cos\theta & 1+2\cos(\theta+2\pi/3) & 1+2\cos(\theta-2\pi/3) \\ 1+2\cos(\theta-2\pi/3) & 1+2\cos\theta & 1+2\cos(\theta+2\pi/3) \\ 1+2\cos(\theta+2\pi/3) & 1+2\cos(\theta-2\pi/3) & 1+2\cos\theta \end{bmatrix}, \end{aligned}$$

[7] Recall the two extreme cases from Part 3. Note: $\|\vec{r}\| = \sqrt{(3a+1)(a+3)}$ or $\sqrt{2+\cos^2(\theta/2)}$

	Set 1 for $Z = A$		Set 2 for $Z = P\,A$		Set 3 for $Z = P^2\,A$	
$A = D_1(\theta) \in \mathbb{D}_1$	$Z = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$		$Z = \begin{bmatrix} b & a & c \\ c & b & a \\ a & c & b \end{bmatrix}$		$Z = \begin{bmatrix} c & b & a \\ a & c & b \\ b & a & c \end{bmatrix}$	
$X \in [A]$	$\mathrm{tr}\,(X)$	$E_1(X)$	$\mathrm{tr}\,(X)$	$E_1(X)$	$\mathrm{tr}\,(X)$	$E_1(X)$
Z	$3a$	$\mathrm{span}\,\{\vec{d}\}$	$3b$	$\mathrm{span}\,\{\vec{d}\}$	$3c$	$\mathrm{span}\,\{\vec{d}\}$
JZ	$-a$	$\mathrm{span}\,\{\vec{r}\}$	$-b$	$\mathrm{span}\,\{\vec{u}\}$	$-c$	$\mathrm{span}\,\{\vec{x}\}$
KZ	$-a$	$\mathrm{span}\,\{\vec{s}\}$	$-b$	$\mathrm{span}\,\{\vec{v}\}$	$-c$	$\mathrm{span}\,\{\vec{y}\}$
LZ	$-a$	$\mathrm{span}\,\{\vec{t}\}$	$-b$	$\mathrm{span}\,\{\vec{w}\}$	$-c$	$\mathrm{span}\,\{\vec{z}\}$
ZJ	$-a$	$\mathrm{span}\,\{J\vec{r}\}$	$-b$	$\mathrm{span}\,\{J\vec{u}\}$	$-c$	$\mathrm{span}\,\{J\vec{x}\}$
JZJ	$3a$	$\mathrm{span}\,\{J\vec{d}\}$	$3b$	$\mathrm{span}\,\{J\vec{d}\}$	$3c$	$\mathrm{span}\,\{J\vec{d}\}$
KZJ	$-a$	$\mathrm{span}\,\{J\vec{t}\}$	$-b$	$\mathrm{span}\,\{J\vec{w}\}$	$-c$	$\mathrm{span}\,\{J\vec{z}\}$
LZJ	$-a$	$\mathrm{span}\,\{J\vec{s}\}$	$-b$	$\mathrm{span}\,\{J\vec{v}\}$	$-c$	$\mathrm{span}\,\{J\vec{y}\}$
ZK	$-a$	$\mathrm{span}\,\{K\vec{s}\}$	$-b$	$\mathrm{span}\,\{K\vec{v}\}$	$-c$	$\mathrm{span}\,\{K\vec{y}\}$
JZK	$-a$	$\mathrm{span}\,\{K\vec{t}\}$	$-b$	$\mathrm{span}\,\{K\vec{w}\}$	$-c$	$\mathrm{span}\,\{K\vec{z}\}$
KZK	$3a$	$\mathrm{span}\,\{K\vec{d}\}$	$3b$	$\mathrm{span}\,\{K\vec{d}\}$	$3c$	$\mathrm{span}\,\{K\vec{d}\}$
LZK	$-a$	$\mathrm{span}\,\{K\vec{r}\}$	$-b$	$\mathrm{span}\,\{K\vec{u}\}$	$-c$	$\mathrm{span}\,\{K\vec{x}\}$
ZL	$-a$	$\mathrm{span}\,\{L\vec{t}\}$	$-b$	$\mathrm{span}\,\{L\vec{w}\}$	$-c$	$\mathrm{span}\,\{L\vec{z}\}$
JZL	$-a$	$\mathrm{span}\,\{L\vec{s}\}$	$-b$	$\mathrm{span}\,\{L\vec{v}\}$	$-c$	$\mathrm{span}\,\{L\vec{y}\}$
KZL	$-a$	$\mathrm{span}\,\{L\vec{r}\}$	$-b$	$\mathrm{span}\,\{L\vec{u}\}$	$-c$	$\mathrm{span}\,\{L\vec{x}\}$
LZL	$3a$	$\mathrm{span}\,\{L\vec{d}\}$	$3b$	$\mathrm{span}\,\{L\vec{d}\}$	$3c$	$\mathrm{span}\,\{L\vec{d}\}$
$\cos^2\theta \neq 1$ or $\dfrac{1}{4}$	$a = \dfrac{1+2\cos\theta}{3},\ b = \dfrac{1+2\cos(\theta-2\pi/3)}{3},\ c = \dfrac{1+2\cos(\theta+2\pi/3)}{3}$					

Table 19: The 48 distinct matrices in $[A]$ for $A \in \mathbb{D}_1$, with $a^2+b^2+c^2 = 1$, $a+b+c = 1$ and a, b, c all distinct; a, b, c also given in terms of the parameter θ, for $\cos^2\theta \neq 1$ and $\cos^2\theta \neq 1/4$.

where we have used some trigonometric identities. The entries in each row of A are all distinct if $\cos^2\theta \neq 1$, $\cos^2\theta \neq 1/4$. Then PA and P^2A correspond to $D_1(\theta - 2\pi/3)$ and $D_1(\theta + 2\pi/3)$, respectively. In terms of the parameter θ, and by making use of some double-angle formulas, we can take

$$\vec{r}(\theta) = \begin{bmatrix} \sqrt{3}\cos(\theta/2) \\ -\sin(\theta/2) \\ \sin(\theta/2) \end{bmatrix}, \vec{s}(\theta) = \begin{bmatrix} \sin(\theta/2) \\ \sqrt{3}\cos(\theta/2) \\ -\sin(\theta/2) \end{bmatrix}, \vec{t}(\theta) = \begin{bmatrix} -\sin(\theta/2) \\ \sin(\theta/2) \\ \sqrt{3}\cos(\theta/2) \end{bmatrix};$$

$$\vec{u}(\theta) = \vec{r}(\theta - 2\pi/3),\ \vec{v}(\theta) = \vec{s}(\theta - 2\pi/3),\ \vec{w} = \vec{t}(\theta - 2\pi/3);$$
$$\vec{x}(\theta) = \vec{r}(\theta + 2\pi/3),\ \vec{y}(\theta) = \vec{s}(\theta + 2\pi/3),\ \vec{z} = \vec{t}(\theta + 2\pi/3).$$

The vector triples

$$\{\vec{r}(\theta), \vec{s}(\theta), \vec{t}(\theta)\}, \{\vec{u}(\theta), \vec{v}(\theta), \vec{w}(\theta)\} \text{ and } \{\vec{x}(\theta), \vec{y}(\theta), \vec{z}(\theta)\}$$

are each linearly independent sets, since

$$\det[\ \vec{r}(\theta) \mid \vec{s}(\theta) \mid \vec{t}(\theta)\ ] = 3\sqrt{3}\cos(\theta/2),$$

and

$$\det[\ \vec{u}(\theta) \mid \vec{v}(\theta) \mid \vec{w}(\theta)\ ] = 3\sqrt{3}\cos(\theta/2 - \pi/3)$$

and

$$\det[\ \vec{x}(\theta) \mid \vec{y}(\theta) \mid \vec{z}(\theta)\ ] = 3\sqrt{3}\cos(\theta/2 + \pi/3),$$

none of which is zero if $\cos^2\theta \neq 1, \cos^2\theta \neq 1/4$. Thus the matrices in Table 19 that represent rotations around the same axis, have different traces, so are distinct matrices; and the matrices in Table 19 that have the same traces, represent rotations around different axes, so are distinct matrices.

In the two remaining cases[8] for $[A]$ we have $\cos^2\theta = 1$. If $\cos\theta = 1$ then $\theta = 0, a = 1, b = c = 0$ and $A = I$, so $[A] = \mathcal{A}$. In this case we can take $\vec{r} = \vec{i}, \vec{s} = \vec{j}, \vec{t} = \vec{k}$ and

$$\vec{u} = \vec{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = -L\vec{d},\ \vec{v} = \vec{z} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = -J\vec{d},\ \vec{w} = \vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = -K\vec{d}.$$

Reading the information off Table 19, but using the above values of a, b, c and $\vec{r}, \vec{s}, \ldots, \vec{z}$, you can conclude that

- J, K and L represent rotations of order 2 around the x-, y- and z-axes, respectively,
- $JP = PK,\ KP = PL,\ LP = PJ$,
- $JP^2 = P^2L,\ KP^2 = P^2J,\ LP^2 = P^2K$.
- P and P^2 represent rotations of order 3 around the axis span $\{\vec{d}\}$,
- JP and P^2J represent rotations of order 3 around the axis span $\{L\vec{d}\}$,
- KP and P^2K represent rotations of order 3 around the axis span $\{J\vec{d}\}$,
- LP and P^2L represent rotations of order 3 around the axis span $\{K\vec{d}\}$,

In particular, *all* the axes of rotations of the matrices in $[I]$ are either parallel to the coordinate axes, or to the diagonals of the cube with vertices $(\pm1, \pm1, \pm1)$.

[8]If A is any orthogonal matrix in $\mathbb{D}_1 \cup \mathbb{D}_2$ with each column containing two 0's, then the number of orthogonal signed variations of each of A, PA and P^2A is $2^3 = 8$, only half of which will have the same determinant as A. Thus $[A]$ will have $3 \times 4 = 12$ matrices. In particular all of $[I]$, $[-P_{12}]$, $[-I]$, $[P_{12}]$ have cardinality 12.

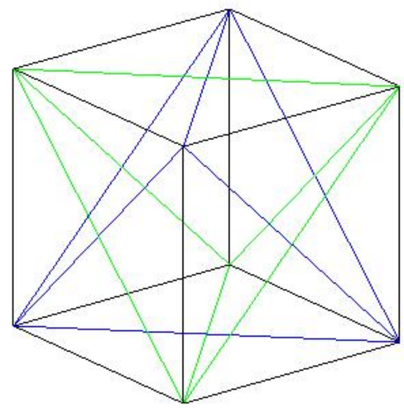


Figure 8: The blue tetrahedron T has vertices $(1,1,1),(1,-1,-1),(-1,1,-1),(-1,-1,1)$; the green tetrahedron T' has vertices $(-1,-1,-1),(-1,1,1),(1,-1,1),(1,1,-1)$.

The significance of $[I]$ is that it is the only group of rotational symmetries included in every symmetry group $S(P)$; see Table 23. Geometrically, the elements of $[I]$ represent rotational symmetries of two interlocked congruent tetrahedra; see Figure 8. If we also include $[-I]$ then we obtain $\mathcal{S} = [I] \cup [-I]$ which is the only group of even matrices common to the symmetry groups $S(C), S(O), S(D), S(D'), S(I)$ and $S(I')$; see Table 23. The additional elements in $[-I]$ are

- $-I$, which represents inversion in the origin,
- $-J, -K, -L$, which represent reflections in the x, y and z planes, respectively,
- $-P, -P^2$ which represent improper rotations of order 6 around the axis span$\{\vec{d}\}$,
- $-JP, -P^2J$ which represent improper rotations of order 6 around the axis span$\{L\vec{d}\}$,
- $-KP, -P^2K$ which represent improper rotations of order 6 around the axis span$\{J\vec{d}\}$,
- $-LP, -P^2L$ which represent improper rotations of order 6 around the axis span$\{K\vec{d}\}$.

Geometrically, $\mathcal{S}$ is the subgroup of even symmetries of two interlocked congruent tetrahedra. The matrices in $[I]$ act on each of these two tetrahedra by fixing them; the matrices in $[-I]$ by interchanging them. That is, if T is the blue tetrahedron of Figure 8 and T' the green one, then: $A \in [I] \Rightarrow A(T) = T$ and $A(T') = T'$; but $A \in [-I] \Rightarrow A(T) = T'$ and $A(T') = T$.

The remaining case is $\cos\theta = -1$ in which case $\theta = \pi$ and

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

A is symmetric and represents a rotation of order 2, and both PA, P^2A represent rotations of order 6, as in Example 6. In this case, we can take[9]

$$\vec{r} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \vec{s} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{t} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

[9]The vectors $\vec{r}(\theta), \vec{s}(\theta), \vec{t}(\theta)$ and so on are valid axes of rotation for $A \in [D_1(\theta)]$ for *all* θ.

	Set 1 for $Z = A$		Set 2 for $Z = PA$		Set 3 for $Z = P^2A$	
$A = D_1(\pi)$	$Z = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$		$Z = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}$		$Z = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$	
$X \in [A]$	$\mathrm{tr}\,(X)$	$E_1(X)$	$\mathrm{tr}\,(X)$	$E_1(X)$	$\mathrm{tr}\,(X)$	$E_1(X)$
Z	-1	$\mathrm{span}\,\{\vec{d}\}$	2	$\mathrm{span}\,\{\vec{d}\}$	2	$\mathrm{span}\,\{\vec{d}\}$
JZ	$1/3$	$\mathrm{span}\,\{\vec{r}\}$	$-2/3$	$\mathrm{span}\,\{\vec{u}\}$	$-2/3$	$\mathrm{span}\,\{\vec{x}\}$
KZ	$1/3$	$\mathrm{span}\,\{\vec{s}\}$	$-2/3$	$\mathrm{span}\,\{\vec{v}\}$	$-2/3$	$\mathrm{span}\,\{\vec{y}\}$
LZ	$1/3$	$\mathrm{span}\,\{\vec{t}\}$	$-2/3$	$\mathrm{span}\,\{\vec{w}\}$	$-2/3$	$\mathrm{span}\,\{\vec{z}\}$
ZJ	$1/3$	$\mathrm{span}\,\{\vec{r}\}$	$-2/3$	$\mathrm{span}\,\{\vec{x}\}$	$-2/3$	$\mathrm{span}\,\{\vec{u}\}$
JZJ	-1	$\mathrm{span}\,\{J\vec{d}\}$	2	$\mathrm{span}\,\{J\vec{d}\}$	2	$\mathrm{span}\,\{J\vec{d}\}$
KZJ	$1/3$	$\mathrm{span}\,\{K\vec{t}\}$	$-2/3$	$\mathrm{span}\,\{J\vec{w}\}$	$-2/3$	$\mathrm{span}\,\{K\vec{w}\}$
LZJ	$1/3$	$\mathrm{span}\,\{J\vec{s}\}$	$-2/3$	$\mathrm{span}\,\{J\vec{v}\}$	$-2/3$	$\mathrm{span}\,\{L\vec{v}\}$
ZK	$1/3$	$\mathrm{span}\,\{\vec{s}\}$	$-2/3$	$\mathrm{span}\,\{\vec{y}\}$	$-2/3$	$\mathrm{span}\,\{\vec{v}\}$
JZK	$1/3$	$\mathrm{span}\,\{K\vec{t}\}$	$-2/3$	$\mathrm{span}\,\{K\vec{w}\}$	$-2/3$	$\mathrm{span}\,\{J\vec{w}\}$
KZK	-1	$\mathrm{span}\,\{K\vec{d}\}$	2	$\mathrm{span}\,\{K\vec{d}\}$	2	$\mathrm{span}\,\{K\vec{d}\}$
LZK	$1/3$	$\mathrm{span}\,\{L\vec{r}\}$	$-2/3$	$\mathrm{span}\,\{K\vec{u}\}$	$-2/3$	$\mathrm{span}\,\{L\vec{u}\}$
ZL	$1/3$	$\mathrm{span}\,\{\vec{t}\}$	$-2/3$	$\mathrm{span}\,\{\vec{z}\}$	$-2/3$	$\mathrm{span}\,\{\vec{w}\}$
JZL	$1/3$	$\mathrm{span}\,\{J\vec{s}\}$	$-2/3$	$\mathrm{span}\,\{L\vec{v}\}$	$-2/3$	$\mathrm{span}\,\{J\vec{v}\}$
KZL	$1/3$	$\mathrm{span}\,\{L\vec{r}\}$	$-2/3$	$\mathrm{span}\,\{L\vec{u}\}$	$-2/3$	$\mathrm{span}\,\{K\vec{u}\}$
LZL	-1	$\mathrm{span}\,\{L\vec{d}\}$	2	$\mathrm{span}\,\{L\vec{d}\}$	2	$\mathrm{span}\,\{L\vec{d}\}$
$\theta = \pi$	$a = -\frac{1}{3},\ b = \frac{2}{3},\ c = \frac{2}{3}$					

Table 20: The 48 distinct matrices in $[A]$ for the special case $a = -1/3, b = c = 2/3$.

Then $J\vec{r} = -\vec{r}, K\vec{s} = -\vec{s}, L\vec{t} = -\vec{t}$ and $K\vec{r} = -L\vec{r}, L\vec{s} = -J\vec{s}, J\vec{t} = -K\vec{t}$, reflecting the fact that pairs of inverses,

$$AJ, JA; AK, KA; AL, LA; KAJ, JAK; LAJ, JAL; LAK, KAL$$

must represent rotations around the same axis. See the second column of Table 20. For the other matrices in $[A]$, take

$$\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}; \vec{x} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \vec{y} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \vec{z} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

Then $J\vec{u} = \vec{x}, K\vec{v} = \vec{y}, L\vec{w} = \vec{z}$, and

$$J\vec{v} = L\vec{y}, J\vec{w} = K\vec{z}, K\vec{u} = L\vec{x}, K\vec{w} = J\vec{z}, L\vec{u} = K\vec{x}, L\vec{v} = J\vec{y}.$$

Now the pairs of inverses representing rotations around the same axes are split over the third and fourth columns of Table 20, because $A^T = A$ and so

$$(PA)^T = A^T P^T = AP^2 = P^2 A.$$

Connections Among the Angles: in terms of the general case, as displayed in Table 19, consider the 12 matrices in Set 1 with trace $-a$, the 12 matrices in Set 2 with trace $-b$, and the 12 matrices in Set 3 with trace $-c$. Using the fact that the trace of a rotation matrix X can be written as $\text{tr}\,(X) = 1 + 2\cos\theta$, where θ is the angle of rotation, we have, for some angles α, β, γ,

$$1 + 2\cos\alpha = -a,\ 1 + 2\cos\beta = -b,\ 1 + 2\cos\gamma = -c.$$

On the other hand, we also know that

$$A = D_1(\theta) = \frac{1}{3}\begin{bmatrix} 1 + 2\cos\theta & 1 - \cos\theta - \sqrt{3}\sin\theta & 1 - \cos\theta + \sqrt{3}\sin\theta \\ 1 - \cos\theta + \sqrt{3}\sin\theta & 1 + 2\cos\theta & 1 - \cos\theta - \sqrt{3}\sin\theta \\ 1 - \cos\theta - \sqrt{3}\sin\theta & 1 - \cos\theta + \sqrt{3}\sin\theta & 1 + 2\cos\theta \end{bmatrix},$$

so that $3a = 1 + 2\cos\theta,\ 3b = 1 - \cos\theta + \sqrt{3}\sin\theta,\ 3c = 1 - \cos\theta - \sqrt{3}\sin\theta$. This gives us a system of equations we can use to solve for α, β, γ in terms of θ :

$$-3 - 6\cos\alpha = 1 + 2\cos\theta \;\Leftrightarrow\; 6\cos\alpha = -4 - 2\cos\theta \quad (2)$$
$$-3 - 6\cos\beta = 1 - \cos\theta + \sqrt{3}\sin\theta \;\Leftrightarrow\; 6\cos\beta = -4 + \cos\theta - \sqrt{3}\sin\theta \quad (3)$$
$$-3 - 6\cos\gamma = 1 - \cos\theta - \sqrt{3}\sin\theta \;\Leftrightarrow\; 6\cos\gamma = -4 + \cos\theta + \sqrt{3}\sin\theta \quad (4)$$

Using sum and difference formulas for cosine, (7) and (8) can be rewritten as

$$6\cos\beta = -4 + 2\cos(\theta + \pi/3) \text{ and } 6\cos\gamma = -4 + 2\cos(\theta - \pi/3).$$

Thus, in compact form,

$$\cos\alpha = -\frac{1}{3}(2 + \cos\theta),\ \cos\beta = -\frac{1}{3}(2 - \cos(\theta + \pi/3)),\ \cos\gamma = -\frac{1}{3}(2 - \cos(\theta - \pi/3)).$$

More: adding equations (6), (7) and (8) and then simplifying gives $\cos\alpha + \cos\beta + \cos\gamma = -2$; squaring equations (6), (7) and (8) and then simplifying gives $2\cos^2\alpha + 2\cos^2\beta + 2\cos^2\gamma = 3$.

Additionally the content of equations (6), (7) and (8) can be recast in matrix form as

$$\begin{bmatrix} \cos\alpha \\ \\ \cos\beta \\ \\ \cos\gamma \end{bmatrix} = \begin{bmatrix} -\dfrac{2}{\sqrt{6}} & 0 & \dfrac{1}{\sqrt{3}} \\ \\ \dfrac{1}{\sqrt{6}} & -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{3}} \\ \\ \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \dfrac{\cos\theta}{\sqrt{6}} \\ \\ \dfrac{\sin\theta}{\sqrt{6}} \\ \\ -\dfrac{2}{\sqrt{3}} \end{bmatrix}.$$

Let R be the coefficient matrix of this system. Check that R is orthogonal with determinant $+1$ and so is itself a rotation matrix. For interest, its eigenvalues are $\lambda_1 = 1$ and

$$\lambda_2, \lambda_3 = \frac{1}{12}\left(-6 - 3\sqrt{2} + 2\sqrt{3} - 2\sqrt{6} \pm i\sqrt{6(9 - 2\sqrt{2} - 2\sqrt{6})}\right).$$

Indeed R represents a rotation of

$$\theta = \arccos\left(\frac{-6 - 3\sqrt{2} + 2\sqrt{3} - 2\sqrt{6}}{12}\right)$$

around the line through the origin parallel to the vector

$$\vec{v} = \begin{bmatrix} (1+\sqrt{2})(\sqrt{6}-2) \\ \sqrt{2} \\ 2+\sqrt{2} \end{bmatrix}.$$

On the other hand, the points with coordinates

$$\begin{bmatrix} \cos\alpha \\ \\ \cos\beta \\ \\ \cos\gamma \end{bmatrix} \text{ and } \begin{bmatrix} \dfrac{\cos\theta}{\sqrt{6}} \\ \\ \dfrac{\sin\theta}{\sqrt{6}} \\ \\ -\dfrac{2}{\sqrt{3}} \end{bmatrix}$$

are both on the sphere with equation

$$x^2 + y^2 + z^2 = \frac{3}{2}$$

since R preserves lengths . . . plot two circles on the sphere . . . with two planes

$$z = -\frac{2}{\sqrt{3}} \text{ and } x + y + z = -2.$$

Three Sets for $[B]$**:** now consider $B = -P_{(12)}A \in \mathbb{D}_2$. In this case B represents a rotation of order 2 about an axis orthogonal to $\vec{d}$. As before, assume a, b, c are all distinct and let $A = aI + cP + bP^2$ with $a^2 + b^2 + c^2 = 1$, $a + b + c = 1$. As in Table 19, all the matrices in $[B]$, along with their traces and axes of rotation are listed in Table 21, where

$$\vec{q}_1 = \begin{bmatrix} c \\ b \\ a-1 \end{bmatrix} \text{ or } \begin{bmatrix} ac \\ ab \\ bc \end{bmatrix}, \vec{r}_1 = \begin{bmatrix} 0 \\ c \\ -a \end{bmatrix}, \vec{s}_1 = \begin{bmatrix} -b \\ 0 \\ a \end{bmatrix}, \vec{t}_1 = \begin{bmatrix} b \\ -c \\ 0 \end{bmatrix}.$$

(Note the alternate entries for $\vec{q}_1$, valid for $abc \neq 0$, since $a^2 - a = bc$.) Check that

$$-P_{(12)}A\,\vec{q}_1 = \vec{q}_1, -P_{(12)}A\,\vec{r}_1 = J\vec{r}_1, -P_{(12)}A\,\vec{s}_1 = K\vec{s}_1, -P_{(12)}A\,\vec{t}_1 = L\vec{t}_1.$$

Again, it is an easy exercise to calculate the traces and axes of rotation of the 16 matrices in Set 1. Note that in the second column of Table 21, some alternatives could have been used because

$$J\vec{r}_1 = -\vec{r}_1, K\vec{s}_1 = -\vec{s}_1, L\vec{t}_1 = -\vec{t}_1; K\vec{r}_1 = -L\vec{r}_1; J\vec{s}_1 = -L\vec{s}_1; K\vec{t}_1 = -J\vec{t}_1,$$

as you may check. Looking at the second column of Table 21, you can see that the 16 matrices are divided into four quartets each with a common trace. One quartet, the four matrices with trace -1, consists of conjugates of B, which represent rotations of order 2 around axes spanned by $\vec{q}_1, J\vec{q}_1, K\vec{q}_1$ and $L\vec{q}_1$. Each of the three remaining quartets consists of rotations around four axes spanned by

$$\vec{r}_1, J\vec{r}_1, K\vec{r}_1, L\vec{r}_1; \vec{s}_1, J\vec{s}_1, K\vec{s}_1, L\vec{s}_1; \text{ or } \vec{t}_1, J\vec{t}_1, K\vec{t}_1, L\vec{t}_1,$$

but because of the above relationships, there are only two distinct axes in each case. For instance, from the second column of Table 21, the matrices KB, BK and LBJ, JBL, all with trace $1-2c$, are pairs of inverses, since $(KB)^T = B^T K^T = BK$ and $(LBJ)^T = J^T B^T L^T = JBL$, with axes of rotations spanned by $\vec{s}_1$ or $L\vec{s}_1$ since

$$E_1(KB) = \text{span}\{\vec{s}_1\} = \text{span}\{K\vec{s}_1\} = E_1(BK);$$

$$E_1(LBJ) = \text{span}\{J\vec{s}_1\} = \text{span}\{L\vec{s}_1\} = E_1(JBL).$$

The matrices for Sets 2 and 3 in $[B]$ are listed in the other two columns of Table 21. The entries are given in terms of the vectors

$$\vec{q}_2 = P^2\vec{q}_1,\ \vec{r}_2 = P^2\vec{r}_1,\ \vec{s}_2 = P^2\vec{s}_1,\ \vec{t}_2 = P^2\vec{t}_1;$$

$$\vec{q}_3 = P\vec{q}_1,\ \vec{r}_3 = P\vec{r}_1,\ \vec{s}_3 = P\vec{s}_1,\ \vec{t}_3 = P\vec{t}_1.$$

As with the elements in Set 1, the entries in Sets 2 and 3 also split up into four quartets with a common trace, and the quartets with traces not equal to -1 comprise two pairs of inverses.

As we did with $[A]$, introduce angles α, β, γ such that

- for the 12 rotation matrices in Sets 1, 2 and 3 with trace $1 - 2a$ their trace is $1 + 2\cos\alpha$,
- for the 12 rotation matrices in Sets 1, 2 and 3 with trace $1 - 2b$ their trace is $1 + 2\cos\beta$,
- for the 12 rotation matrices in Sets 1, 2 and 3 with trace $1 - 2c$ their trace is $1 + 2\cos\gamma$.

$B = -P_{(12)}\,A$	Set 1 for $Z = B$		Set 2 for $Z = P\,B$		Set 3 for $Z = P^2\,B$	
$A = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$	$Z = -\begin{bmatrix} b & a & c \\ a & c & b \\ c & b & a \end{bmatrix}$		$Z = -\begin{bmatrix} a & c & b \\ c & b & a \\ b & a & c \end{bmatrix}$		$Z = -\begin{bmatrix} c & b & a \\ b & a & c \\ a & c & b \end{bmatrix}$	
$X \in [B]$	$\mathrm{tr}\,(X)$	$E_1(X)$	$\mathrm{tr}\,(X)$	$E_1(X)$	$\mathrm{tr}\,(X)$	$E_1(X)$
Z	-1	$\mathrm{span}\,\{\vec{q}_1\}$	-1	$\mathrm{span}\,\{\vec{q}_2\}$	-1	$\mathrm{span}\,\{\vec{q}_3\}$
JZ	$1-2b$	$\mathrm{span}\,\{\vec{r}_1\}$	$1-2a$	$\mathrm{span}\,\{\vec{t}_2\}$	$1-2c$	$\mathrm{span}\,\{\vec{s}_3\}$
KZ	$1-2c$	$\mathrm{span}\,\{\vec{s}_1\}$	$1-2b$	$\mathrm{span}\,\{\vec{r}_2\}$	$1-2a$	$\mathrm{span}\,\{\vec{t}_3\}$
LZ	$1-2a$	$\mathrm{span}\,\{\vec{t}_1\}$	$1-2c$	$\mathrm{span}\,\{\vec{s}_2\}$	$1-2b$	$\mathrm{span}\,\{\vec{r}_3\}$
ZJ	$1-2b$	$\mathrm{span}\,\{J\vec{r}_1\}$	$1-2a$	$\mathrm{span}\,\{J\vec{t}_2\}$	$1-2c$	$\mathrm{span}\,\{J\vec{s}_3\}$
JZJ	-1	$\mathrm{span}\,\{J\vec{q}_1\}$	-1	$\mathrm{span}\,\{J\vec{q}_2\}$	-1	$\mathrm{span}\,\{J\vec{q}_3\}$
KZJ	$1-2a$	$\mathrm{span}\,\{J\vec{t}_1\}$	$1-2c$	$\mathrm{span}\,\{J\vec{s}_2\}$	$1-2b$	$\mathrm{span}\,\{J\vec{r}_3\}$
LZJ	$1-2c$	$\mathrm{span}\,\{J\vec{s}_1\}$	$1-2b$	$\mathrm{span}\,\{J\vec{r}_2\}$	$1-2a$	$\mathrm{span}\,\{J\vec{t}_3\}$
ZK	$1-2c$	$\mathrm{span}\,\{K\vec{s}_1\}$	$1-2b$	$\mathrm{span}\,\{K\vec{r}_2\}$	$1-2a$	$\mathrm{span}\,\{K\vec{t}_3\}$
JZK	$1-2a$	$\mathrm{span}\,\{K\vec{t}_1\}$	$1-2c$	$\mathrm{span}\,\{K\vec{s}_2\}$	$1-2b$	$\mathrm{span}\,\{K\vec{r}_3\}$
KZK	-1	$\mathrm{span}\,\{K\vec{q}_1\}$	-1	$\mathrm{span}\,\{K\vec{q}_2\}$	-1	$\mathrm{span}\,\{K\vec{q}_3\}$
LZK	$1-2b$	$\mathrm{span}\,\{K\vec{r}_1\}$	$1-2a$	$\mathrm{span}\,\{K\vec{t}_2\}$	$1-2c$	$\mathrm{span}\,\{K\vec{s}_3\}$
ZL	$1-2a$	$\mathrm{span}\,\{L\vec{t}_1\}$	$1-2c$	$\mathrm{span}\,\{L\vec{s}_2\}$	$1-2b$	$\mathrm{span}\,\{L\vec{r}_3\}$
JZL	$1-2c$	$\mathrm{span}\,\{L\vec{s}_1\}$	$1-2b$	$\mathrm{span}\,\{L\vec{r}_2\}$	$1-2a$	$\mathrm{span}\,\{L\vec{t}_3\}$
KZL	$1-2b$	$\mathrm{span}\,\{L\vec{r}_1\}$	$1-2a$	$\mathrm{span}\,\{L\vec{t}_2\}$	$1-2c$	$\mathrm{span}\,\{L\vec{s}_3\}$
LZL	-1	$\mathrm{span}\,\{L\vec{q}_1\}$	-1	$\mathrm{span}\,\{L\vec{q}_2\}$	-1	$\mathrm{span}\,\{L\vec{q}_3\}$
$\cos^2\theta \neq 1$ or $\frac{1}{4}$	$a = \frac{1+2\cos\theta}{3},\ b = \frac{1+2\cos(\theta-2\pi/3)}{3},\ c = \frac{1+2\cos(\theta+2\pi/3)}{3}$					

Table 21: The 48 distinct matrices in $[-P_{(12)}A]$ for $A = D_1(\theta)$, with $a^2+b^2+c^2 = 1$, $a+b+c = 1$, a, b, c all distinct; a, b, c also given in terms of the parameter θ, for $\cos^2\theta \neq 1$ and $\cos^2\theta \neq 1/4$.

Then

$$1+2\cos\alpha = 1-2a,\ 1+2\cos\beta = 1-2b,\ 1+2\cos\gamma = 1-2c,$$

implying

$$\cos\alpha = -a,\ \cos\beta = -b,\ \cos\gamma = -c.$$

So for the equivalence class $[B]$, in general, we have

$$\cos\alpha+\cos\beta+\cos\gamma = -1 \text{ and } \cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1.$$

More Connections: it follows in this case that

$$B = -P_{(12)}, A = \begin{bmatrix} \cos\beta & \cos\alpha & \cos\gamma \\ \cos\alpha & \cos\gamma & \cos\beta \\ \cos\gamma & \cos\beta & \cos\alpha \end{bmatrix}.$$

So B affords a direct way to find the angles of rotation, α, β, γ, for each matrix in Table 21 with trace $1-2a, 1-2b, 1-2c$, respectively. Note also that B is itself a symmetric deranged matrix. Since $\operatorname{tr}(B) = -1$, B represents a rotation of order 2 about

$$q_1 = \begin{bmatrix} ac \\ ab \\ bc \end{bmatrix} = \begin{bmatrix} \cos\alpha\,\cos\gamma \\ \cos\beta\,\cos\alpha \\ \cos\beta\,\cos\gamma \end{bmatrix}$$

with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, repeated. As for JB, KB and LB :

- JB is a rotation of β about $\vec{r}_1 = \begin{bmatrix} 0 \\ c \\ -a \end{bmatrix}$ with eigenvalues 1, $e^{\pm i\,\beta}$.
- KB is a rotation of γ about $\vec{s}_1 = \begin{bmatrix} -b \\ 0 \\ a \end{bmatrix}$ with eigenvalues 1, $e^{\pm i\,\gamma}$.
- LB is a rotation of α about $\vec{t}_1 = \begin{bmatrix} b \\ -c \\ 0 \end{bmatrix}$ with eigenvalues 1, $e^{\pm i\,\alpha}$.

On the other hand, since $B = -P_{(12)}A$, with $A = D_1(\theta)$, we know that

$$3a = 1+2\cos\theta,\ 3b = 1-\cos\theta+\sqrt{3}\sin\theta,\ 3c = 1-\cos\theta-\sqrt{3}\sin\theta.$$

Making use of sum and difference formulas for cosine, we have

$$\cos\alpha = -\frac{1}{3}(1+2\cos\theta),\ \cos\beta = -\frac{1}{3}\left(1-2\cos(\theta+\pi/3)\right),\ \cos\gamma = -\frac{1}{3}\left(1-2\cos(\theta-\pi/3)\right).$$

The two remaining cases occur if $\sin\theta = 0$. If $\theta = 0$, then $a = 1, b = c = 0$, $A = I$ and $[-P_{(12)}A] = [-P_{(12)}]$. Just as with $[I]$, there are only 12 matrices in $[-P_{(12)}]$. In this case we can take $\vec{r}_1 = \vec{s}_1 = \vec{k}, \vec{r}_2 = \vec{s}_2 = \vec{i}, \vec{r}_3 = \vec{s}_3 = \vec{j}$, which are all parallel to coordinate axes, and

$$\vec{q}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ \vec{q}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},\ \vec{q}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix};\ \vec{t}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \vec{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ \vec{t}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

which are all parallel to lines joining the midpoints of pairs of opposite edges of the cube. Then the 12 matrices in $[-P_{(12)}]$ can be read off the first four rows of Table 21. So $[-P_{(12)}]$ includes

- six rotations of order 2, $-P_{(12)}, -LP_{(12)}, -P_{(23)}, -JP_{(23)}, -P_{(13)}, -KP_{(13)}$, one around each of the six axes, $\vec{q}_1, \vec{t}_1, \vec{q}_2, \vec{t}_2, \vec{q}_3, \vec{t}_3$, respectively; and
- six rotations of order 4, $-KP_{(23)}, -LP_{(23)}; -JP_{(13)}, -LP_{(13)}; -JP_{(12)}, -KP_{(12)}$, one pair around each of the coordinate axes, $\vec{\imath}, \vec{\jmath}, \vec{k}$ respectively.

Since

$$L\vec{q}_1 = -\vec{q}_1, J\vec{q}_1 = \vec{t}_1 = -K\vec{q}_1;\ J\vec{q}_2 = -\vec{q}_2, K\vec{q}_2 = \vec{t}_2 = -L\vec{q}_2;\ K\vec{q}_3 = -\vec{q}_3, L\vec{q}_3 = \vec{t}_3 = -J\vec{q}_3$$

and

$$J\vec{r}_1 = K\vec{r}_1 = -\vec{r}_1, L\vec{r}_1 = \vec{r}_1; J\vec{r}_2 = \vec{r}_2, K\vec{r}_2 = L\vec{r}_2 = -\vec{r}_2; J\vec{r}_3 = L\vec{r}_3 = -\vec{r}_3, K\vec{r}_3 = \vec{r}_3,$$

the remaining matrices in $[-P_{(12)}]$ corresponding to the remaining 12 rows of Table 21 simply duplicate the matrices in the first four rows of Table 21, thus implying

- $JP_{(12)} = P_{(12)}K, KP_{(12)} = P_{(12)}J, LP_{(12)} = P_{(12)}L,$
- $JP_{(23)} = P_{(23)}J, KP_{(23)} = P_{(23)}L, LP_{(23)} = P_{(23)}K,$
- $JP_{(13)} = P_{(13)}L, KP_{(13)} = P_{(13)}K, LP_{(13)} = P_{(13)}J.$

In particular the 24 matrices in $[I] \cup [-P_{(12)}]$ represent all 24 rotations fixing the cube.

The remaining case is $\theta = \pi$, with $a = -1/3, b = c = 2/3$. In this case we can take

$$\vec{q}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \vec{r}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \vec{s}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \vec{t}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

$[B]$ still has 48 matrices; see Table 22. The main difference between this case and the general case is that in this case there are only three possible values for $\operatorname{tr}(X),\ X \in [B]$, whereas in general there are four possible values for $\operatorname{tr}(X)$; see Table 21. Comparing Tables 20 and 22, you can see that for every matrix $X \in [B]$ its inverse is in the same column, since $B^T = B$; whereas in the third and fourth columns of Table 20, inverses of the matrices with trace $-2/3$ are in different columns.

Elements of Infinite Order: we claim that in both Tables 20 and 22 there are only 12 matrices with finite order, namely the conjugates of the three matrices at the top of each table. For the other matrices X in Tables 20 and 22 the options are

$$\operatorname{tr}(X) = \frac{1}{3}, -\frac{2}{3}, -\frac{1}{3} \text{ or } \frac{5}{3}.$$

Let X represent a rotation of α about its axis of rotation, so that $\operatorname{tr}(X) = 1 + 2\cos\alpha$. Then

$$\cos\alpha = -\frac{1}{3}, -\frac{5}{6}, -\frac{2}{3} \text{ or } \frac{1}{3},$$

respectively. Using the fact that $\arccos(-x) = \pi - \arccos(x)$, so $\arccos(-x)$ is a rational multiple of π if and only if $\arccos(x)$ is, we only need to show

$$\arccos\left(\frac{1}{3}\right), \arccos\left(\frac{2}{3}\right) \text{ and } \arccos\left(\frac{5}{6}\right)$$

are not rational multiples of π. Instead of now proceeding as in Example 12 we can use the following two results[10]

1. $\arccos(\sqrt{n}/3) \notin \pi\mathbb{Q}$ if $\gcd(n, 3) \le 3$, using $n = 1$ or 4, and
2. $\arccos(\sqrt{m/n}) \notin \pi\mathbb{Q}$ for $m, n \in \mathbb{N}^*$ if $n > 1, \gcd(m, n) = 1$ and n is not a power of 2, using $m = 25, n = 36$.

[10] Beni Bogosel's blog

$B = -P_{(12)}\,A$	Set 1 for $Z = B$		Set 2 for $Z = P\,B$		Set 3 for $Z = P^2\,B$	
$A = D_1(\pi)$	$-\dfrac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$		$-\dfrac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$		$-\dfrac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$	
$X \in [B]$	$\mathrm{tr}\,(X)$	$E_1(X)$	$\mathrm{tr}\,(X)$	$E_1(X)$	$\mathrm{tr}\,(X)$	$E_1(X)$
Z	-1	$\mathrm{span}\,\{\vec{q}_1\}$	-1	$\mathrm{span}\,\{\vec{q}_2\}$	-1	$\mathrm{span}\,\{\vec{q}_3\}$
JZ	$-1/3$	$\mathrm{span}\,\{\vec{r}_1\}$	$5/3$	$\mathrm{span}\,\{\vec{t}_2\}$	$-1/3$	$\mathrm{span}\,\{\vec{s}_3\}$
KZ	$-1/3$	$\mathrm{span}\,\{\vec{s}_1\}$	$-1/3$	$\mathrm{span}\,\{\vec{r}_2\}$	$5/3$	$\mathrm{span}\,\{\vec{t}_3\}$
LZ	$5/3$	$\mathrm{span}\,\{\vec{t}_1\}$	$-1/3$	$\mathrm{span}\,\{\vec{s}_2\}$	$-1/3$	$\mathrm{span}\,\{\vec{r}_3\}$
ZJ	$-1/3$	$\mathrm{span}\,\{J\vec{r}_1\}$	$5/3$	$\mathrm{span}\,\{J\vec{t}_2\}$	$-1/3$	$\mathrm{span}\,\{J\vec{s}_3\}$
JZJ	-1	$\mathrm{span}\,\{J\vec{q}_1\}$	-1	$\mathrm{span}\,\{J\vec{q}_2\}$	-1	$\mathrm{span}\,\{J\vec{q}_3\}$
KZJ	$5/3$	$\mathrm{span}\,\{J\vec{t}_1\}$	$-1/3$	$\mathrm{span}\,\{J\vec{s}_2\}$	$-1/3$	$\mathrm{span}\,\{J\vec{r}_3\}$
LZJ	$-1/3$	$\mathrm{span}\,\{J\vec{s}_1\}$	$-1/3$	$\mathrm{span}\,\{J\vec{r}_2\}$	$5/3$	$\mathrm{span}\,\{J\vec{t}_3\}$
ZK	$-1/3$	$\mathrm{span}\,\{K\vec{s}_1\}$	$-1/3$	$\mathrm{span}\,\{K\vec{r}_2\}$	$5/3$	$\mathrm{span}\,\{K\vec{t}_3\}$
JZK	$5/3$	$\mathrm{span}\,\{K\vec{t}_1\}$	$-1/3$	$\mathrm{span}\,\{K\vec{s}_2\}$	$-1/3$	$\mathrm{span}\,\{K\vec{r}_3\}$
KZK	-1	$\mathrm{span}\,\{K\vec{q}_1\}$	-1	$\mathrm{span}\,\{K\vec{q}_2\}$	-1	$\mathrm{span}\,\{K\vec{q}_3\}$
LZK	$-1/3$	$\mathrm{span}\,\{K\vec{r}_1\}$	$5/3$	$\mathrm{span}\,\{K\vec{t}_2\}$	$-1/3$	$\mathrm{span}\,\{K\vec{s}_3\}$
ZL	$5/3$	$\mathrm{span}\,\{L\vec{t}_1\}$	$-1/3$	$\mathrm{span}\,\{L\vec{s}_2\}$	$-1/3$	$\mathrm{span}\,\{L\vec{r}_3\}$
JZL	$-1/3$	$\mathrm{span}\,\{L\vec{s}_1\}$	$-1/3$	$\mathrm{span}\,\{L\vec{r}_2\}$	$5/3$	$\mathrm{span}\,\{L\vec{t}_3\}$
KZL	$-1/3$	$\mathrm{span}\,\{L\vec{r}_1\}$	$5/3$	$\mathrm{span}\,\{L\vec{t}_2\}$	$-1/3$	$\mathrm{span}\,\{L\vec{s}_3\}$
LZL	-1	$\mathrm{span}\,\{L\vec{q}_1\}$	-1	$\mathrm{span}\,\{L\vec{q}_2\}$	-1	$\mathrm{span}\,\{L\vec{q}_3\}$
$\theta = \pi$	$a = -\dfrac{1}{3},\ b = \dfrac{2}{3},\ c = \dfrac{2}{3}$					

Table 22: The 48 distinct matrices in $[-P_{(12)}A]$ for the special case $a = -1/3, b = c = 2/3$.

$[A] \subset \mathbb{P}$	det	order of $[A]$	# matrices in $[A]$ with order 1	2	3	4	5	6	10	symmetries of T, T'	C, O	D, I	D', I'	sgn
$[I]$	1	12	1	3	8	0	0	0	0	Yes	Yes	Yes	Yes	1
$[-I]$	-1	12	0	4	0	0	0	8	0	No	Yes	Yes	Yes	1
$[-P_{(12)}]$	1	12	0	6	0	6	0	0	0	No	Yes	No	No	-1
$[P_{(12)}]$	-1	12	0	6	0	6	0	0	0	Yes	Yes	No	No	-1
$[-S]$	1	48	0	12	12	0	24	0	0	No	No	Yes	No	1
$[S]$	-1	48	0	12	0	0	0	12	24	No	No	Yes	No	1
$[-S']$	1	48	0	12	12	0	24	0	0	No	No	No	Yes	1
$[S']$	-1	48	0	12	0	0	0	12	24	No	No	No	Yes	1
# of rotations		120	1	33	32	6	48	0	0	12	24	60	60	
non-rotations		120	0	34	0	6	0	32	48	12	24	60	60	

Table 23: The eight equivalence classes in $\mathbb{P}$; see also Table 15. $S(T) = S(T') = [I] \cup [P_{(12)}]$; $S(C) = S(O) = [I] \cup [P_{(12)}] \cup [-I] \cup [-P_{(12)}]$; $S(D) = S(I) = [I] \cup [S] \cup [-I] \cup [-S]$; and $S(D') = S(I') = [I] \cup [S'] \cup [-I] \cup [-S']$. Of the 240 matrices, 216 are even and only 24 are odd.

Table 23 summarizes the 240 matrices which represent symmetries of the Platonic solids.

Conjecture 1 *Let $M \in \mathbb{SD}$. Then $M \in \mathbb{P}$ if and only if $[M]$ consists entirely of matrices with finite order. That is, all the matrices in the equivalence class $[M]$ have finite order if and only if*

$$[M] = [I], [P_{(12)}], [S], [S'], [-I], [-P_{(12)}], [-S] \text{ or } [-S'],$$

where

$$P_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S = \frac{1}{2}\begin{pmatrix} \phi & 1 & \psi \\ 1 & \psi & \phi \\ \psi & \phi & 1 \end{pmatrix}, \quad S' = \frac{1}{2}\begin{pmatrix} \psi & 1 & \phi \\ 1 & \phi & \psi \\ \phi & \psi & 1 \end{pmatrix}.$$

Example 17: consider the following three matrices in $\mathbb{D}_1$,

$$A_1 = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}, \quad A_2 = \frac{1}{3}\begin{bmatrix} 1 & 1-\sqrt{3} & 1+\sqrt{3} \\ 1+\sqrt{3} & 1 & 1-\sqrt{3} \\ 1-\sqrt{3} & 1+\sqrt{3} & 1 \end{bmatrix},$$

$$A_3 = \frac{1}{6}\begin{bmatrix} 2\phi & 3-\sqrt{6+3\phi}-\phi & 3+\sqrt{6+3\phi}-\phi \\ 3+\sqrt{6+3\phi}-\phi & 2\phi & 3-\sqrt{6+3\phi}-\phi \\ 3-\sqrt{6+3\phi}-\phi & 3+\sqrt{6+3\phi}-\phi & 2\phi \end{bmatrix},$$

which have orders 6, 4 and 5, respectively. None of these matrices is in $\mathbb{P}$. So in each case we claim $[A_i]$ contains at least one matrix that does not have finite order. For instance,

$$\operatorname{tr}(JA_1) = -\frac{2}{3}, \ \operatorname{tr}(A_2K) = -\frac{1}{3} \text{ and } \operatorname{tr}(LA_3) = -\frac{\phi}{3};$$

whence the rotation angles $\theta_1, \theta_2, \theta_3$ of JA_1, A_2K, LA_3, respectively, satisfy

$$\cos\theta_1 = -\frac{5}{6},\ \cos\theta_2 = -\frac{2}{3},\ \cos\theta_3 = -\frac{3+\phi}{6}$$

$$\Leftrightarrow \cos(\pi+\theta_1) = \frac{5}{6},\ \cos(\pi+\theta_2) = \frac{2}{3},\ \cos(\pi+\theta_3) = \frac{3+\phi}{6}.$$

As in Example 12 it immediately follows that angles θ_1 and θ_2 are not rational multiple of π. As for θ_3 : let $r = \dfrac{3+\phi}{6}$. Then

$$\begin{aligned} & 6r = 3 + \frac{1+\sqrt{5}}{2} \\ \Rightarrow\ & 12r = 6 + 1 + \sqrt{5} \\ \Rightarrow\ & 12r - 7 = \sqrt{5} \\ \Rightarrow\ & 144r^2 - 168r + 49 = 5 \\ \Rightarrow\ & 144r^2 - 168r + 44 = 0 \\ \Rightarrow\ & 36r^2 - 42r + 11 = 0. \end{aligned}$$

Since the leading coefficient of this last irreducible polynomial is not a power of 2, θ_3 is not a rational multiple of π.

Or consider the orthogonal deranged matrix

$$A_4 = \frac{1}{7}\begin{bmatrix} -3 & 2 & -6 \\ 2 & -6 & -3 \\ -6 & -3 & 2 \end{bmatrix},$$

which represents a rotation of order 2, but is not in $\mathbb{D}_1$. A_4 is also not in $\mathbb{P}$, and

$$\operatorname{tr}(JA_4) = \frac{1}{7} \Leftrightarrow 1 + 2\cos\theta_4 = \frac{1}{7} \Leftrightarrow \cos\theta_4 = -\frac{3}{7} \Leftrightarrow \cos(\pi+\theta_4) = \frac{3}{7},$$

whence JA_4 does not have finite order. So not all the matrices in $[A_4]$ have finite order. □

Attempting to Prove the Conjecture: we can assume that the order of M is finite; otherwise there is nothing to prove since $M \in [M]$ and every element in $\mathbb{P}$ has finite order. Secondly, as pointed out in the simplifications above, we can assume $[M] = [A]$ for some matrix $A \in \mathbb{D}$. Let

$$A = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} \in \mathbb{D}_1,$$

with $a^2 + b^2 + c^2 = 1$ and $a + b + c = 1$. Then by Theorem 10, $-P_{(12)}A \in \mathbb{D}_2$, $-A \in \mathbb{D}_3$, and $P_{(12)}A \in \mathbb{D}_4$. If the values of a, b and c are not all distinct, then there are two possibilities:

$$\{a, b, c\} = \{0, 0, 1\} \text{ or } \{a, b, c\} = \left\{-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\}.$$

In the first case, as shown in the Ninth Pass of Part 3 and as described in Three Sets for $[A]$, $[I], [-I], [P_{(12)}]$ and $[-P_{(12)}]$ are all subsets of $S(C)$ and

$$S(C) = [I] \cup [-I] \cup [P_{(12)}] \cup [-P_{(12)}].$$

In the second case, $[A]$ consists of the matrices in Table 20 and $[-P_{(12)}A]$ consists of the matrices in Table 22, both of which contain elements of infinite order, as shown above. Thus we can assume a, b, c are all distinct.

First part of proof: now consider $[A]$ with $A = D_1(\theta) \in \mathbb{D}_1$ such that a, b, c are all distinct and A has finite order with

$$\mathrm{tr}\,(A) = 3a = 1 + 2\cos\theta.$$

Let X be a matrix in Set 1 of $[A]$ such that $\mathrm{tr}\,(X) = -a$ and $\mathrm{tr}\,(X) = 1 + 2\cos\alpha$. Then

$$\cos\alpha = -\frac{2+\cos\theta}{3} \Leftrightarrow \cos\theta = -2 - 3\cos\alpha.$$

Suppose the order of A is m. Then $\cos\theta$ is a root of the irreducible polynomial $\Psi_m(x)$. Substituting for $\cos\theta$ we get that $\Psi_m(-2-3\cos\alpha) = 0$. For example, if $m = 5$ we have

$$\begin{aligned}\Psi_5(-2-3\cos\alpha) = 0 \quad &\Rightarrow \quad 4(-2-3\cos\alpha)^2 + 2(-2-3\cos\alpha) - 1 = 0\\ &\Rightarrow \quad 4(4 + 12\cos\alpha + 9\cos^2\alpha) - 4 - 6\cos\alpha - 1 = 0\\ &\Rightarrow \quad 36\cos^2\alpha + 42\cos\alpha + 11 = 0,\end{aligned}$$

which means $\cos\alpha$ is a root of the irreducible polynomial $36x^2 + 42x + 11$. Since the leading coefficient of this polynomial is divisible by 3, α cannot be a rational multiple of π and the matrix X does not have finite order. As a second example, consider $m = 9$:

$$\begin{aligned}\Psi_9(-2-3\cos\alpha) = 0 \quad &\Rightarrow \quad 8(-2-3\cos\alpha)^3 - 6(-2-3\cos\alpha) + 1 = 0\\ &\Rightarrow \quad -3(72\cos^3\alpha + 144\cos^2\alpha + 90\cos\alpha + 17) = 0,\end{aligned}$$

which means $\cos\alpha$ is a root of the irreducible polynomial $72x^3 + 144x^2 + 90x + 17$. Again, the leading coefficient is divisible by 3, so α cannot be a rational multiple of π and the matrix X does not have finite order. As in Part 1, we can find the irreducible factors $\Psi_m(-2-3x)$ by factoring $T_n(-2-3x) - 1$. For instance, using *Maple* or *Mathematica*,

$$\begin{aligned}T_9(-2-3x) - 1 &= -243(x+1)(2x+1)^2(72x^3 + 144x^2 + 90x + 17)^2\\ &= (-3(x+1))\,(-3(2x+1))^2\,(-3(72x^3 + 144x^2 + 90x + 17))^2\\ &= (\Psi_1(-2-3x))^2\,(\Psi_3(-2-3x))^2\,(\Psi_9(-2-3x))^2\end{aligned}$$

Table 24 lists the (simplified) equations $\Psi_m(-2-3x) = 0$ for $1 \le m \le 30$, that $x = \cos\alpha$ must satisfy if the order of the matrix A with $\mathrm{tr}\,(A) = -3 - 6x = 1 + 2\cos\theta$ is m, and X is a rotation of α. Observe that the leading coefficient of the polynomial on the left side of each equation of degree $q > 1$ is 6^q, unless m, the order of the corresponding matrix A, is a power of 2 or 3. In the first case, the leading coefficient is $2^{q-1}3^q$; in the second case, the leading coefficient is $2^q 3^{q-1}$. But in the case $q = 1$, the leading coefficient is 1. In other words, the only polynomial equations with leading coefficients not divisible by 3 have $m = 1$ or 3.

- if $m = 1$ then $x + 1 = 0 \Rightarrow x = -1$, so $\mathrm{tr}\,(A) = -3 + 6 = 3$, and $A = I$, in which case a, b, c are not all distinct. And as pointed out above, $[I] \subset \mathbb{P}$.
- if $m = 3$ then $2x + 1 = 0 \Rightarrow x = -1/2$, so $\mathrm{tr}\,(A) = -3 + 3 = 0$, and $\cos\theta = -1/2$. Then $A = P_{(123)}$ or $P_{(132)}$, and a, b, c are not all distinct. But $[P_{(123)}] = [P_{(132)}] = [I] \subset \mathbb{P}$.

Otherwise the irreducible polynomial on the left side of the equation $\Psi_m(-2-3x) = 0$ has leading coefficient divisible by 3. This means $\alpha = \arccos x$ can't be a rational multiple of π, and so X can't have finite order. Finally, by symmetry, the same arguments apply if

- $\mathrm{tr}\,(PA) = 3b$, PA has finite order, and X is in Set 2 of $[A]$ with $\mathrm{tr}\,(X) = -b$, or
- $\mathrm{tr}\,(P^2A) = 3c$, P^2A has finite order, and X is in Set 3 of $[A]$ with $\mathrm{tr}\,(X) = -c$.

Hence for all $A \in \mathbb{D}_1 \backslash \{I, P_{(123)}, P_{(132)}\}$, $A \notin \mathbb{P}$ and $[A]$ contains matrices with infinite order. $\square$

order of A m	Simplified equation $\Psi_m(-2-3x)=0$ if $\cos\theta=-2-3x$ and $\Psi_m(\cos\theta)=0$
1	$x+1=0$
2	$3x+1=0$
3	$2x+1=0$
4	$3x+2=0$
5	$36x^2+42x+11=0$
6	$6x+5=0$
7	$216x^3+396x^2+228x+41=0$
8	$18x^2+24x+7=0$
9	$72x^3+144x^2+90x+17=0$
10	$36x^2+54x+19=0$
11	$7776x^5+24624x^4+30240x^3+17964x^2+5154x+571=0$
12	$36x^2+48x+13=0$
13	$46656x^6+178848x^5+278640x^4+225504x^3+99864x^2+22926x+2131=0$
14	$216x^3+468x^2+324x+71=0$
15	$1296x^4+3672x^3+3744x^2+1608x+241=0$
16	$648x^4+1728x^3+1656x^2+672x+97=0$
17	$1679616x^8+8678016x^7+19268928x^6+23996736x^5$ $+18318960x^4+8772624x^3+2572344x^2+422136x+29681=0$
18	$216x^3+432x^2+270x+53=0$
19	$10077696x^9+58786560x^8+150045696x^7+219796416x^6+203521248x^5$ $+123468624x^4+49053600x^3+12303144x^2+1767294x+110771=0$
20	$1296x^4+3456x^3+3276x^2+1296x+181=0$
21	$46656x^6+194400x^5+329184x^4+289008x^3+138240x^2+34032x+3361=0$
22	$7776x^5+27216x^4+37152x^3+24660x^2+7938x+989=0$
23	$362797056x^{11}+2600045568x^{10}+8364487680x^9+15937876224x^8$ $+19977352704x^7+17289780480x^6+10539341568x^5+4523603760x^4$ $+1339452936x^3+260542836x^2+29959788x+1542841=0$
24	$1296x^4+3456x^3+3312x^2+1344x+193=0$
25	$60466176x^{10}+403107840x^9+1192527360x^8+2060328960x^7$ $+2300840640x^6+1734413472x^5+893319840x^4+310280760x^3$ $+69526260x^2+9072690x+523451=0$
26	$46656x^6+194400x^5+330480x^4+292896x^3+142488x^2+36018x+3691=0$
27	$3359232x^9+20155392x^8+52907904x^7+79688448x^6+75839328x^5$ $+47262528x^4+19275408x^3+4958496x^2+729810x+46817=0$
28	$46656x^6+186624x^5+301968x^4+252288x^3+114552x^2+26784x+2521=0$
29	$78364164096x^{14}+718338170880x^{13}+3027904229376x^{12}+7776917692416x^{11}$ $+13593884755968x^{10}+17102988891648x^9+15968336060160x^8+11237148361728x^7$ $+5988190197888x^6+2404315700928x^5+715887188640x^4+153265227840x^3$ $+22301305200x^2+1974053910x+80198051=0$
30	$1296x^4+3240x^3+2880x^2+1080x+145=0$

Table 24: Simplified equation $\Psi_m(-2-3x)=0$, for $1\le m\le 30$, if m is the order of the rotation matrix A with $\mathrm{tr}\,(A)=-3-6x$ and $\cos\theta=-2-3x$, with $x=\cos\alpha\in\mathbb{R}$ and X a rotation of α. Except for $m=1$ or 3, the leading coefficient of the irreducible polynomial on the left side of each equation is divisible by 3. Can you prove it for $m>30$? **Note:** these calculations apply equally well to any 3×3 rotation matrices with traces $3a$ and $-a$: if the former has finite order the latter cannot have finite order unless $m=1$ or 3.

Prequel to rest of the proof: let $B = -P_{(12)}A$ and suppose a, b, c are all distinct and every matrix in $[B]$ has finite order. We claim $\{a, b, c\} = \{1/2, \phi/2, \psi/2\}$. To see this, recall from Table 21 that matrices in $[B]$ which are not of order 2 are rotations of α, β or γ with

$$\cos\alpha = -a, \ \cos\beta = -b \text{ and } \cos\gamma = -c.$$

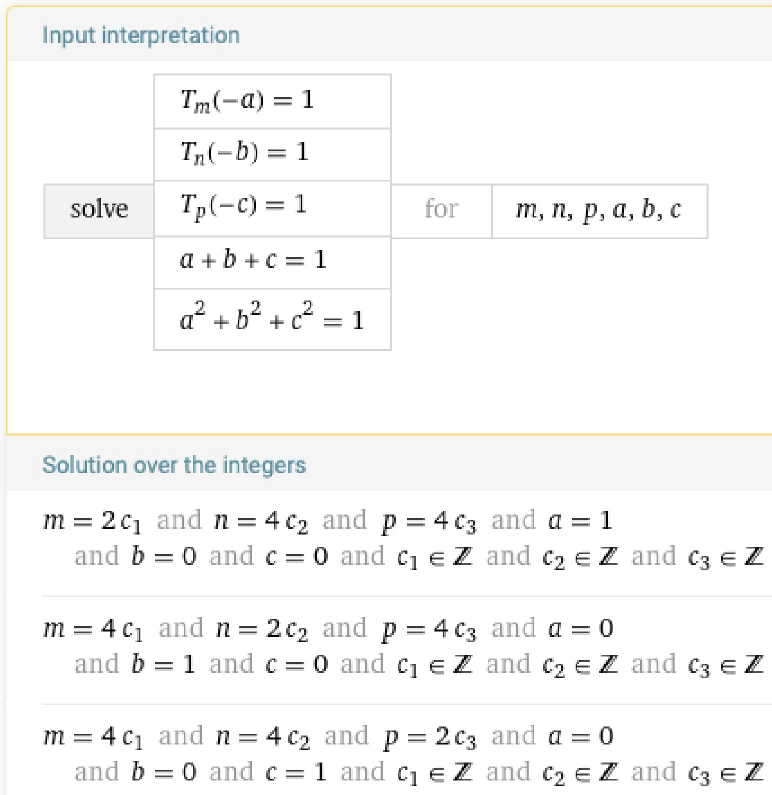
Suppose n_1, n_2, n_3 are positive integers such that

$$T_{n_1}(\cos\alpha) = 1, \ T_{n_2}(\cos\beta) = 1 \text{ and } T_{n_3}(\cos\gamma) = 1.$$

Then we attempt to solve the system of equations

$$T_{n_1}(-a) = 1, \ T_{n_2}(-b) = 1, \ T_{n_3}(-c) = 1, \ a^2 + b^2 + c^2 = 1, \ a + b + c = 1$$

for n_1, n_2, n_3, a, b, c. Naively feeding these equations into *Mathematica* results in the following:



Mathematica, at least as accessed on Wolfram Alpha, only supplies solutions over the integers. Even so, this solution is revealing: if one of n_1, n_2, n_3 is congruent to $0 \bmod 2$ and the other two are congruent to $0 \bmod 4$, then the only solutions for a, b, c are

$$\{a, b, c\} = \{1, 0, 0\},$$

and a, b, c are *not* all distinct. As for non-integer solutions, we have already displayed some in Tables 13 and 14 in which a, b, c are all distinct and

$$\{a, b, c\} = \left\{\frac{1}{2}, \frac{\phi}{2}, \frac{\psi}{2}\right\}.$$

In Tables 13 and 14, the elements which are rotations of α and have trace 0 have order 3, the elements which are rotations of β or γ and have trace ψ or ϕ have order 5. We now compute some more examples, using *Maple.* If

$$T_3(-a) = 1, T_{15}(-b) = 1, T_{15}(-c) = 1, a^2 + b^2 + c^2 = 1, a + b + c = 1,$$

then there are two solutions

$$a = \frac{1}{2}, b = \frac{\phi}{2}, c = \frac{\psi}{2} \text{ or } a = \frac{1}{2}, b = \frac{\psi}{2}, c = \frac{\phi}{2}.$$

But if $n_1 = n_2 = n_3 = 15$ then there are six solutions, two with $a = 1/2$, two with $b = 1/2$ and two with $c = 1/2$. Or consider the system

$$T_{12}(-a) = 1, T_{20}(-b) = 1, T_{10}(-c) = 1, a^2 + b^2 + c^2 = 1, a + b + c = 1,$$

which has solutions $a = 0, b = 0, c = 1$, corresponding to $n_1, n_2 \equiv 0 \bmod 4, \ n_3 \equiv 0 \bmod 2$, or

$$a = \frac{1}{2}, b = \frac{\psi}{2}, c = \frac{\phi}{2} \text{ or } a = \frac{1}{2}, b = \frac{\phi}{2}, c = \frac{\psi}{2},$$

corresponding to $n_1 \equiv 0 \bmod 3$ and $n_2, n_3 \equiv 0 \bmod 5$.

j	solution(s) to $\Psi_j(-a) = 0$	value(s) of a, $-1/3 \le a \le 1$
1	$-a - 1 = 0 \Rightarrow a = -1$	NA
2	$-a + 1 = 0 \Rightarrow a = 1$	1
3	$-2a + 1 = 0 \Rightarrow a = 1/2$	1/2
4	$-a = 0 \Rightarrow a = 0$	0
5	$4(-a)^2 - 2a - 1 = 0 \Rightarrow a = \phi/2$ or $\psi/2$	$\phi/2,\ \psi/2$
6	$-2a - 1 = 0 \Rightarrow a = -1/2$	NA
8	$2(-a)^2 - 1 = 0 \Rightarrow a = 1/\sqrt{2}$ or $-1/\sqrt{2}$	$1/\sqrt{2}$
10	$4(-a)^2 + 2a - 1 = 0 \Rightarrow a = -\psi/2$ or $-\phi/2$	$-\psi/2$
12	$4(-a)^2 - 3 = 0 \Rightarrow a = \sqrt{3}/2$ or $-\sqrt{3}/2$	$\sqrt{3}/2$

Table 25: Solutions to $\Psi_j(-a) = 0$ for nine cases in which the degree of Ψ_j is less than 3.

Of course we have already seen that if $m|n$ then $T_m(x) - 1$ divides $T_n(x) - 1$. Thus we can simplify the conditions that a, b, and c must satisfy by restricting j, k and l to be the *orders* of the matrices in $[B]$ with traces $1 - 2a, 1 - 2b$ and $1 - 2c$, respectively. That is,

$$\Psi_j(-a) = 0,\ \Psi_k(-b) = 0,\ \Psi_l(-c) = 0,\ a^2 + b^2 + c^2 = 1, a + b + c = 1.$$

We also know that $-1/3 \le a, b, c \le 1$. One way to solve the above system is to consider values of j, say, and then find possible values of a, b and c. See Table 25 for nine cases in which the degree of Ψ_j is less than 3. What we want to show is that $a \in \{0, 1, 1/2, \phi/2, \psi/2\}$; and so it is for $j = 2, 3, 4$ or 5. But if $j = 8, 10$ or 12 we obtain other values of a.

In general, we can solve for b and c in terms of a :

$$b = \frac{1 - a \pm \sqrt{-3a^2 + 2a + 1}}{2},\ c = \frac{1 - a \mp \sqrt{-3a^2 + 2a + 1}}{2}.$$

Then

$$(2b - 1 + a)^2 = -3a^2 + 2a + 1 \text{ and } (2c - 1 + a)^2 = -3a^2 + 2a + 1.$$

Expanding and simplifying shows that both b and c must satisfy the polynomial equation

$$x^2 - x = a(1 - x) - a^2,$$

where $a = -\cos\alpha$. Since $\cos\beta = -b$ and $\cos\gamma = -c$, both $\cos\beta$ and $\cos\gamma$ must satisfy the polynomial equation $x^2 + x = a(1 + x) - a^2$, where $a = -\cos\alpha$. We can use this result to show that in the cases $j = 8, 10$ or 12, the angles β and γ cannot be rational multiples of π.

- If $j = 8$ and $a = 1/\sqrt{2}$ then $\cos\beta$ and $\cos\gamma$ must satisfy

$$\begin{aligned} x^2 + x = \frac{1 + x}{\sqrt{2}} - \frac{1}{2} \quad &\Rightarrow \quad \left(x^2 + x + \frac{1}{2}\right)^2 = \frac{(1 + x)^2}{2} \\ &\Rightarrow \quad x^4 + x^2 + \frac{1}{4} + 2x^3 + x^2 + x = \frac{1 + 2x + x^2}{2} \\ &\Rightarrow \quad 4x^4 + 8x^3 + 6x^2 - 1 = 0. \end{aligned}$$

  This equation cannot be $\Psi_n(x) = 0$ for any n, since every polynomial $\Psi_n(x)$ of degree $k > 1$ has leading coefficient 2^k, unless n is a power of 2, in which case the coefficients of $\Psi_n(x)$ have common factor 2 so the leading coefficient in the equation could be 2^{k-1}.

- If $j = 10$ and $a = -\psi/2$ then $\cos\beta$ and $\cos\gamma$ must satisfy

$$\begin{aligned}
& x^2 + x = -\frac{1-\sqrt{5}}{4}(1+x) - \frac{3-\sqrt{5}}{8} \\
\Rightarrow\ & \left(x^2 + \frac{5}{4}x + \frac{5}{8}\right)^2 = \frac{5}{64}(3-2x)^2 \\
\Rightarrow\ & x^4 + \frac{25}{16}x^2 + \frac{25}{64} + \frac{5}{2}x^3 + \frac{5}{4}x^2 + \frac{25}{16}x = \frac{5}{64}(9 - 12x + 4x^2) \\
\Rightarrow\ & x^4 + \frac{5}{2}x^3 + \frac{5}{2}x^2 + \frac{5}{8}x - \frac{5}{16} = 0 \\
\Rightarrow\ & 16x^4 + 40x^3 + 40x^2 + 10x - 5 = 0.
\end{aligned}$$

  The polynomial $16x^4 + 40x^3 + 40x^2 + 10x - 5$ is irreducible, and both $\cos\beta$ and $\cos\gamma$ are roots. Can β and γ be rational multiples of π?

- If $j = 12$ and $a = \sqrt{3}/2$ then $\cos\beta$ and $\cos\gamma$ must satisfy

$$\begin{aligned}
x^2 + x = \frac{\sqrt{3}}{2}(1+x) - \frac{3}{4} \quad & \Rightarrow\ \left(x^2 + x + \frac{3}{4}\right)^2 = \frac{3}{4}(1+x)^2 \\
& \Rightarrow\ x^4 + x^2 + \frac{9}{16} + 2x^3 + \frac{3}{2}x^2 + \frac{3}{2}x = \frac{3}{4}(1 + 2x + x^2) \\
& \Rightarrow\ x^4 + 2x^3 + \frac{7}{4}x^2 - \frac{3}{16} = 0 \\
& \Rightarrow\ 16x^4 + 32x^3 + 28x^2 - 3 = 0.
\end{aligned}$$

  Again, the polynomial $16x^4 + 32x^3 + 28x^2 - 3$ is irreducible, and both $\cos\beta$ and $\cos\gamma$ are roots. Can β and γ be rational multiples of π?

The answer in both cases, $j = 10$ and 12, is No, but the answer is not as obvious as it was in the case $j = 8$. Consider Table 26, which is an extension of Table 1. We make the following

Observation: the constant term of each polynomial $\Psi_d(x)$ is ± 1 unless $d = 2^k$ for $k > 0$ or $d = 4 \cdot p^i$, where p is an odd prime and $i > 0$. If $d = 2^k$ then the constant term is ± 2, unless $k = 2$ in which case the constant term is 0. If $d = 4 \cdot p^i$, where p is an odd prime the constant term of $\Psi_d(x)$ is $\pm p$. Actually, things are even more interesting: if $d = 4 \cdot p^i$, for i a positive integer, then $\Psi_d(x)$ is a polynomial of degree $\phi(d)/2 = p^{i-1}(p-1)$ with constant term $\pm p$. See the lines with $d = 12, 20, 28, 36, 44$ and 52 in Table 26, for some examples. For another example, with $i > 1$, consider $d = 4 \cdot 3^3 = 108$. Then the three polynomials $\Psi_d(x)$ corresponding to $d = 4 \cdot 3^i$ for $i = 1, 2, 3$, are:

$\Psi_d(x)$	d
$4x^2 - 3$	$12 = 4 \cdot 3$
$64x^6 - 96x^4 + 36x^2 - 3$	$36 = 4 \cdot 3^2$
$262144x^{18} - 1179648x^{16} + 2211840x^{14} - 2236416x^{12}$ $+1317888x^{10} - 456192x^8 + 88704x^6 - 8640x^4 + 324x^2 - 3$	$108 = 4 \cdot 3^3$

You can confirm with *Maple* or *Mathematica* that $\Psi_d(x)$ for $d = 4 \cdot 3^4 = 324$ has degree $54 = 3^3(3-1)$ and constant term -3. For yet another example, consider $d = 4 \cdot 5^2 = 100$. The two polynomials $\Psi_d(x)$ with $d = 4 \cdot 5^i$ for $i = 1, 2$ are:

$\Psi_d(x)$	d
$16x^4 - 20x^2 + 5$	$20 = 4 \cdot 5$
$1048576x^{20} - 5242880x18 + 11141120x^{16} - 13107200x^{14} + 9318400x^{12}$ $-4101120x^{10} + 1100800x^8 - 171200x^6 + 14000x^4 - 500x^2 - 5$	$100 = 4 \cdot 5^2$

Irreducible polynomial $\Psi_d(x)$, of degree $\phi(d)/2$ for $d > 2$	d
$x-1$	1
$2x+2$	2
$2x+1$	3
$2x$	4
$4x^2+2x-1$	5
$2x-1$	6
$8x^3+4x^2-4x-1$	7
$4x^2-2$	8
$8x^3-6x+1$	9
$4x^2-2x-1$	10
$32x^5+16x^4-32x^3-12x^2+6x+1$	11
$4x^2-3$	12
$64x^6+32x^5-80x^4-32x^3+24x^2+6x-1$	13
$8x^3-4x^2-4x+1$	14
$16x^4-8x^3-16x^2+8x+1$	15
$16x^4-16x^2+2$	16
$256x^8+128x^7-448x^6-192x^5+240x^4+80x^3-40x^2-8x+1$	17
$8x^3-6x-1$	18
$512x^9+256x^8-1024x^7-448x^6+672x^5+240x^4-160x^3-40x^2+10x+1$	19
$16x^4-20x^2+5$	20
$64x^6-32x^5-96x^4+48x^3+32x^2-16x+1$	21
$32x^5-16x^4-32x^3+12x^2+6x-1$	22
$2048x^{11}+1024x^{10}-5120x^9-2304x^8+4608x^7$ $+1792x^6-1792x^5-560x^4+280x^3+60x^2-12x-1$	23
$16x^4-16x^2+1$	24
$1024x^{10}-2560x^8+2240x^6+32x^5-800x^4-40x^3+100x^2+10x-1$	25
$64x^6-32x^5-80x^4+32x^3+24x^2-6x-1$	26
$512x^9-1152x^7+864x^5-240x^3+18x+1$	27
$64x^6-112x^4+56x^2-7$	28
$16384x^{14}+8192x^{13}-53248x^{12}-24576x^{11}+67584x^{10}+28160x^9-42240x^8$ $-15360x^7+13440x^6+4032x^5-2016x^4-448x^3+112x^2+14x-1$	29
$16x^4+8x^3-16x^2-8x+1$	30
$32768x^{15}+16384x^{14}-114688x^{13}-53248x^{12}+159744x^{11}+67584x^{10}-112640x^9$ $-42240x^8+42240x^7+13440x^6-8064x^5-2016x^4+672x^3+112x^2-16x-1$	31
$256x^8-512x^6+320x^4-64x^2+2$	32
$64x^6-96x^4+36x^2-3$	36
$1024x^{10}-2816x^8+2816x^6-1232x^4+220x^2-11$	44
$4096x^{12}-13312x^{10}+16640x^8-9984x^6+2912x^4-364x^2+13$	52
$256x^8-448x^6+224x^4-32x^2+1$	60
$65536x^{16}-262144x^{14}+425984x^{12}-360448x^{10}$ $+168960x^8-43008x^6+5376x^4-256x^2+2$	64
$4096x^{12}-12288x^{10}+13824x^8-7168x^6+1680x^4-144x^2+1$	72

Table 26: Some irreducible polynomials $\Psi_d(x)$ and the order d of the corresponding rotation matrix X with $\mathrm{tr}\,(X) = 1 + 2x$ and $x = \cos\theta$, where θ is the rotation angle, if x is a root of Ψ_d, for $1 \leq d \leq 32$, with $d = 36, 44, 52, 60, 64$ & 72 thrown in for interest. For $X \in [B]$, a, b, c all distinct, we can take $x = \cos\alpha, \cos\beta$ or $\cos\gamma$ with $a = -\cos\alpha, b = -\cos\beta$ or $c = -\cos\gamma$, for $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$ and $a^2 + b^2 + c^2 = 1$.

Again, you can confirm with *Maple* or *Mathematica* that $\Psi_d(x)$ with $d = 4 \cdot 5^3 = 500$ has degree $100 = 5^2(5-1)$ and constant term 5.

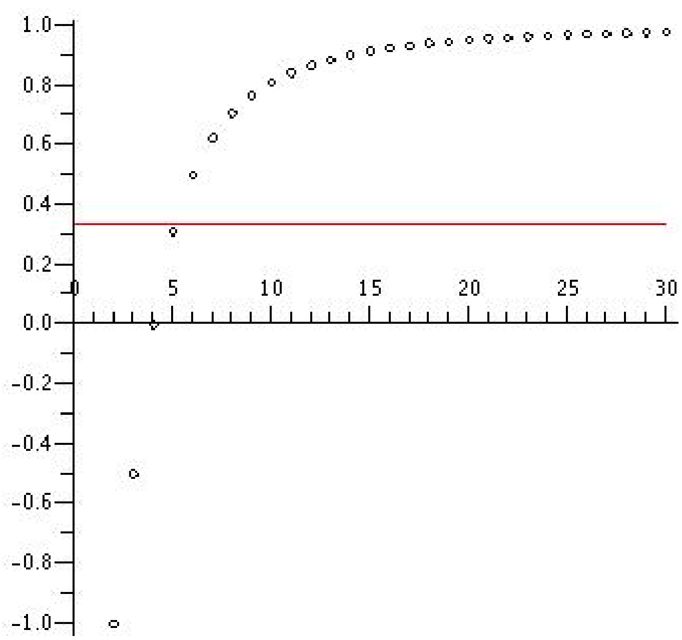
So, according to these observations, the polynomials in the cases $j = 10$ or 12, with $p = 5$ and $p = 3$, respectively, can't be $\Psi_d(x)$, for any d. Consequently β and γ can't be rational multiples of π.

An alternate approach: notice that in the above examples we skipped the cases $j = 7, 9$ and 11 because in those cases $\Psi_d(x)$ has degree 3 or 5 and the algebra becomes unwieldy. How to solve such systems in general? One way is to use the roots of $\Psi_d(x) = 0$, for $d > 2$, which are

$$x = \cos\left(2\pi \frac{k}{d}\right)$$

for $(k, d) = 1$ and $1 \leq k < d/2$. Let us do some calculations if $\alpha = 2\pi/m$, remembering that we require $-1 \leq \cos\alpha \leq 1/3$. We find that the only acceptable values of m are $m = 2, 3, 4$ or 5, and in each case $a \in \{0, 1, 1/2, \phi/2, \psi/2\}$. See the chart below. Also, the plot below illustrates $\lim_{m\to\infty} \cos\left(\frac{2\pi}{m}\right) = 1$. The red line is $y = \frac{1}{3}$.

m	α	$\cos\alpha$	a
2	π	-1	1
3	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{1}{2}$
4	$\frac{\pi}{2}$	0	0
5	$\frac{2\pi}{5}$	$-\frac{\psi}{2}$	$\frac{\psi}{2}$
6	$\frac{\pi}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$



However there is always the possibility that $j = m$ and $\alpha = \dfrac{2\pi k}{m}$ for some $k, 1 < k < m/2$, and k relatively prime to m. For example, if $\alpha = \dfrac{2\pi\, 2}{5} = \dfrac{4\pi}{5}$ then $\cos\alpha = -\dfrac{\phi}{2}$, which is similar to the case $m = 5$ above, so the results are basically the same. But how do we know in general that such cases are not problematic? In particular,

$$\lim_{k\to m/2} \cos\left(\frac{2\pi k}{m}\right) = \cos\pi = -1,$$

thus many values of k could provide acceptable values of $\alpha = 2\pi k/m$; i.e. $-1 \leq \cos\alpha \leq 1/3$.

Example 18: consider $m = 7$ and $\alpha = 2\pi\frac{2}{7} = \frac{4\pi}{7}$. Then $\cos\alpha \approx -0.222521$. The problem is that there is no simple exact formula for $\cos\alpha$, so none of our previous methods can be applied to determine if β and γ are rational multiples of π or not. However we can solve for $\cos\alpha$ in the equation $x^2 + x = -\cos\alpha\,(1+x) - \cos^2\alpha$ to find (not surprisingly) that

$$\cos\alpha = \frac{1}{2}\left(-x - 1 \pm \sqrt{-3x^2 - 2x + 1}\right).$$

Since $\Psi_7(\cos\alpha) = 0$ we know that

$$\Psi_7\left(\frac{1}{2}\left(-x - 1 \pm \sqrt{-3x^2 - 2x + 1}\right)\right) = 0.$$

Choosing the positive root and using *Mathematica* to perform the algebra we find

$$8x^3+3x^2\sqrt{-3x^2-2x+1}+10x^2+4x\sqrt{-3x^2-2x+1}+(-3x^2-2x+1)^{3/2}-\sqrt{-3x^2-2x+1}+2x-1=0$$

$$\begin{aligned}
&\Rightarrow\quad 8x^3 + 10x^2 + 2x - 1 = -(3x^2 + 4x - 3x^2 - 2x + 1 - 1)\sqrt{-3x^2 - 2x + 1}\\
&\Rightarrow\quad 8x^3 + 10x^2 + 2x - 1 = -(2x)\sqrt{-3x^2 - 2x + 1}\\
&\Rightarrow\quad (8x^3 + 10x^2 + 2x - 1)^2 = (2x)^2\,(-3x^2 - 2x + 1)\\
&\Rightarrow\quad 64x^6 + 160x^5 + 144x^4 + 32x^3 - 20x^2 - 4x + 1 = 0.
\end{aligned}$$

(You can check that choosing the negative root results in the same irreducible polynomial.) Now $\cos\beta, \cos\gamma$ must be roots of the polynomial $S_7(x) = 64x^6 + 160x^5 + 144x^4 + 32x^3 - 20x^2 - 4x + 1$. Consider the six approximate roots to this equation, as supplied by *Mathematica*, namely

$$-0.958058 \text{ and } 0.180579,\ -0.352296 \text{ and } 0.253264,\ -0.811745 \pm 0.594390i.$$

1. The first pair corresponds to the approximate values of $\cos\beta$ and $\cos\gamma$ if $\alpha = 4\pi/7$.
2. The second pair corresponds to the approximate values of $\cos\beta$ and $\cos\gamma$ if $\alpha = 6\pi/7$.
3. And thirdly, the pair of complex numbers correspond to the approximate values of $\cos\beta$ and $\cos\gamma$ if $\alpha = 2\pi/7$.

But all roots of Ψ_d are real. So $S_7 \neq \Psi_m$ for any m and β, γ can't be rational multiples of π. □

Simplification: there is a more straightforward way to find $S_7(x)$, namely

$$S_7(x) = \Psi_7\left(\frac{1}{2}\left(-x - 1 + \sqrt{-3x^2 - 2x + 1}\right)\right)\Psi_7\left(\frac{1}{2}\left(-x - 1 - \sqrt{-3x^2 - 2x + 1}\right)\right),$$

since both expressions

$$\Psi_7\left(\frac{1}{2}\left(-x - 1 + \sqrt{-3x^2 - 2x + 1}\right)\right) \text{ and } \Psi_7\left(\frac{1}{2}\left(-x - 1 - \sqrt{-3x^2 - 2x + 1}\right)\right)$$

equal zero. In general we define

$$S_n(x) = \Psi_n\left(\frac{1}{2}\left(-x - 1 + \sqrt{-3x^2 - 2x + 1}\right)\right)\Psi_n\left(\frac{1}{2}\left(-x - 1 - \sqrt{-3x^2 - 2x + 1}\right)\right).$$

In this way we have calculated polynomials S_n for $2 \leq n \leq 14$; see Table 27. The degree of $S_n(x)$ for $n > 2$ is $\phi(n)$, where ϕ is Euler's phi function. But $\deg S_n = 2\deg\Psi_n$, for $n \geq 2$. Aside: there are other ways to calculate $S_n(x)$, as we shall see.

n with $\Psi_n(\cos\alpha)=0$	$S_n(x)=\Psi_n\left(\frac{1}{2}\left(-x-1+\sqrt{-3x^2-2x+1}\right)\right) \times \Psi_n\left(\frac{1}{2}\left(-x-1-\sqrt{-3x^2-2x+1}\right)\right)$	k,l such that $\Psi_k(\cos\beta)=0$ or $\Psi_l(\cos\gamma)=0$
2 ($a=1$)	$4x^2=(\Psi_4(x))^2$	4, 4 ($b=c=0$)
3 ($a=1/2$)	$4x^2+2x-1=\Psi_5(x)$	5, 5 ($b=\phi/2, c=\psi/2$)
4 ($a=0$)	$4x^2+4x=4x(x+1)=\Psi_2(x)\cdot\Psi_4(x)$	2, 4 ($b=1, c=0$)
5 ($a=\psi/2$)	$16x^4+24x^3+8x^2-2x-1=(2x+1)^2(4x^2+2x-1)$ $=(\Psi_3(x))^2\cdot\Psi_5(x)$	3, 5 ($b=1/2, c=\phi/2$)
6 ($a=-1/2$)	$4x^2+6x+3$	(2 complex roots)
7	$64x^6+160x^5+144x^4+32x^3-20x^2-4x+1$	(2 complex roots)
8	$16x^4+32x^3+24x^2-4$	(2 complex roots)
9	$64x^6+192x^5+240x^4+128x^3+12x^2-6x-1$	(2 complex roots)
10	$16x^4+40x^3+40x^2+10x-5$	(2 complex roots)
11	$1024x^{10}+4608x^9+8960x^8+8960x^7+3968x^6-288x^5$ $-864x^4-152x^3+48x^2+6x-1$	(4 complex roots)
12	$16x^4+32x^3+28x^2-3$	(2 complex roots)
13	$4096x^{12}+22528x^{11}+55296x^{10}+75264x^9+$ $56576x^8+17024x^7-5696x^6-5472x^5$ $-624x^4+368x^3+60x^2-6x-1$	(4 complex roots)
14	$64x^6+224x^5+336x^4+224x^3+28x^2-28x-7$	(2 complex roots)

Table 27: Polynomials $S_n(x)$, for $2\le n\le 14$, that $\cos\beta,\cos\gamma$ must satisfy if $\Psi_n(\cos\alpha)=0$. Known factors of $S_n(x)$ are included plus values of $k.l$ s.t. $\Psi_k(\cos\beta)=0$ or $\Psi_l(\cos\gamma)=0$. If $S_n(x)$ is irreducible and has complex roots then it is not Ψ_d for any d, and both β and γ are not rational multiples of π. For the cases $n=2,3,4,5$ and 6, sample values of a,b,c are included.

Example 19: let $n = 20$. Use $\Psi_{20}(x) = 16x^4 - 20x^2 + 5$ and *Mathematica* to check that

$$S_{20}(x) = 256x^8 + 1024x^7 + 1856x^6 + 1664x^5 + 576x^4 - 160x^3 - 140x^2 + 5.$$

Its approximate roots are

$$-0.975528 \pm 0.950742\,i; -0.793893 \pm 0.550465\,i; -0.739750, 0.327535; -0.241605, 0.192662,$$

pairs of which correspond to possible values of $\cos\beta$ and $\cos\gamma$ if

$$\alpha = \frac{2\pi}{20}, \frac{6\pi}{20}, \frac{14\pi}{20}, \frac{18\pi}{20},$$

respectively, as you can check. As in Example 18, β and γ cannot be rational multiples of π. □

Example 20: let $n = 72$ and suppose that $\Psi_{72}(\cos\alpha) = 0$. The polynomial Ψ_{72} is listed in Table 26. Making use of *Maple,* we find that $\cos\beta$ and $\cos\gamma$ must be roots of the degree 24 irreducible polynomial

$$S_{72}(x) = 16777216x^{24} + 201326592x^{23} + 1157627904x^{22} + 4194304000x^{21} + 10613686272x^{20} +$$

$$19654508544x^{19} + 27088912384x^{18} + 27556577280x^{17} + 19738066944x^{16} + 8447066112x^{15} +$$

$$278986752x^{14} - 2258239488x^{13} - 1401700352x^{12} - 229785600x^{11} + 142381056x^{10} + 78307328x^9$$

$$+4836096x^8 - 5790720x^7 - 1206528x^6 + 142848x^5 + 49008x^4 - 576x^3 - 432x^2 + 1.$$

As in Examples 18 and 19 we can break down the (approximate) solutions to $S_{72}(x) = 0$ into 12 pairs corresponding to the 12 values of k s.t. $0 < k < 36$ and k is relatively prime to 72, namely for

$$k = 1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31 \text{ or } 35.$$

That is, the approximate values of $\cos\beta, \cos\gamma$ are

1. $-0.9980973490 \pm 0.9961928810\,i$ if $k = 1$ and $\alpha = 2 \cdot 1 \cdot \pi/72$,
2. $-0.9531538935 \pm 0.9050962640\,i$ if $k = 5$ and $\alpha = 2 \cdot 5 \cdot \pi/72$,
3. $-0.9095760220 \pm 0.8141459180\,i$ if $k = 7$ and $\alpha = 2 \cdot 7 \cdot \pi/72$,
4. $-0.7867882180 \pm 0.5324759755\,i$ if $k = 11$ and $\alpha = 2 \cdot 11 \cdot \pi/72$,
5. $-0.7113091310 \pm 0.3086483067\,i$ if $k = 13$ and $\alpha = 2 \cdot 13 \cdot \pi/72$,
6. -0.9916013485 or -0.09555439430 if $k = 17$ and $\alpha = 2 \cdot 17 \cdot \pi/72$,
7. -0.9929673540 or 0.08012309665 if $k = 19$ and $\alpha = 2 \cdot 19 \cdot \pi/72$,
8. -0.8608398745 or 0.2834581364 if $k = 23$ and $\alpha = 2 \cdot 23 \cdot \pi/72$,
9. -0.7517707590 or 0.3253471954 if $k = 25$ and $\alpha = 2 \cdot 25 \cdot \pi/72$,
10. -0.4857952823 or 0.3049473266 if $k = 29$ and $\alpha = 2 \cdot 29 \cdot \pi/72$,
11. -0.3419876656 or 0.2482954526 if $k = 31$ and $\alpha = 2 \cdot 31 \cdot \pi/72$,
12. 0.05969638855 or -0.06350169045 if $k = 35$ and $\alpha = 2 \cdot 35 \cdot \pi/72$.

And again, as in Examples 18 and 19, the irreducible polynomial $S_{72} \neq \Psi_d$ for any d since it has complex roots. Thus the angles β and γ cannot be rational multiples of π. □

Calculating S_n **:** before we complete the proof of the Conjecture, we describe two different ways to calculate $S_n(x)$, one using Chebyshev polynomials of the first kind, and one using a factorization of $S_n(x)$ into a product of quadratic factors.

S **and** T**:** since the factors of $T_n(z) - 1$ are of the form $\Psi_d(z)$ where $d|n$ we might well expect something similar with $S_n(x)$ and

$$T_n\left(\frac{1}{2}\left(-x-1\pm\sqrt{-3x^2-2x+1}\right)\right).$$

Example 21: to this end let us use *Maple* to calculate and factor the product

$$T_n^*(x) = \left[T_n\left(\frac{-x-1+\sqrt{-3x^2-2x+1}}{2}\right)-1\right]\left[T_n\left(\frac{-x-1-\sqrt{-3x^2-2x+1}}{2}\right)-1\right]$$

for $1 \leq n \leq 20$, thereby also extending the results of Table 27, as we shall see.

1. $T_1^*(x) = x^2 + 2x + 2$. This is $S_1(x)$ which we have not mentioned previously. In this case $\Psi_1(\cos\alpha) = 0$ so $\cos\alpha = 1$ and algebraically $\cos\beta$, $\cos\gamma = -1 \pm i$, the roots of $S_1(x)$.

2. $T_2^*(x) = 4(x^2 + 2x + 2)\, x^2 = S_1(x)\, S_2(x)$.

3. $T_3^*(x) = (x^2 + 2x + 2)(4x^2 + 2x - 1)^2 = S_1(x)(S_3(x))^2$.

4. $T_4^*(x) = 64\,(x^2 + 2x + 2)\, x^4\, (x + 1)^2 = S_1(x)\, S_2(x)\, (S_4(x))^2$.

5. $T_5^*(x) = (x^2 + 2x + 2)(4x^2 + 2x - 1)^2(2x + 1)^4 = S_1(x)(S_5(x))^2$.

6. $T_6^*(x) = 4(x^2 + 2x + 2)x^2(4x^2 + 2x - 1)^2(4x^2 + 6x + 3)^2 = S_1(x)S_2(x)(S_3(x))^2(S_6(x))^2$.

7. $T_7^*(x) = (x^2 + 2x + 2)(64x^6 + 160x^5 + 144x^4 + 32x^3 - 20x^2 - 4x + 1)^2 = S_1(x)(S_7(x))^2$.

8. $T_8^*(x) = 1024(x^2+2x+2)\, x^4\, (x+1)^2\, (4x^4+8x^3+6x^2-1)^2 = S_1(x)S_2(x)(S_4(x))^2(S_8(x))^2$.

9. $T_9^*(x) = (x^2 + 2x + 2)(4x^2 + 2x - 1)^2(64x^6 + 192x^5 + 240x^4 + 128x^3 + 12x^2 - 6x - 1)^2$
   $= S_1(x)\, (S_3(x))^2\, (S_9(x))^2$.

10. $T_{10}^*(x) = 4\,(x^2 + 2x + 2)\, x^2\, (2x + 1)^4\, (4x^2 + 2x - 1)^2\, (16x^4 + 40x^3 + 40x^2 + 10x - 5)^2$
    $= S_1(x)\, S_2(x)\, (S_5(x))^2\, (S_{10}(x))^2$.

11. $T_{11}^*(x) = (x^2 + 2x + 2)\times$

    $(1024x^{10} + 4608x^9 + 8960x^8 + 8960x^7 + 3968x^6 - 288x^5 - 864x^4 - 152x^3 + 48x^2 + 6x - 1)^2$

    $= S_1(x)\, (S_{11}(x))^2$.

12.
$$T_{12}^*(x) = 64(x^2 + 2x + 2)\, x^4\, (4x^2 + 2x - 1)^2\, (x + 1)^2\, (4x^2 + 6x + 3)^2$$
$$\times(16x^4 + 32x^3 + 28x^2 - 3)^2 = S_1(x)\, S_2(x)\, (S_3(x))^2\, (S_4(x))^2\, (S_6(x))^2\, (S_{12}(x))^2$$

13.
$$T_{13}^*(x) = (x^2 + 2x + 2)\, (4096x^{12} + 22528x^{11} + 55296x^{10} + 75264x^9 + 56576x^8$$
$$+17024x^7 - 5696x^6 - 5472x^5 - 624x^4 + 368x^3 + 60x^2 - 6x - 1)^2 = S_1(x)(S_{13}(x))^2.$$

14.

$$T^*_{14}(x) = 4\,(x^2+2x+2)\,x^2\,(64x^6+160x^5+144x^4+32x^3-20x^2-4x+1)^2\times$$
$$(64x^6+224x^5+336x^4+224x^3+28x^2-28x-7)^2 = S_1(x)\,S_2(x)\,(S_7(x))^2\,(S_{14}(x))^2.$$

15.

$$T^*_{15}(x) = (x^2+2x+2)\,(4x^2+2x-1)^4\,(2x+1)^4$$
$$\times(256x^8+1152x^7+2368x^6+2624x^5+1520x^4+304x^3-72x^2-8x+1)^2$$
$$= S_1(x)\,(S_3(x))^2\,(S_5(x))^2\,(S_{15}(x))^2.$$

16.

$$T^*_{16}(x) = (x^2+2x+2)\,4x^2\,16x^2\,(x+1)^2\,(16x^4+32x^3+24x^2-4)^2$$
$$\times(256x^8+1024x^7+1792x^6+1536x^5+480x^4-128x^3-96x^2+4)^2$$
$$= S_1(x)\,S_2(x)\,(S_4(x))^2\,(S_8(x))^2\,(S_{16}(x))^2.$$

17.

$$T^*_{17}(x) = (x^2+2x+2)\,(65536x^{16}+491520x^{15}+1703936x^{14}+3514368x^{13}$$
$$+4628480x^{12}+3829760x^{11}+1680384x^{10}-24576x^9-435712x^8-177536x^7+8576x^6$$
$$+20672x^5+2320x^4-800x^3-104x^2+8x+1)^2 = S_1(x)\,(S_{17}(x))^2.$$

18.

$$T^*_{18}(x) = 4\,(x^2+2x+2)\,x^2\,(4x^2+2x-1)^2\,(4x^2+6x+3)^2\times$$
$$(64x^6+192x^5+240x^4+128x^3+12x^2-6x-1)^2(64x^6+192x^5+240x^4+96x^3-36x^2-18x+3)^2$$
$$= S_1(x)\,S_2(x)\,(S_3(x))^2\,(S_6(x))^2\,(S_9(x))^2\,(S_{18}(x))^2.$$

19.

$$T^*_{19}(x) = (x^2+2x+2)(262144x^{18}+2228224x^{17}+8847360x^{16}+21299200x^{15}$$
$$+33816576x^{14}+35889152x^{13}+23969792x^{12}+7366656x^{11}-2231296x^{10}-3036160x^9$$
$$-872960x^8+168320x^7+139840x^6+9952x^5-7120x^4-880x^3+140x^2+10x-1)^2$$
$$= S_1(x)\,(S_{19})^2.$$

20.

$$T^*_{20}(x) = 64\,(x^2+2x+2)\,x^4\,(x+1)^2\,(4x^2+2x-1)^2\,(2x+1)^4\,(16x^4+40x^3+40x^2+10x-5)^2$$
$$\times(256x^8+1024x^7+1856x^6+1664x^5+576x^4-160x^3-140x^2+5)^2$$
$$= S_1(x)\,S_2(x)\,(S_4(x))^2\,(S_5(x))^2\,(S_{10}(x))^2\,(S_{20}(x))^2.$$

Check that $S_{15}(x), S_{16}(x), S_{17}(x), S_{18}(x), S_{19}(x)$ and $S_{20}(x)$, as found above, are the irreducible polynomials that $\cos\beta$ and $\cos\gamma$ satisfy if $\Psi_n(\cos\alpha) = 0$, for $n = 15, 16, 17, 18, 19$ and 20, respectively. □ Moreover, the degree of $T^*_n(x)$ is $2n$, and similar to the factorization of $T_n(x)-1$, we have:

$$\text{if } n \text{ is even, } T^*_n(x) = S_1(x)S_2(x) \prod_{\substack{d\,|\,n \\ d>2}} (S_d(x))^2$$

$$\text{and if } n \text{ is odd, } T_n^*(x) = S_1(x) \prod_{\substack{d \mid n \\ d > 2}} (S_d(x))^2.$$

A Factorization of $S_n(x)$ Into a Product of Quadratics: if $\cos\alpha, \cos\beta$ and $\cos\gamma$ satisfy

$$\cos\alpha + \cos\beta + \cos\gamma = -1 \text{ and } \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

then

$$\begin{aligned}(x - \cos\beta)(x - \cos\gamma) &= x^2 - (\cos\beta + \cos\gamma)x + \cos\beta\ \cos\gamma \\ &= x^2 - (-1 - \cos\alpha)x + (-\cos\beta\ \cos\alpha - \cos\gamma\ \cos\alpha) \\ &= x^2 + x\ \cos\alpha + x + \cos\alpha\,(1 + \cos\alpha) \\ &= x^2 + x\ \cos\alpha + x + \cos^2\alpha + \cos\alpha\end{aligned}$$

and both $\cos\beta, \cos\gamma$ are roots of this quadratic. Now suppose $\Psi_n(\cos\alpha) = 0$. Then $\alpha = \dfrac{2\pi\, p}{n}$, for p relatively prime to n and $1 \le p < \dfrac{n}{2}$. Then for $n \ge 2$, after judicious multiplication by 4,

$$S_n(x) = \prod_{\substack{(p,n)=1 \\ 1 \le p < n/2}} \left(4x^2 + 4x\cos\left(\frac{2\pi p}{n}\right) + 4x + 4\cos^2\left(\frac{2\pi p}{n}\right) + 4\cos\left(\frac{2\pi p}{n}\right)\right).$$

Example 22: some calculations using *Maple*. First we define the appropriate quadratic.

```
> quad2 := (x,p,n) -> 4*x^2+4*x*cos(2*Pi*p/n)+4*x+4*cos^2(2*Pi*p/n)+4*cos(2*Pi*p/n);
```

Then, considering the cases when α is a rational multiple of π and $\cos\alpha$ is a rational number:

```
> quad2(x,1,2);
```

$$4x^2$$

```
> quad2(x,1,3);
```

$$4x^2 + 2x - 1$$

```
> quad2(x,1,4);
```

$$4x^2 + 4x$$

```
> quad2(x,1,6);
```

$$4x^2 + 6x + 3.$$

These are, respectively, $S_2(x), S_3(x), S_4(x)$ and $S_6(x)$—the only cases[11] with $n > 1$ for which $S_n(x)$ is a quadratic. As an example for which $S_n(x)$ does not have degree 2, consider $n = 5$:

```
> quad2(x,1,5);
```

$$4x^2 + 4x\cos\left(\frac{2\pi}{5}\right) + 4x + 4\cos^2\left(\frac{2\pi}{5}\right) + 4\cos\left(\frac{2\pi}{5}\right)$$

```
> quad2(x,2,5):
```

$$4x^2 - 4x\cos\left(\frac{\pi}{5}\right) + 4x + 4\cos^2\left(\frac{\pi}{5}\right) - 4\cos\left(\frac{\pi}{5}\right)$$

[11]If $n = 1$ then $\alpha = 0$ and $x^2 + x\cos 0 + x + \cos^2 0 + \cos 0 = x^2 + 2x + 2 = S_1(x)$, so multiplication by 4 is not required.

Now we should multiply these two quadratics together, but unfortunately *Maple* does not simplify the product. However, we can, making use of the vales

$$\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4} \text{ and } \cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}.$$

Then computation shows that `quad2(x,1,5)*quad2(x,2,5)` does indeed simplify to

$$16x^4 + 24x^3 + 8x^2 - 2x - 1 = S_5(x).$$

Or consider $n = 8$. In this case, *Maple* does simplify the product.

```
> quad2(x,1,8)*quad2(x,3,8)
```

$$\left(4x^2 + 2x\sqrt{2} + 4x + 2 + 2\sqrt{2}\right)\left(4x^2 - 2x\sqrt{2} + 4x + 2 - 2\sqrt{2}\right)$$

```
> expand(%);
```

$$16x^4 + 32x^3 + 24x^2 - 4$$

which is $S_8(x)$. Similarly with $n = 12$:

```
> quad2(x,1,12)*quad2(x,5,12);
```

$$\left(4x^2 + 2x\sqrt{3} + 4x + 3 + 2\sqrt{3}\right)\left(4x^2 - 2x\sqrt{3} + 4x + 3 - 2\sqrt{3}\right)$$

```
> expand(%)
```

$$16x^4 + 32x^3 + 28x^2 - 3,$$

which is $S_{12}(x)$. As for $n = 10$, we have to simplify things manually, as we did in the case $n = 5$.

```
> quad2(x,1,10)*quad2(x,3,10);
```

$$\begin{aligned}&\left(4x^2 + 4x\cos(\pi/5) + 4x + 4\cos^2(\pi/5) + 4\cos(\pi/5)\right) \times\\&\left(4x^2 - 4x\cos(2\pi/5) + 4x + 4\cos^2(2\pi/5) - 4\cos(2\pi/5)\right)\\&= 16x^4 + 40x^3 + 40x^2 + 10x - 5,\end{aligned}$$

which is $S_{10}(x)$. Finally, let us consider $n = 20$ for which we will have to calculate four quadratic expressions.

```
> quad2(x,1,20);
```

$$4x^2 + 4x\cos(\pi/10) + 4x + 4\cos^2(\pi/10) + 4\cos(\pi/10)$$

```
> quad2(x,3,20);
```

$$4x^2 + 4x\cos(3\pi/10) + 4x + 4\cos^2(3\pi/10) + 4\cos(3\pi/10)$$

```
> quad2(x,7,20);
```

$$4x^2 - 4x\cos(3\pi/10) + 4x + 4\cos^2(3\pi/10) - 4\cos(3\pi/10)$$

```
> quad2(x,9,20);
```

$$4x^2 - 4x\cos(\pi/10) + 4x + 4\cos^2(\pi/10) - 4\cos(\pi/10)$$

We can simplify the product of these four quadratics using

$$\cos\left(\frac{\pi}{10}\right) = \sqrt{\frac{5+\sqrt{5}}{8}} \text{ and } \cos\left(\frac{3\pi}{10}\right) = \sqrt{\frac{5-\sqrt{5}}{8}},$$

to find

```
> expand(quad2(x,1,20)*quad2(x,3,20)*quad2(x,7,20)*quad2(x,9,20));
```

$$256x^8 + 1024x^7 + 1856x^6 + 1664x^5 + 576x^4 - 160x^3 - 140x^2 + 5,$$

which is $S_{20}(x)$. □

We now state two key results about $S_n(x)$, as illustrated in Table 27 and Examples 19 and 20:

Theorem 17 *Let $n > 1$. Then $S_n(x)$ has complex roots if $n \geq 6$.*

Proof: recall that for $n > 2$,

$$\Psi_n(x) = \prod_{\substack{(k,n)=1 \\ 1 \leq k < n/2}} 2\left(x - \cos\left(\frac{2\pi\, k}{n}\right)\right).$$

In particular, if $k = 1$ and $n \geq 6$ then

$$\frac{2\pi\, k}{n} \leq \frac{\pi}{3} \text{ and } \cos\left(\frac{2\pi\, k}{n}\right) \geq \frac{1}{2}.$$

So if $\alpha = 2\,\pi/n$ and $x = \cos\alpha$, then $\Psi_n(x) = 0$ and

$$\cos\alpha > \frac{1}{3} \Leftrightarrow a < -\frac{1}{3},$$

which means the corresponding values of b, c, and consequently $\cos\beta, \cos\gamma$, must be complex:

$$b = \frac{1}{2}\left(-a + 1 - \sqrt{-3a^2 + 2a + 1}\right) \text{ and } c = \frac{1}{2}\left(-a + 1 + \sqrt{-3a^2 + 2a + 1}\right)$$

are real only if $-\dfrac{1}{3} \leq a \leq 1$. □ Moreover, the complex roots of $S_n(x)$ come in conjugate pairs, since the coefficients of $S_n(x)$ are all real; and the number of complex roots is twice the cardinality of the set

$$\{k \mid (k,n) = 1,\ 1 \leq k < n/2 \text{ and } \cos(2\pi\, k/n) > 1/3\}.$$

Theorem 18 *Let $n > 1$. Then $S_n(x)$ is irreducible if $n = 3$ or $n \geq 6$.*

Proof: ELFTS. Just kidding. Even with two different approaches for calculating $S_n(x)$ there seems to be no easy way to establish this. In addition to the evidence supplied by the above examples, calculations with *Maple* confirm the result for $n \leq 100$. □

Completing the Proof of Conjecture 1: this is easy if we assume the above two theorems. Assume $\cos\alpha, \cos\beta$ and $\cos\gamma$ are solutions to the system of equations

$$\cos\alpha + \cos\beta + \cos\gamma = -1,\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

such that $\Psi_j(\cos\alpha) = 0, \Psi_k(\cos\beta) = 0, \Psi_l(\cos\gamma) = 0$. That is, suppose *all* the matrices in $[B] = [-P_{(12)}A]$ have finite order. See Table 28 for the case a, b, c are all distinct. (Otherwise $B = -P_{12}$ and there is nothing to prove, or the elements of $[B]$ as listed in Table 22 contain elements of infinite order.) Then both Ψ_k and Ψ_l must divide S_j.

- Suppose first of all that $k \neq l$. Then $S_j(x)$ is reducible and, by Theorem 18 and from Table 27, the only possibilities are $j = 4$ and $\{k, l\} = \{2, 4\}$ or $j = 5$ and $\{k, l\} = \{3, 5\}$.
- But if $k = l$ then there are three possibilities:

    1. $S_j(x) = \Psi_k(x)$, in which case $S_j(x)$ is irreducible with real roots, so (from Table 27) $j = 3$ and $k = l = 5$.

$B=-P_{(12)}\,A$	Set 1 for $Z=B$		Set 2 for $Z=P\,B$		Set 3 for $Z=P^2\,B$	
$A=\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$	$Z=-\begin{bmatrix} b & a & c \\ a & c & b \\ c & b & a \end{bmatrix}$		$Z=-\begin{bmatrix} a & c & b \\ c & b & a \\ b & a & c \end{bmatrix}$		$Z=-\begin{bmatrix} c & b & a \\ b & a & c \\ a & c & b \end{bmatrix}$	
$a+b+c=1$ $a^2+b^2+c^2=1$	with $a=-\cos\alpha,\ b=-\cos\beta,\ c=-\cos\gamma$ and $\Psi_j(\cos\alpha)=0,\ \Psi_k(\cos\beta)=0,\ \Psi_l(\cos\gamma)=0.$					
$X\in[B]$	$\mathrm{tr}\,(X)$	order of X	$\mathrm{tr}\,(X)$	order of X	$\mathrm{tr}\,(X)$	order of X
Z	-1	2	-1	2	-1	2
JZ	$1-2b$	k	$1-2a$	j	$1-2c$	l
KZ	$1-2c$	l	$1-2b$	k	$1-2a$	j
LZ	$1-2a$	j	$1-2c$	l	$1-2b$	k
ZJ	$1-2b$	k	$1-2a$	j	$1-2c$	l
JZJ	-1	2	-1	2	-1	2
KZJ	$1-2a$	j	$1-2c$	l	$1-2b$	k
LZJ	$1-2c$	l	$1-2b$	k	$1-2a$	j
ZK	$1-2c$	l	$1-2b$	k	$1-2a$	j
JZK	$1-2a$	j	$1-2c$	l	$1-2b$	k
KZK	-1	2	-1	2	-1	2
LZK	$1-2b$	k	$1-2a$	j	$1-2c$	l
ZL	$1-2a$	j	$1-2c$	l	$1-2b$	k
JZL	$1-2c$	l	$1-2b$	k	$1-2a$	j
KZL	$1-2b$	k	$1-2a$	j	$1-2c$	l
LZL	-1	2	-1	2	-1	2

Table 28: The 48 distinct matrices in $[-P_{(12)}A]$ if $A\in\mathbb{D}_1$, a,b,c are all distinct; $a^2+b^2+c^2=1$; $a+b+c=1$; $\cos\alpha=-a, \cos\beta=-b, \cos\gamma=-c$; $\Psi_j(\cos\alpha)=0,\ \Psi_k(\cos\beta)=0,\ \Psi_l(\cos\gamma)=0$. All matrices in $[-P_{(12)}A]$ have finite order; α,β,γ are the angles of rotation for the 12 rotation matrices with trace $1-2a, 1-2b, 1-2c$, respectively.

2. $S_j(x) = (\Psi_k(x))^2$ in which case $S_j(x)$ is reducible with real roots, so (from Table 27) $j = 2$ and $k = l = 4$.
3. Otherwise, $S_j(x)$ is divisible by $\Psi_k(x)$ but not equal to either $\Psi_k(x)$ or $(\Psi_k(x))^2$. So by Theorem 18, $j < 6$ but $j \neq 3$. From Table 27 this is impossible, since for $j = 4$ or 5, $k \neq l$.

Finally, looking at Table 27 we can see that for the above possible combinations of j, k, l the only possible values of a, b and c are $\{a, b, c\} = \{1, 0, 0\}$ or $\left\{\frac{1}{2}, \frac{\phi}{2}, \frac{\psi}{2}\right\}$. $\square$

Completing the Proof of Conjecture 1 Without Assuming Theorem 18: as before, assume $\cos\alpha, \cos\beta$ and $\cos\gamma$ are solutions to the system of equations

$$\cos\alpha + \cos\beta + \cos\gamma = -1,\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

such that $\Psi_j(\cos\alpha) = 0, \Psi_k(\cos\beta) = 0, \Psi_l(\cos\gamma) = 0$. Then

- $S_j(\cos\beta) = 0$ and $S_j(\cos\gamma) = 0$, so both $\Psi_k(x)$ and $\Psi_l(x)$ divide $S_j(x)$,
- $S_k(\cos\alpha) = 0$ and $S_k(\cos\gamma) = 0$, so both $\Psi_j(x)$ and $\Psi_l(x)$ divide $S_k(x)$,
- $S_l(\cos\alpha) = 0$ and $S_l(\cos\beta) = 0$, so both $\Psi_j(x)$ and $\Psi_k(x)$ divide $S_l(x)$.

Case 1: suppose j, k and l are all distinct. Then Ψ_j, Ψ_k and Ψ_l are all distinct and

$$\Psi_k(x) \cdot \Psi_l(x) \,|\, S_j(x),\ \Psi_j(x) \cdot \Psi_l(x) \,|\, S_k(x), \Psi_j(x) \cdot \Psi_k(x) \,|\, S_l(x),$$

so

$$\begin{cases} \deg\Psi_k + \deg\Psi_l & \leq & \deg S_j & = & 2\deg\Psi_j \\ \deg\Psi_j + \deg\Psi_l & \leq & \deg S_k & = & 2\deg\Psi_k \\ \deg\Psi_j + \deg\Psi_k & \leq & \deg S_l & = & 2\deg\Psi_l \end{cases}.$$

The only solution to this system of inequalities is $\deg\Psi_j = \deg\Psi_k = \deg\Psi_l$, so we must have

$$S_j = \Psi_k \cdot \Psi_l,\ S_k = \Psi_j \cdot \Psi_l,\ S_l = \Psi_j \cdot \Psi_k.$$

This means all the roots of S_j, S_k and S_l are *real*. Consequently $2 \leq j, k, l \leq 5$. But Table 27 shows that for $2 \leq j, k, l \leq 5$ two of Ψ_j, Ψ_k and Ψ_l *are* the same. So Case 1 is impossible.
Case 2: suppose two of j, k, l are equal, say $k = l$. Then

- all of $\cos\alpha, \cos\beta$ and $\cos\gamma$ are roots of S_k, so both Ψ_j and $\Psi_l = \Psi_k$ divide S_k.
- both $\cos\beta$ and $\cos\gamma$ are roots of S_j, so Ψ_k divides S_j.

In this case we only get two inequalities,

$$\deg\Psi_j + \deg\Psi_k \leq \deg S_k = 2\deg\Psi_k \Leftrightarrow \deg\Psi_j \leq \deg\Psi_k \text{ and } \deg\Psi_k \leq \deg S_j = 2\deg\Psi_j,$$

from which we can conclude $\deg\Psi_j \leq \deg\Psi_k \leq 2\deg\Psi_j$.

Consider the extremes first. If $\deg\Psi_j = \deg\Psi_k$, then

$$\Psi_j \cdot \Psi_k \mid S_k \text{ and } \deg(\Psi_j \cdot \Psi_k) = \deg\Psi_j + \deg\Psi_k = 2\deg\Psi_k = \deg S_k.$$

Therefore $\Psi_j \cdot \Psi_k = S_k$, which means the roots of S_k are all real. So $k \leq 5$ and the only possibility, from Table 27, is $j = 2$ and $k = 4$. If, on the other hand, $\deg\Psi_k = 2\deg\Psi_j$, then

$$\Psi_k \mid S_j \text{ and } \deg\Psi_k = 2\deg\Psi_j = \deg S_j.$$

Therefore $\Psi_k = S_j$, which means the roots of S_j are all real. So $j \leq 5$ and the only possibility, from Table 27, is $j = 3$ and $k = 5$.

In general, though, it is simpler to consider the divisions listed above:

$$\Psi_k \text{ divides } S_j,\ \Psi_k \text{ divides } S_k,\ \Psi_j \text{ divides } S_k.$$

In particular, if Ψ_k divides S_k then the root $x = \cos(2\pi/k)$ of Ψ_k must satisfy one of the quadratic factors of S_k. That is, there must be an integer p such that $(p, k) = 1,\ 1 \le p < k/2$ and $\texttt{quad}(\cos(2\pi/k), p, k) = 0$. That is,

$$\begin{aligned}
&\cos^2(2\pi/k) + \cos(2\pi/k)\cos(2\pi p/k) + \cos(2\pi/k) + \cos^2(2\pi p/k) + \cos(2\pi p/k) = 0\\
&\Leftrightarrow \cos^2\theta + \cos\theta\cos(p\,\theta) + \cos\theta + \cos^2(p\,\theta) + \cos(p\,\theta) = 0,\ \text{for } \theta = 2\pi/k,\\
&\Leftrightarrow \cos^2\theta + \cos\theta\, T_p(\cos\theta) + \cos\theta + (T_p(\cos\theta))^2 + T_p(\cos\theta) = 0,\\
&\Leftrightarrow x^2 + x\,T_p(x) + x + (T_p(x))^2 + T_p(x) = 0,\ \text{for } x = \cos\theta,
\end{aligned}$$

where T_p is a Chebyshev polynomial of the first kind. Moreover,

$$1 \le p < \frac{k}{2} \Rightarrow 2 \le 2p < k \Rightarrow \frac{1}{2} \ge \frac{1}{2p} > \frac{1}{k} \Rightarrow \pi \ge \frac{\pi}{p} > \frac{2\pi}{k} \Rightarrow \cos\pi \le \cos\frac{\pi}{p} < \cos\frac{2\pi}{k}.$$

That is, $x > \cos(\pi/p)$, which is a severe restriction on x. Should there be a solution x then we must find a positive integer k such that $(k, p) = 1, k > 2p$ and $\cos(2\pi/k) = x$, if possible. Solving the equation we find:

$$\begin{aligned}
x^2 + x\,T_p(x) + x + (T_p(x))^2 + T_p(x) = 0 \quad &\Rightarrow \quad T_p(x) = \frac{-x - 1 \pm \sqrt{-3x^2 - 2x + 1}}{2}\\
&\Rightarrow \quad -1 \le x \le \frac{1}{3},\ \text{since } T_p : \mathbb{R} \longrightarrow \mathbb{R}\\
\text{(using restriction from above)} \quad &\Rightarrow \quad \cos\frac{\pi}{p} < x \le \frac{1}{3}\\
&\Rightarrow \quad p = 1 \text{ or } 2,\ \text{if there is to be any solution for } x.
\end{aligned}$$

We now use *Mathematica* to solve the above equation for x in the two possible cases:

- $p = 1 : x = -2/3$ or $x = 0$. In the first case, $2\pi/k = \arccos(-2/3)$, for which there is no integer solution. In the second case, the only positive integer solution is $k = 4$. This corresponds to the triple $(j, k, l) = (2, 4, 4)$.
- $p = 2$: the only solution is $x = -\frac{\psi}{2}$, and the only positive integer solution is $k = 5$. This corresponds to the triple $(j, k, l) = (3, 5, 5)$.

Case 3: now suppose $j = k = l$. So $\cos\alpha, \cos\beta$ and $\cos\gamma$ are roots of the same polynomial Ψ_j. In particular, $k = l$, so we can use the result of Case 2. The only two solutions from Case 2, with $k = l$, have

$$\Psi_4(\cos\beta) = 0,\ \Psi_4(\cos\gamma) = 0,\ \text{but } \Psi_2(\cos\alpha) = 0$$

or

$$\Psi_5(\cos\beta) = 0,\ \Psi_5(\cos\gamma) = 0,\ \text{but } \Psi_3(\cos\alpha) = 0.$$

Thus Case 3 is impossible.

Putting it all together we conclude that the only matrices in $B \in \mathbb{D}_2$ for which all matrices in $[B]$ have finite order are

- $-P_{(12)}, -P_{(23)}$ or $-P_{(13)}$, for which $[-P_{(12)}] = [-P_{(23)}] = [-P_{(13)}]$,
- $-S, -P_{(123)}S$ or $-P_{(132)}S$, for which $[-S] = [-P_{(123)}S] = [-P_{(132)}S]$,
- $-S', -P_{(123)}S'$ or $-P_{(132)}S'$, for which $[-S'] = [-P_{(123)}S'] = [-P_{(132)}S']$,

where S and S' are as given in Conjecture 1. □