Asymptotically improved solvers for the variable coefficient Helmholtz equation

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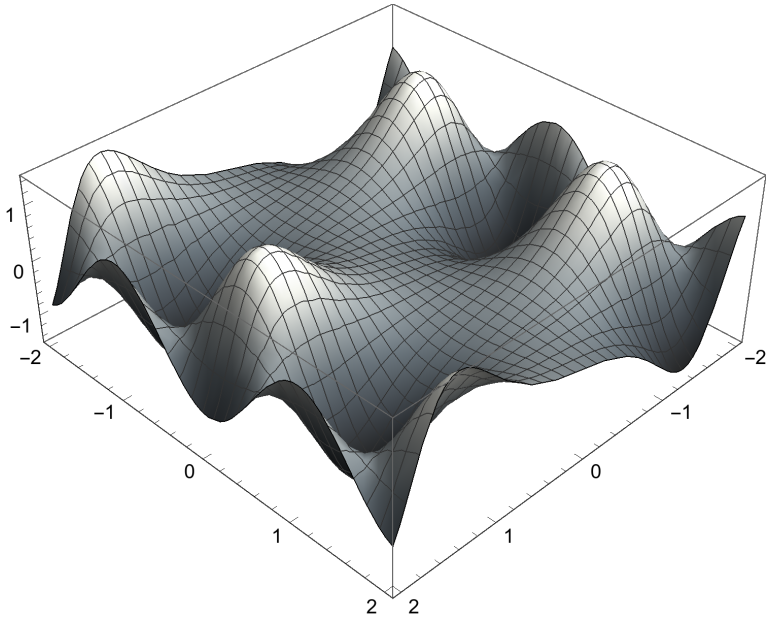
The variable coefficient Helmholtz equation

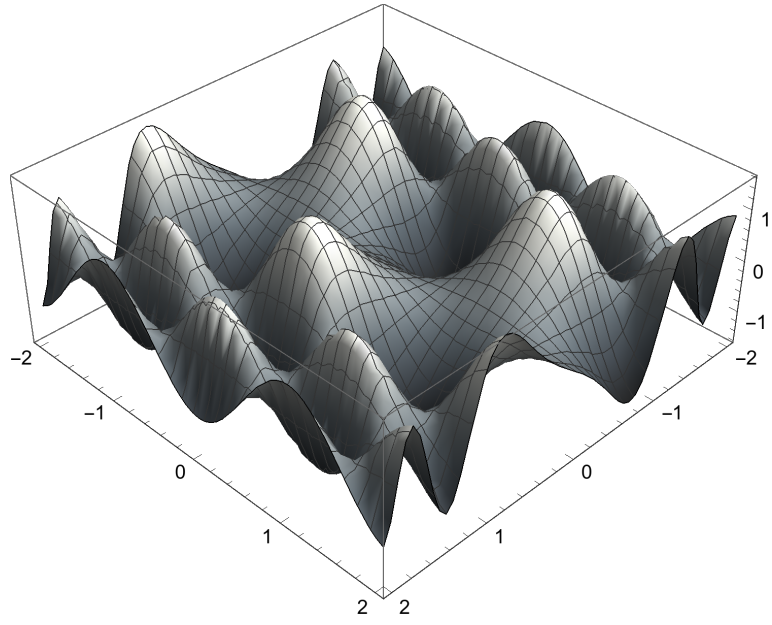
$$\Delta u(x) + k^2 q(x) u(x) = 0$$

can be used to model the scattering of waves from inhomogeneous media.

When the coefficient q is piecewise smooth and positive, its solutions are oscillatory with the wavenumber k controlling their frequency of oscillation.



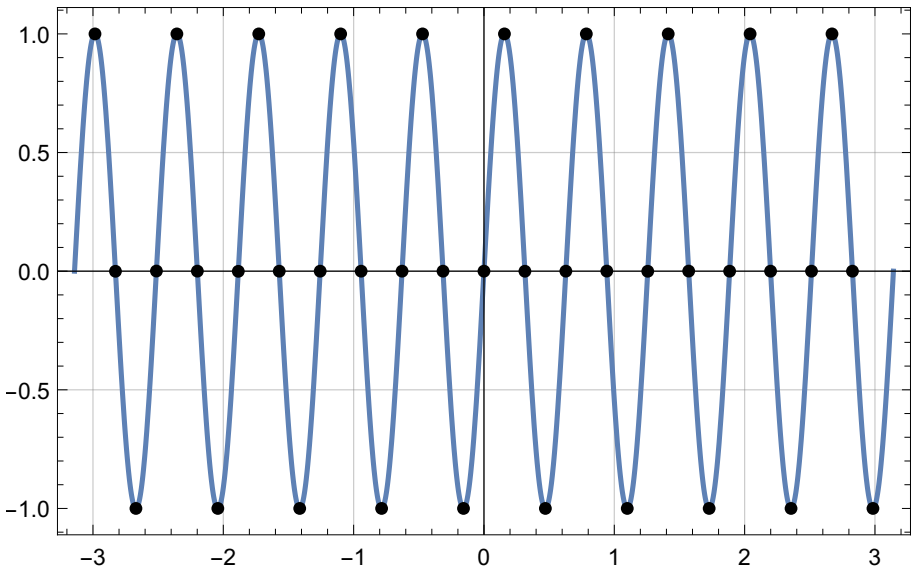
A solution $u(x_1, x_2)$ when
$q(x_1, x_2) = x_1^2/4, \quad k = 6$



A solution $u(x_1, x_2)$ when
$q(x_1, x_2) = x_1^2/4, \quad k = 12$

Problem

In d dimensions, discretizing the solutions of the variable coefficient Helmholtz equation using standard techniques (e.g., collocation, orthogonal polynomial bases, finite element bases) requires $\mathcal{O}\left(k^d\right)$ points.



This is a consequence of the Nyquist sampling theorem.

Problem

In d dimensions, discretizing the solutions of the variable coefficient Helmholtz equation using standard techniques (e.g., collocation, orthogonal polynomial bases, finite element bases) requires $\mathcal{O}\left(k^d\right)$ points.

One consequence of this is that any solver using such methods to represent solutions must have a running time which grow at least as fast as $\mathcal{O}\left(k^d\right)$.

This makes them prohibitively expensive in the high-frequency regime, when the wavenumber k is large.

Central observation

In certain cases and under mild conditions on the coefficient q, the variable coefficient Helmholtz equation

$$\Delta u(x) + k^2 q(x) u(x) = 0$$

admits solutions whose **logarithms are nonoscillatory**, even though the solutions themselves are highly oscillatory.

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- One spatial dimension
- Radially symmetric coefficients in two spatial dimensions
- **Conjecture:** true for general (nonsymmetric) coefficients in two dimensions

Solution

In those cases, we can solve the variable coefficient Helmholtz equation by constructing a basis in its space of solutions whose **logarithms are nonoscillatory functions**. We then represent the desired (oscillatory) solutions — which is usually specified via boundary conditions — with respect to this basis.

This allows us to construct extremely efficient solvers whose asymptotic running times are grow more slowly than $\mathcal{O}\left(k^d\right)$.

Outline of this talk

I will describe three solvers based on this idea:

- A solver for the one-dimensional problem which runs in time independent of the wavenumber k
- A solver for the Dirichlet problem in the case of a **radially symmetric** coefficient in two dimensions whose running time is a $\mathcal{O}(\log(k)m)$ where m is a measure of the complexity of the boundary data
- A method for simulating scattering from a **radially symmetric** potential in two spatial dimensions which runs in $\mathcal{O}(k\log(k))$ time

The variable coefficient Helmholtz equation in one dimension

If $u(x) = \exp(\psi(x))$ satisfies the Helmholtz equation

$$u''(x) + k^2 q(x) u(x) = 0,$$

then ψ is a solution of the **Riccati equation**

$$\psi''(x) + (\psi'(x))^2 + k^2 q(x) = 0.$$

The one-dimensional solver operates by numerically constructing a nonoscillatory solution of the Riccati equation.

WKB Approximation

There are many asymptotic methods for the Helmholtz equation that operate by constructing a nonoscillatory function which approximates a solution of the Riccati equation.

For instance, the WKB approximations of the solutions of

$$u''(x) + k^2 q(x) u(x) = 0$$

are obtained by inserting the ansatz

$$\psi'(x) \approx \sum_{n=-1}^{\infty} k^{-n} \alpha_n(x)$$

into the Riccati equation

$$\psi''(x) + \left(\psi'(x)\right)^2 + k^2 q(x) = 0.$$

WKB Approximation

There is a solution ψ of the Riccati equation such that

$$\psi(x) = \sum_{n=-1}^{N-1} k^{-n} \int_{x_0}^{x} \alpha_n(t)\, dt + \mathcal{O}\left(k^{-N}\right)$$

with

$$\begin{aligned}
\alpha_{-1}(x) &= i\sqrt{q(x)} \\
\alpha_0(x) &= -\frac{q'(x)}{4q(x)} \\
\alpha_1(x) &= -\frac{i}{8}\frac{q''(x)}{(q(x))^{3/2}} + \frac{5i}{32}\frac{(q'(x))^2}{(q(x))^{5/2}} \\
&\vdots \\
\alpha_{n+1}(x) &= \frac{i}{2\sqrt{q(x)}}\left(\alpha_n'(x) + \sum_{j=0}^{n}\alpha_j(x)\alpha_{n-j}(x)\right) \\
&\vdots
\end{aligned}$$

WKB Approximation

There is a solution u of the Helmholtz equation u such that

$$u(x) = \exp\left(\sum_{n=-1}^{N-1} k^{-n} \int_{x_0}^{x} \alpha_n(t)\, dt\right)\left(1 + \mathcal{O}\left(k^{-N}\right)\right)$$

with

$$\begin{aligned}
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&\vdots
\end{aligned}$$

WKB Approximation

The expansion

$$\psi(x) \approx \sum_{n=-1}^{N-1} k^{-n} \int_{x_0}^{x} \alpha_n(t)\, dt$$

used to approximate a solution of the Riccati equation do not depend on k, but only on q and its derivatives. In particular, they are nonoscillatory in the sense that they can be evaluated in time independent of k.

But while WKB approximations and related methods are excellent theoretical tools, they do not lead to viable numerical methods. Indeed, they have at least three critical flaws:

- The coefficients in high order expansions involve high order derivatives of q
- These expansions is asymptotic in nature rather than convergent
- $\mathcal{O}\left(\frac{1}{k^n}\right)$ accuracy at $\mathcal{O}(n)$ cost

An improved result

When q is positive, splitting the Riccati equation into real and imaginary components shows that its solutions must be of the form

$$\psi(x) = i\alpha(x) - \frac{1}{2}\log\left(\alpha'(x)\right)$$

with α a real-valued solution of **Kummer's equation**

$$(\alpha'(x))^2 = k^2 q(x) + \frac{3}{4}\left(\frac{\alpha''(x)}{\alpha'(x)}\right)^2 - \frac{1}{2}\frac{\alpha'''(x)}{\alpha'(x)}.$$

The functions

$$\frac{\cos\left(\alpha(x)\right)}{\sqrt{\alpha'(t)}} \quad \text{and} \quad \frac{\sin\left(\alpha(x)\right)}{\sqrt{\alpha'(t)}}$$

form a basis in the space of solutions of the Helmholtz equation and we call α a **phase function** for the Helmholtz equation.

An improved result

Under mild conditions on q, there exist a solution α of Kummer's equation and a function α_0 which is roughly as oscillatory as the coefficient q such that

$$\alpha(x) = \alpha_0(x) + \mathcal{O}\left(\exp(-Ck)\right).$$

More explicitly, if, after a suitable transformation is applied, q has a rapidly decaying Fourier transform, then, after a similar transformation is applied, α_0 has a Fourier transform which decays at roughly the same rate.

In particular, α_0 can be represented using a number of discretization nodes which does not increase with k so that we get $\mathcal{O}\left(\exp(-Ck)\right)$ accuracy with $\mathcal{O}\left(1\right)$ cost.

An improved result

Under mild conditions on q, there exist a solution α of Kummer's equation and a function α_0 which is roughly as oscillatory as the coefficient q such that

$$\alpha(x) = \alpha_0(x) + \mathcal{O}\left(\exp(-Ck)\right).$$

Although this improved efficiency is nice, it is perhaps more important that there is a robust numerical method for computing the phase function α.

Note that we do not compute the function α_0! We exploit the fact that the exponential decay in the error means that α is nonoscillatory from the point of view of numerical analysis even at quite small values of k.

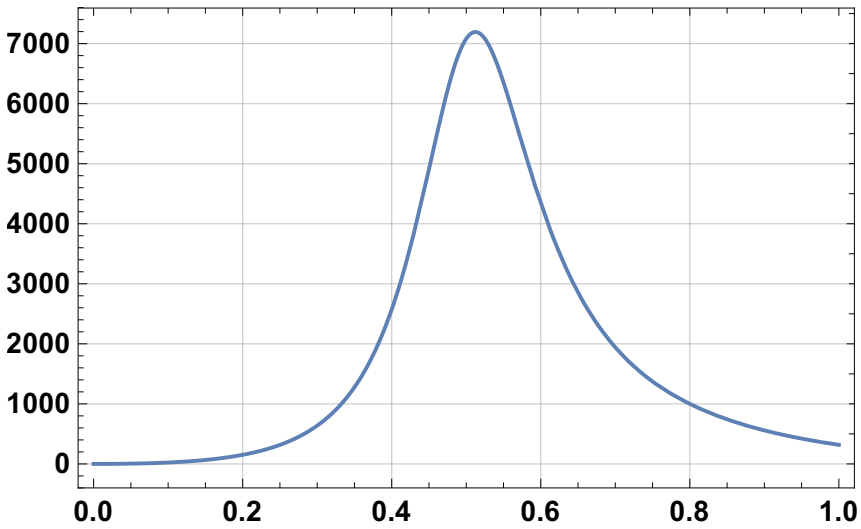
Nonoscillatory phase functions

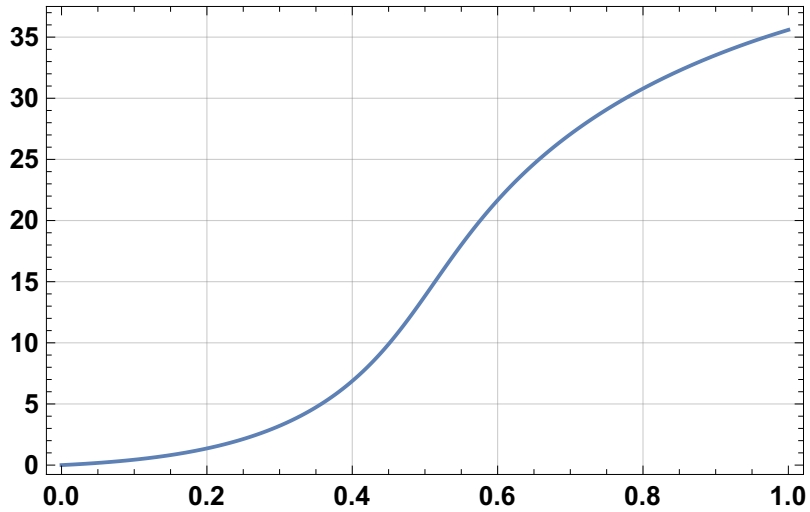
Under mild conditions on q, including that it is smooth and positive, the second order differential equation

$$u''(x) + k^2 q(x) u(x) = 0$$

admits a phase function which is roughly as oscillatory as q.

$$q(x) = \frac{\sin(2x)^2}{0.1 + (x - 0.5)^2} \quad \text{with } k = 10$$





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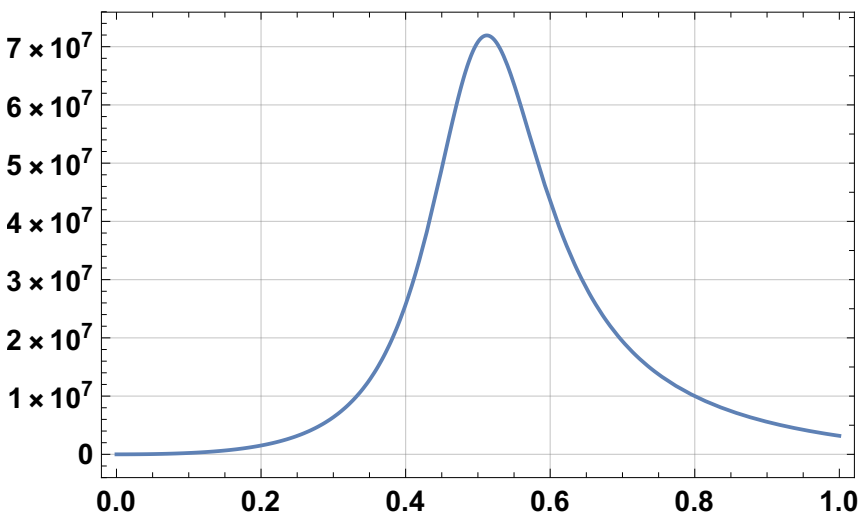

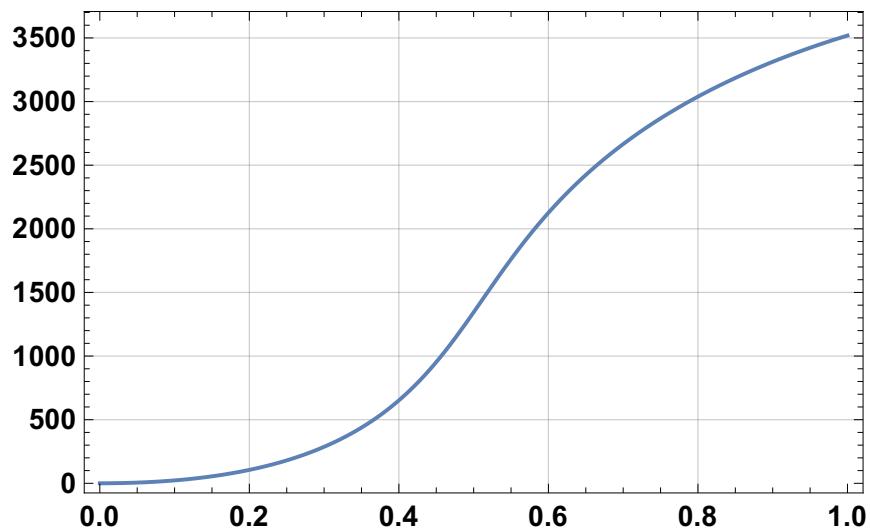

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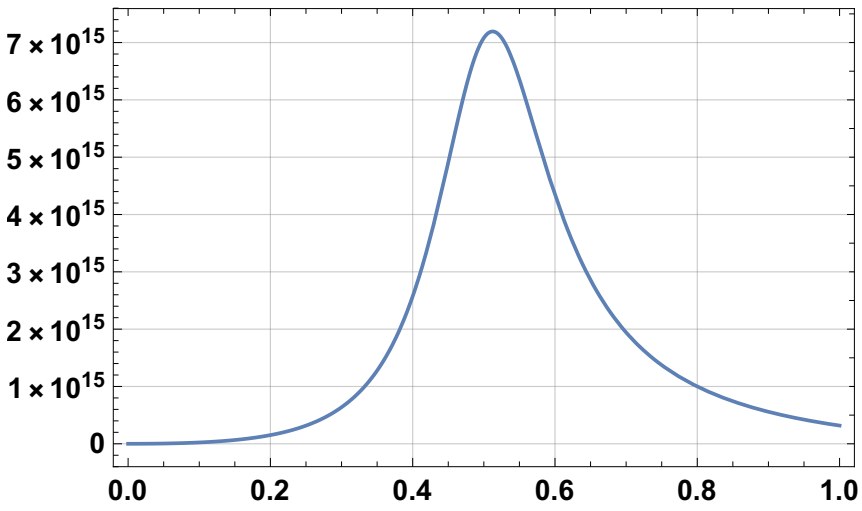
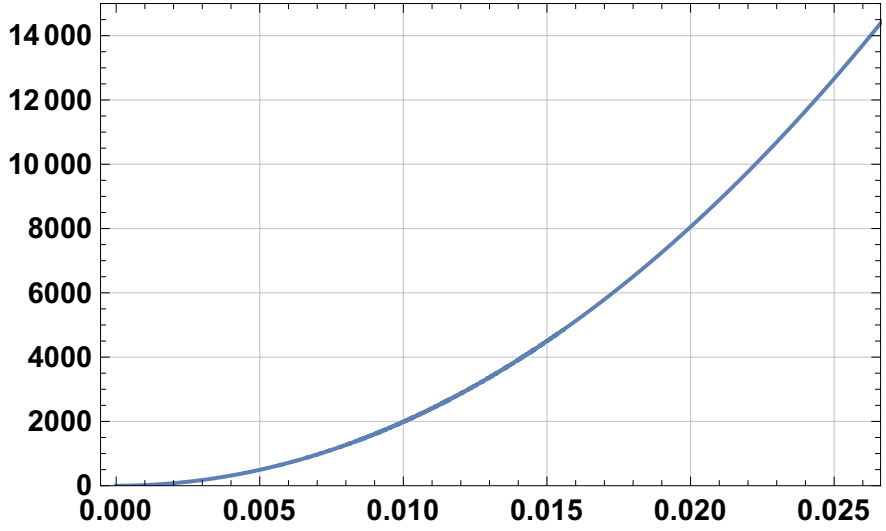
$$q(x) = \frac{\sin(2x)^2}{0.1 + (x - 0.5)^2} \quad \text{with } k = 10^7$$





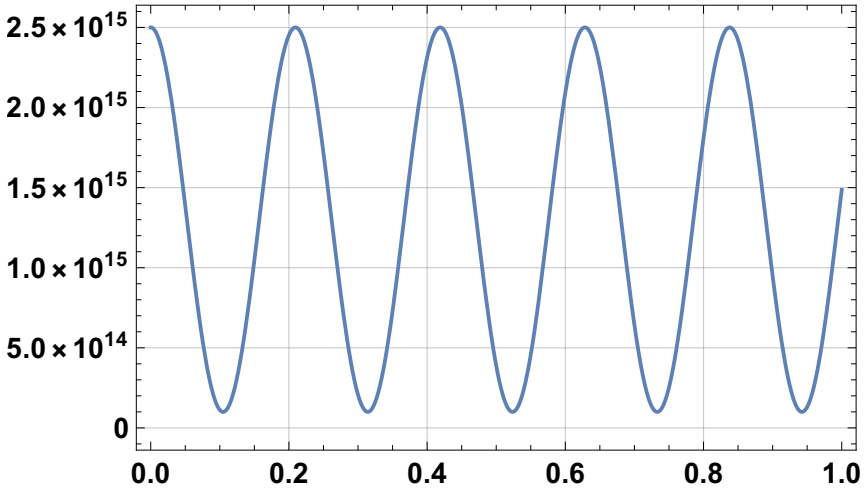
Nonoscillatory phase functions

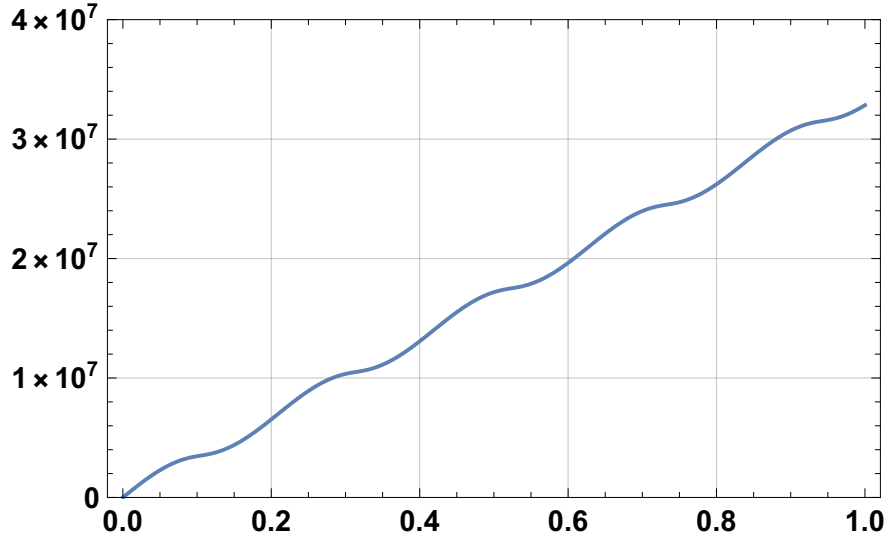
Under mild conditions on q, including that it is smooth and positive, the second order differential equation

$$u''(x) + k^2 q(x) u(x) = 0$$

admits a phase function which is roughly as oscillatory as q.

$$q(x) = (13 + 12\cos(30x)) \quad \text{with } k = 10^7$$





A numerical algorithm for the calculation of nonoscillatory phase functions

In order to construct a nonoscillatory phase function α for the Helmholtz equation

$$u''(x) + q(x)u(x) = 0,$$

I solve Kummer's equation numerically

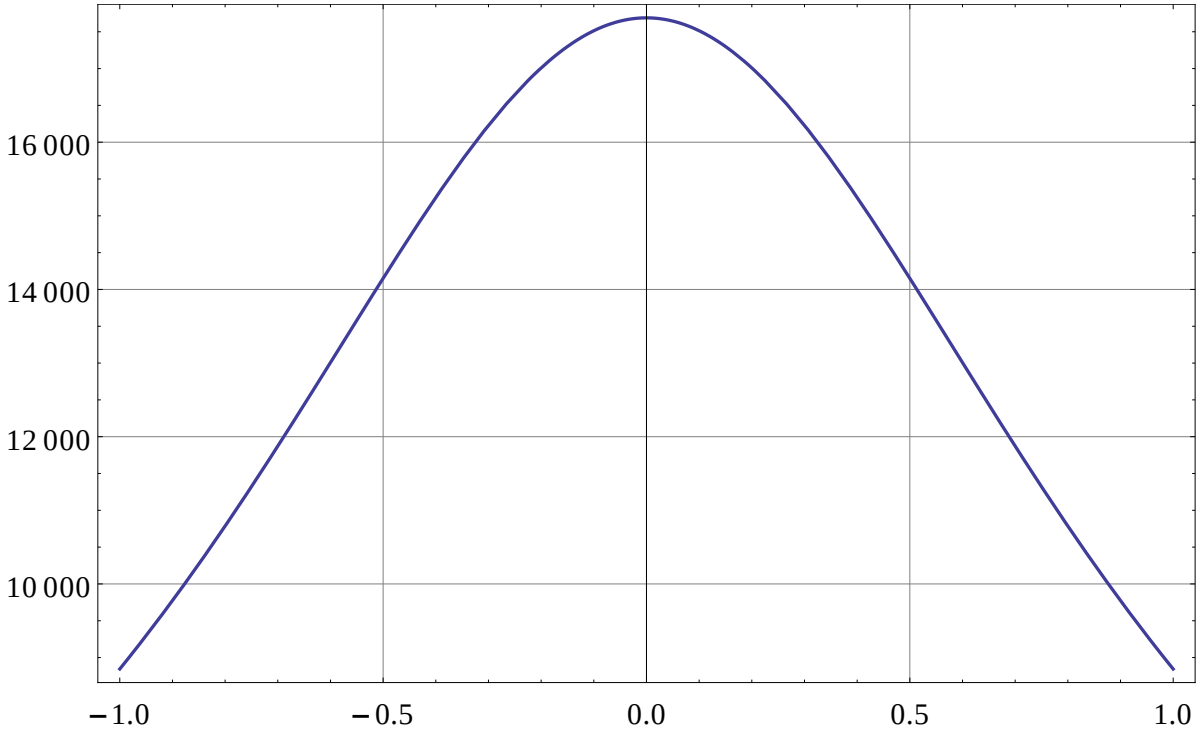
$$\left(\alpha'(x)\right)^2 = q(x) - \frac{1}{2}\frac{\alpha'''(x)}{\alpha'(x)} + \frac{3}{4}\left(\frac{\alpha''(x)}{\alpha'(x)}\right)^2.$$

Of course, most of its solutions are highly oscillatory. In order to choose a solution of Kummer's equation which is not, I will rely on the fact that when $q(x) = \eta^2$,

$$\alpha(x) = \eta x$$

is a nonoscillatory phase function for the Helmholtz equation.

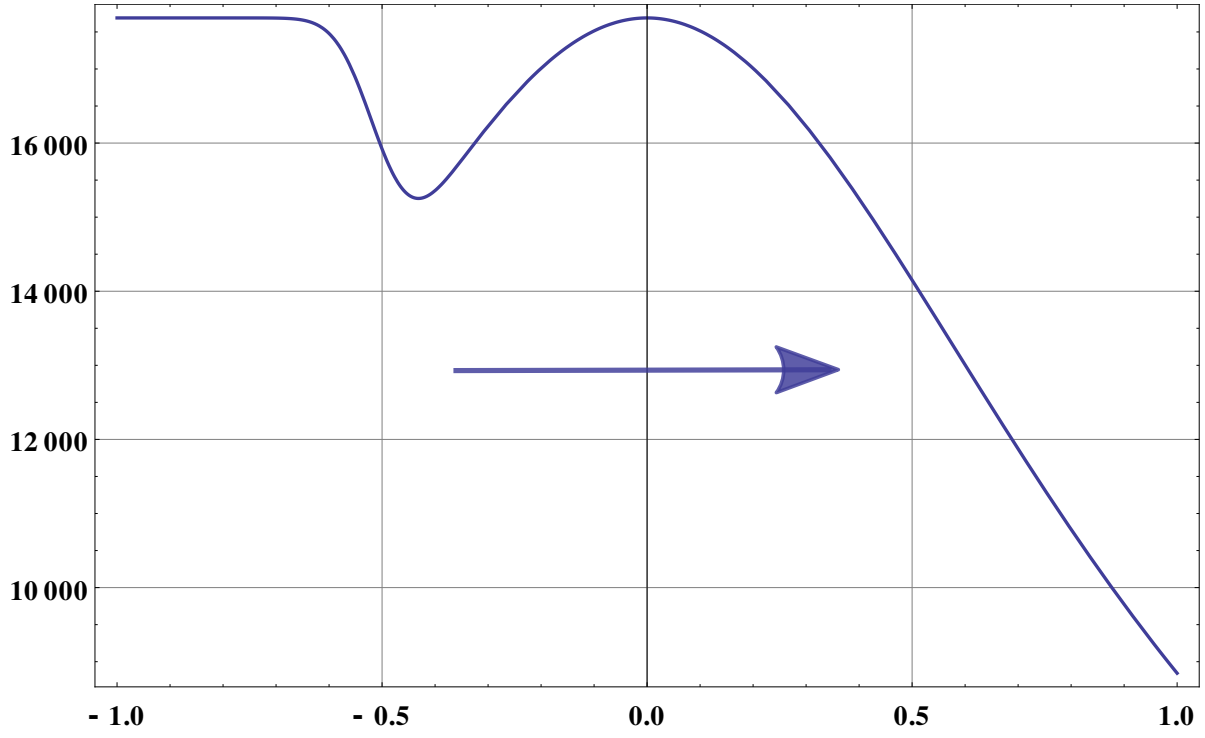
A numerical algorithm for the calculation of nonoscillatory phase functions



The original coefficient $q(x)$ on the interval $[a, b]$.

$$\left(\alpha'(x)\right)^2 = q(x) - \frac{1}{2}\frac{\alpha'''(x)}{\alpha'(x)} + \frac{3}{4}\left(\frac{\alpha''(x)}{\alpha'(x)}\right)^2$$

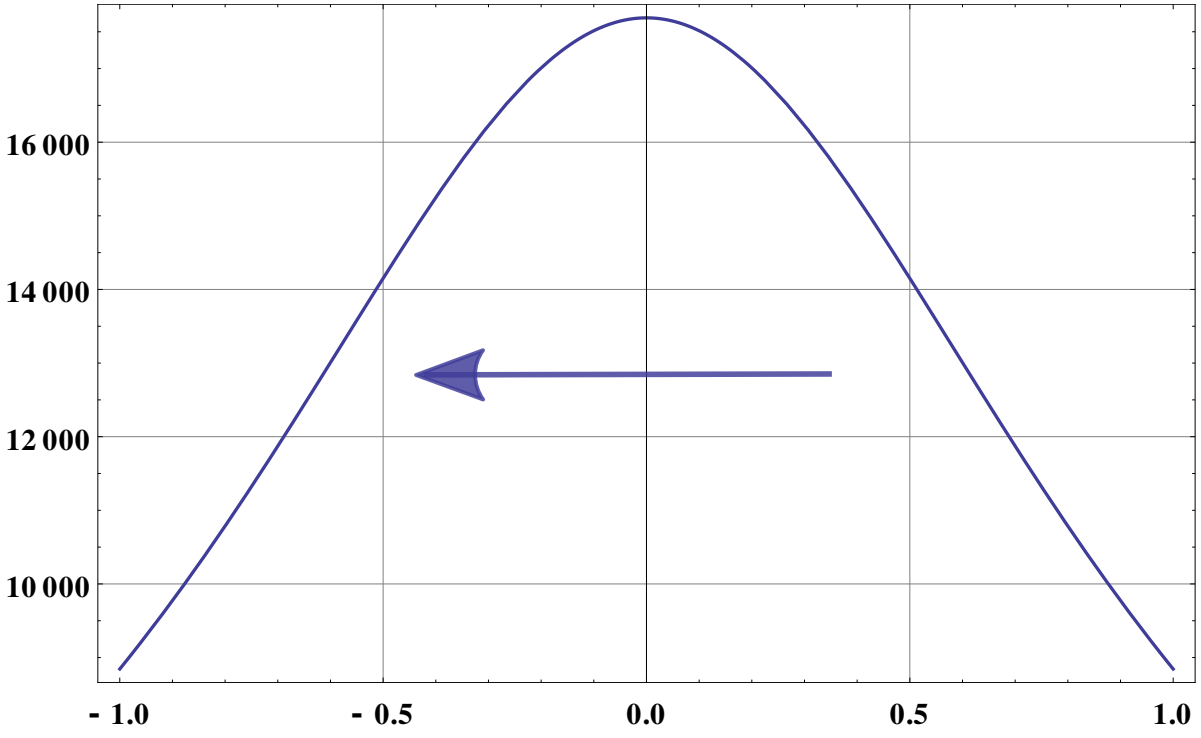
A numerical algorithm for the calculation of nonoscillatory phase functions



A windowed version $\tilde{q}(x)$ of $q(x)$ which is equal to η^2 near the left endpoint a of the interval and equal to $q(x)$ near the right endpoint b of the interval.

$$\begin{cases} \left(\tilde{\alpha}'(x)\right)^2 = \tilde{q}(x) - \dfrac{1}{2}\dfrac{\tilde{\alpha}'''(x)}{\tilde{\alpha}'(x)} + \dfrac{3}{4}\left(\dfrac{\tilde{\alpha}''(x)}{\tilde{\alpha}'(x)}\right)^2 \\ \tilde{\alpha}'(a) = \eta \\ \tilde{\alpha}''(a) = 0 \end{cases}$$

A numerical algorithm for the calculation of nonoscillatory phase functions



The original coefficient q.

$$\begin{cases} \left(\alpha'(x)\right)^2 = q(x) - \dfrac{1}{2}\dfrac{\alpha'''(x)}{\alpha'(x)} + \dfrac{3}{4}\left(\dfrac{\alpha''(x)}{\alpha'(x)}\right)^2 \\ \alpha'(b) = \tilde{\alpha}'(b) \\ \alpha''(b) = \tilde{\alpha}''(b) \end{cases}$$

Numerical results for the one-dimensional solver

$$q(x) = \frac{k^2 \sin(2x)^2}{0.1 + (x - 0.5)^2}$$

k	Phase function construction time	Average time to evaluate solution	Maximum absolute error
10^1	1.79×10^{-03}	5.51×10^{-07}	6.39×10^{-14}
10^2	3.10×10^{-03}	5.39×10^{-07}	3.51×10^{-13}
10^3	2.90×10^{-03}	5.39×10^{-07}	6.16×10^{-12}
10^4	3.06×10^{-03}	5.61×10^{-07}	3.78×10^{-11}
10^5	3.80×10^{-03}	4.77×10^{-07}	1.83×10^{-10}
10^6	2.91×10^{-03}	5.12×10^{-07}	1.33×10^{-09}
10^7	4.33×10^{-03}	5.38×10^{-07}	5.66×10^{-09}
10^8	3.46×10^{-03}	4.92×10^{-07}	2.54×10^{-08}
10^9	1.87×10^{-03}	5.22×10^{-07}	7.32×10^{-07}

The obtained accuracy is on the order of ϵk with ϵ equal to machine precision. This is close to the optimal level of accuracy which can be achieved.

When $\sin(kx)$ is evaluated numerically, the optimal obtainable accuracy is roughly $k\epsilon$ with ϵ equal to machine precision.

The Dirichlet problem in two spatial dimensions

A similar approach can be used to solve the variable coefficient Helmholtz equation in two dimensions.

I will first illustrate it using the relatively simple boundary value problem

$$\begin{cases} \Delta u(x) + k^2 q(x) u(x) = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^2 \\ u(x) = f(x) \quad \text{on} \quad \partial\Omega, \end{cases}$$

where:

- the coefficient q is piecewise smooth and positive
- Ω is the disk of radius R centered at 0

For a countable number of values of k (called resonant frequencies), this problem is ill-posed. I will shortly discuss a different boundary value problem that does not suffer from this difficulty and which is more relevant in applications.

The Dirichlet problem in two spatial dimensions

The first step of the algorithm is to calculate the truncated Fourier expansion of the boundary data:

$$f(R\exp(i\theta)) \approx \sum_{n=-m}^{m} a_n \exp(in\theta).$$

Next, for each integer $n = -m, -m+1, \ldots, m$ we construct a solution u_n of the Helmholtz equation such that

$$u_n\left(R\exp(i\theta)\right) = \exp(in\theta).$$

The desired solution of the Helmholtz equation is then

$$u(x) = \sum_{n=-m}^{m} a_n u_n(x).$$

The Dirichlet problem in two spatial dimensions

The trick is, of course, that we construct u_n by solving the Riccati equation

$$\Delta\psi(x) + \nabla\psi(x) \cdot \nabla\psi(x) + k^2 q(x) = 0$$

satisfied by the logarithms of the solutions of the Helmholtz equation.

In particular, for each n, we find a solution ψ_n of the Riccati equation such that $u_n(x) = \exp(\psi_n(x))$ has the desired behavior on the boundary:

$$u_n(R\exp(i\theta)) = \exp(in\theta).$$

The cost of this algorithm is on the order of

$$\underbrace{m\log(m)}_{\text{FFT}} \quad + \quad \underbrace{m \cdot (\text{Cost of solving the Riccati equation})}_{\text{Compute basis}}$$

The Dirichlet problem in the case of a radially symmetric coefficient

The method of separation of variables shows that if u satisfies

$$\Delta u(x) + k^2 q(r) u(x) = 0$$

then it admits an expansion of the form

$$u(r\exp(i\theta)) = \sum_{n=-\infty}^{\infty} a_n \underbrace{\varphi_{|n|}(r)\exp(in\theta)}_{u_n}$$

where, for each nonnegative integer n, $\varphi_n(r)\sqrt{r}$ is a solution of the second order differential equation

$$y''(r) + \left(k^2 q(r) + \frac{\frac{1}{4} - n^2}{r^2} \right) y(r) = 0.$$

The Dirichlet problem in the case of a radially symmetric coefficient

We cannot just solve this differential equation using the approach described earlier because, except possibly when $n = 0$, the coefficient in the equation

$$\varphi_n''(r) + \left(k^2 q(r) + \frac{\frac{1}{4} - n^2}{r^2} \right) \varphi_n(r) = 0,$$

changes sign (i.e., the differential equation has turning points).

The results I mentioned earlier apply in the case of coefficients which are positive. Near turning points, the behavior of phase functions is somewhat more complicated.

The Dirichlet problem in the case of a radially symmetric coefficient

This isn't a significant problem, though — the cost of solving

$$y''(r) + \left(k^2 q(r) + \frac{\frac{1}{4} - n^2}{r^2} \right) y(r) = 0.$$

only grows logarithmically with k.

So the running time of our algorithm is on the order of

$$\underbrace{m \log(m)}_{\text{FFT}} + \underbrace{m \log(k)}_{\text{Compute basis}}$$

The Dirichlet problem in the case of a radially symmetric coefficient

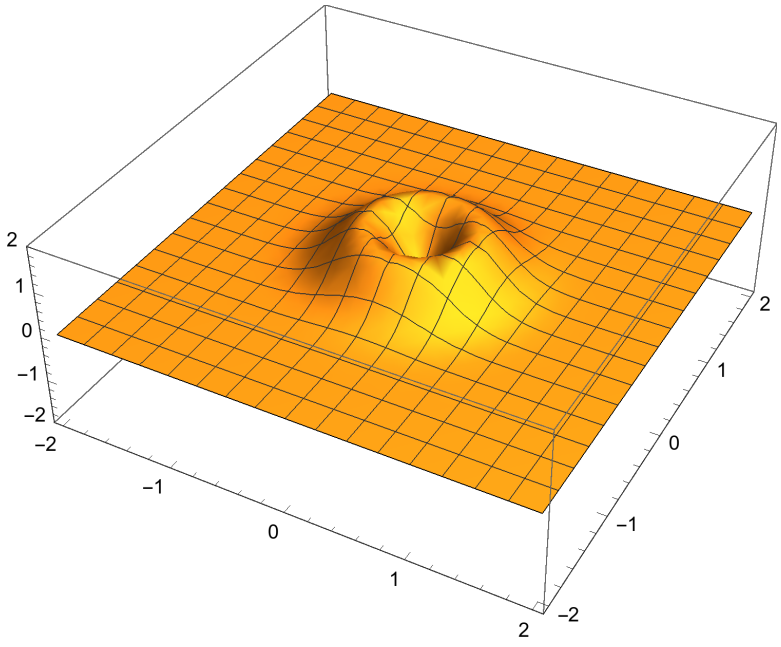
This isn't a significant problem, though — the cost of solving

$$y''(r) + \left(k^2 q(r) + \frac{\frac{1}{4} - n^2}{r^2} \right) y(r) = 0.$$

only grows logarithmically with k.

In most physically relevant cases (i.e., scattering problems), $m = \mathcal{O}(k)$ so that the running time is on the order of

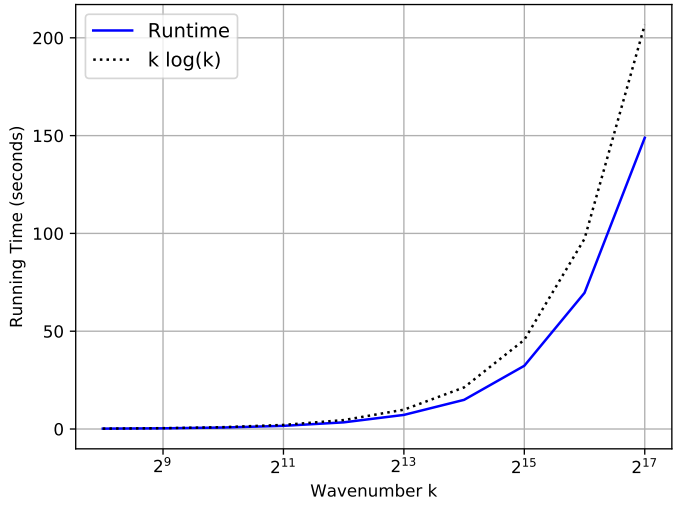
$$\underbrace{k \log(k)}_{\text{FFT}} + \underbrace{k \log(k)}_{\text{Compute basis}}$$



$$\begin{cases} \Delta u(x) + k^2 q(x) u(x) = 0 \\ u(x) = f(x) \end{cases}$$

$$q(r) = 14 r^2 \exp(-5r^2)$$

$$f(x) = H_0(k|x-4|)$$



k	m	Maximum absolute error	Running time (in seconds)
2^8	603	1.85×10^{-13}	1.89×10^{-01}
2^9	1206	1.63×10^{-12}	3.54×10^{-01}
2^{10}	2412	1.85×10^{-12}	7.57×10^{-01}
2^{11}	4825	7.16×10^{-12}	$1.61\times10^{+00}$
2^{12}	9650	2.73×10^{-11}	$3.40\times10^{+00}$
2^{13}	19301	6.80×10^{-11}	$7.20\times10^{+00}$
2^{14}	38603	4.16×10^{-10}	$1.49\times10^{+01}$
2^{15}	77207	6.92×10^{-10}	$3.23\times10^{+01}$
2^{16}	154415	3.94×10^{-09}	$6.96\times10^{+01}$
2^{17}	308831	5.74×10^{-09}	$1.48\times10^{+02}$

Scattering from a radially symmetric potential in two dimensions

The boundary problem

$$\begin{cases} \Delta u(x) + k^2 (1 + q(x))\, u(x) = 0 \quad \text{for all } x \in \mathbb{R}^2 \\ u = u_i + u_s \\ \Delta u_i(x) + k^2 u_i(x) = 0 \quad \text{for all } x \in \mathbb{R}^2 \\ \lim\limits_{r \to \infty} \sup\limits_{0 \le \theta \le 2\pi} \sqrt{r} \left| \dfrac{\partial u_s}{\partial r}(r \exp(i\theta)) \; - iku_s(r \exp(i\theta)) \right| = 0 \end{cases}$$

is used to model the scattering of an incoming wave u_i from an inhomogeneous medium.

The function u, which is called the total field, is the sum of the known incident wave u_i and the scattered field u_s, which is to be determined.

We will assume that the scattering potential q is a piecewise smooth and **radially symmetric** with compact support in the disk Ω of radius R centered at 0.

Scattering from a radially symmetric potential in two dimensions

The scattered field u_s is uniquely determined by the boundary value problem

$$\begin{cases} \Delta u_s(x) + k^2 (1 + q(x)) u_s(x) = -k^2 q(x) u_i(x) \quad \text{for all} \quad x \in \mathbb{R}^2 \\ \lim_{r \to \infty} \sup_{0 \le \theta \le 2\pi} \sqrt{r} \left| \frac{\partial u_s}{\partial r}(r \exp(i\theta)) - iku_s(r \exp(i\theta)) \right| = 0, \end{cases}$$

which can be easily obtained from the preceding formulation of the problem.

In particular, since q has support contained in Ω, u_s satisfies the constant coefficient Helmholtz equation

$$\Delta u_s(x) + k^2 q(x) u_s(x) = 0$$

in the exterior of Ω.

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Because u_s satisfies the constant coefficient Helmholtz equation in the exterior of Ω and the Sommerfeld condition, it admits the expansion

$$u_s(r\exp(i\theta)) = \sum_{n=-\infty}^{\infty} b_n H_n(kr)\exp(in\theta)$$

in the exterior of Ω. Here, H_n is the Hankel function of order n.

The total field u, which satisfies the variable coefficient Helmholtz equation in $\mathbb{R}^2$, can be represented via the expansion

$$u(r\exp(i\theta)) = \sum_{n=-\infty}^{\infty} a_n \underbrace{\varphi_{|n|}(r)\exp(in\theta)}_{u_n}$$

where, for each nonnegative integer n, $\varphi_n(r)\sqrt{r}$ is a solution of

$$y''(r) + \left(k^2\left(1+q(r)\right) + \frac{\frac{1}{4}-n^2}{r^2}\right) y(r) = 0.$$

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In the interior of Ω:

$$u(r\exp(i\theta)) = \sum_{n=-\infty}^{\infty} a_n \varphi_{|n|}(kr)\exp(in\theta)$$

In the exterior of Ω:

$$u(r\exp(i\theta)) = \underbrace{\sum_{n=-\infty}^{\infty} b_n H_n(r)\exp(in\theta)}_{u_s(r\exp(i\theta))} + u_i(r\exp(i\theta))$$

Scattering from a radially symmetric potential in two dimensions

By standard elliptic regularity results, u and its radial derivative $\dfrac{\partial u}{\partial r}$ must be continuous across the boundary $\partial\Omega$:

$$\underbrace{\sum_{n=-\infty}^{\infty} a_n \varphi_{|n|}(kR)\exp(in\theta)}_{u(R\exp(i\theta))} = \underbrace{\sum_{n=-\infty}^{\infty} b_n H_n(R)\exp(in\theta)}_{u_s(R\exp(i\theta))} + u_i(R\exp(i\theta))$$

$$\underbrace{\sum_{n=-\infty}^{\infty} a_n \varphi'_{|n|}(R)\exp(in\theta)}_{\frac{\partial u}{\partial r}(R\exp(i\theta))} = \underbrace{\sum_{n=-\infty}^{\infty} b_n kH'_n(kR)\exp(in\theta)}_{\frac{\partial u_s}{\partial r}(R\exp(i\theta))} + \frac{\partial u_i}{\partial r}(R\exp(i\theta))$$

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$$\underbrace{\sum_{n=-\infty}^{\infty} a_n \varphi_{|n|}(kR)\exp(in\theta)}_{u(R\exp(i\theta))} = \underbrace{\sum_{n=-\infty}^{\infty} b_n H_n(R)\exp(in\theta)}_{u_s(R\exp(i\theta))} + \underbrace{\sum_{n=-\infty}^{\infty} c_n \exp(in\theta)}_{u_i(R\exp(i\theta))}$$

$$\underbrace{\sum_{n=-\infty}^{\infty} a_n \varphi'_{|n|}(R)\exp(in\theta)}_{\frac{\partial u}{\partial r}(R\exp(i\theta))} = \underbrace{\sum_{n=-\infty}^{\infty} b_n k H'_n(kR)\exp(in\theta)}_{\frac{\partial u_s}{\partial r}(R\exp(i\theta))} + \underbrace{\sum_{n=-\infty}^{\infty} d_n \exp(in\theta)}_{\frac{\partial u_i}{\partial r}(R\exp(i\theta))}$$

We can easily solve for the coefficients a_n and b_n in terms of c_n and d_n.

Scattering from a radially symmetric potential in two dimensions

Since the incident u_i satisfies the constant coefficient Helmholtz equation at wavenumber k, we can actually represent it and its normal derivative using $\mathcal{O}(k)$ Fourier modes. That is, we will have high accuracy approximations

$$u_i(R\exp(i\theta)) \approx \sum_{n=-m}^{m} c_n \exp(in\theta)$$

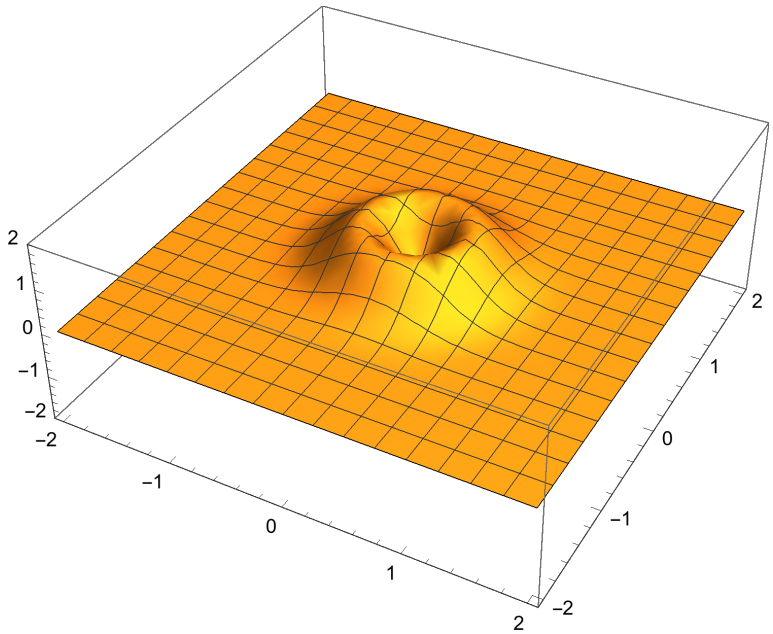
$$\frac{\partial u_i}{\partial r}(R\exp(i\theta)) \approx \sum_{n=-m}^{m} d_n \exp(in\theta)$$

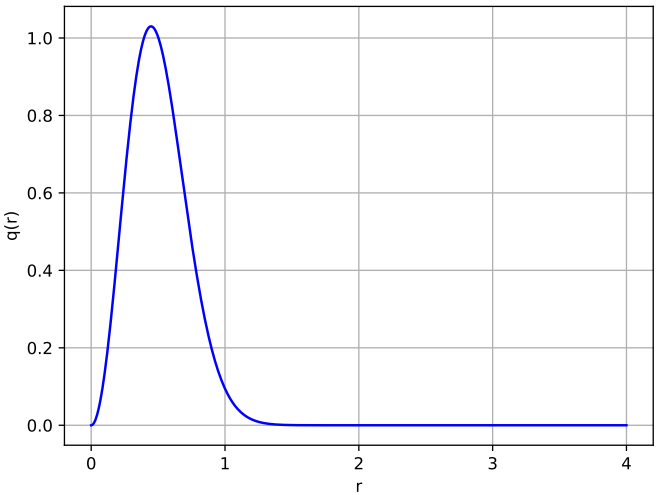
with $m = \mathcal{O}(k)$. Moreover, the coefficients in these expansions can be computed via the fast Fourier transform in $\mathcal{O}(k\log(k))$ operations.

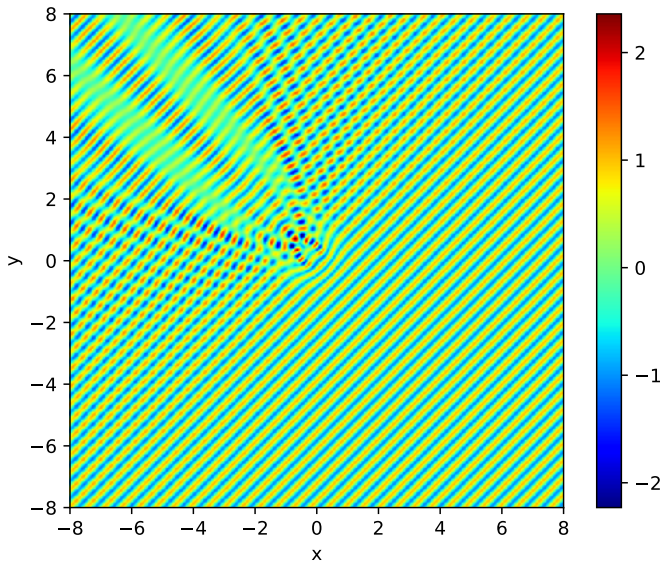
So the running time of this algorithm is

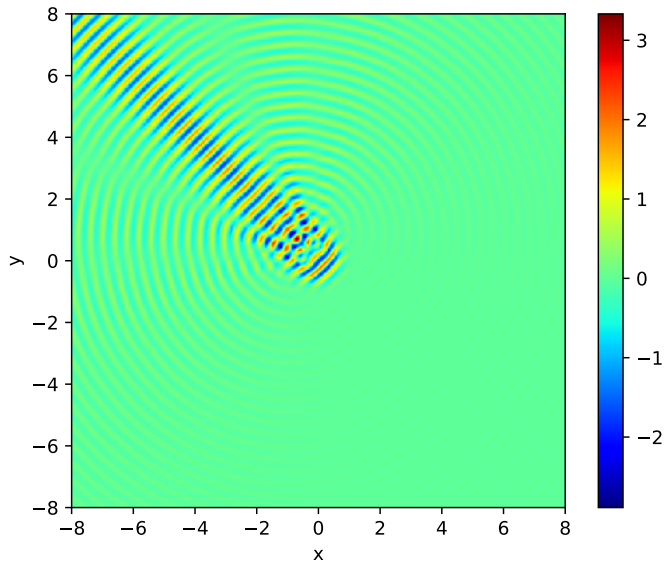
$$\underbrace{k\log(k)}_{\text{FFT}} + \underbrace{k\log(k)}_{\text{Compute basis}}$$

$$q(r) = 14r^2 \exp(-5r^2), \quad u_i(r\exp(i\theta)) = \exp\left(ikr\cos\left(\theta - \frac{\pi}{4}\right)\right)$$

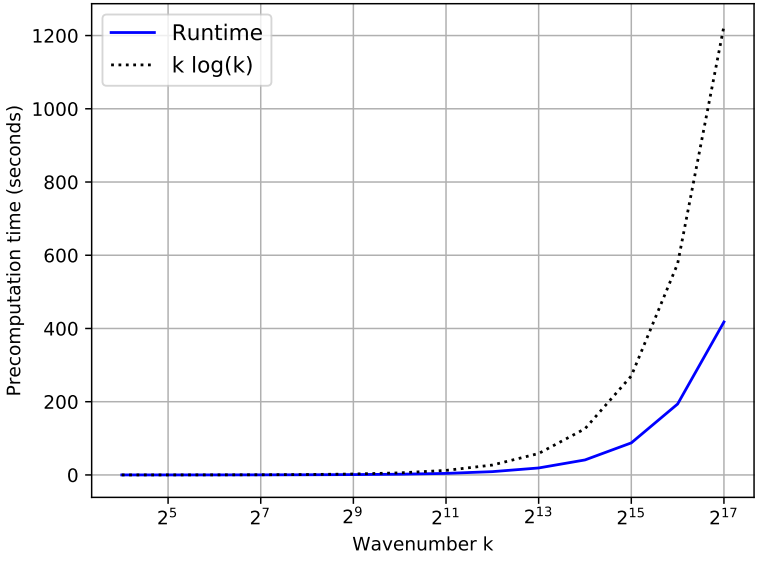


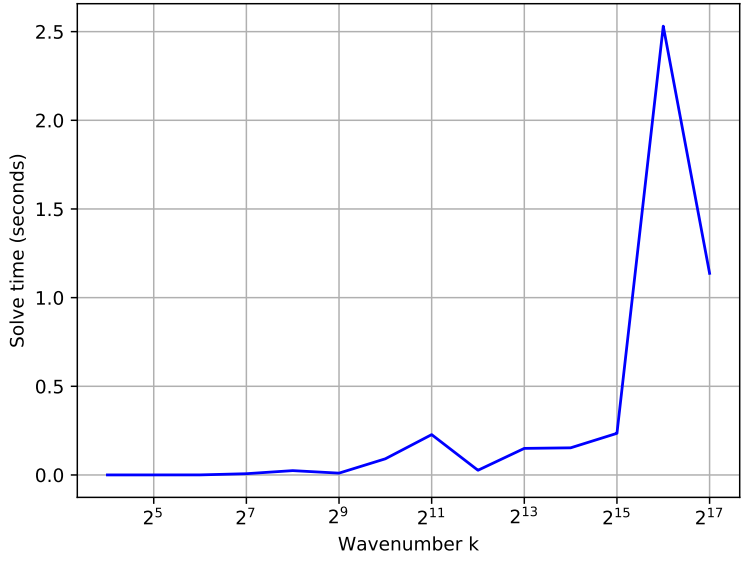




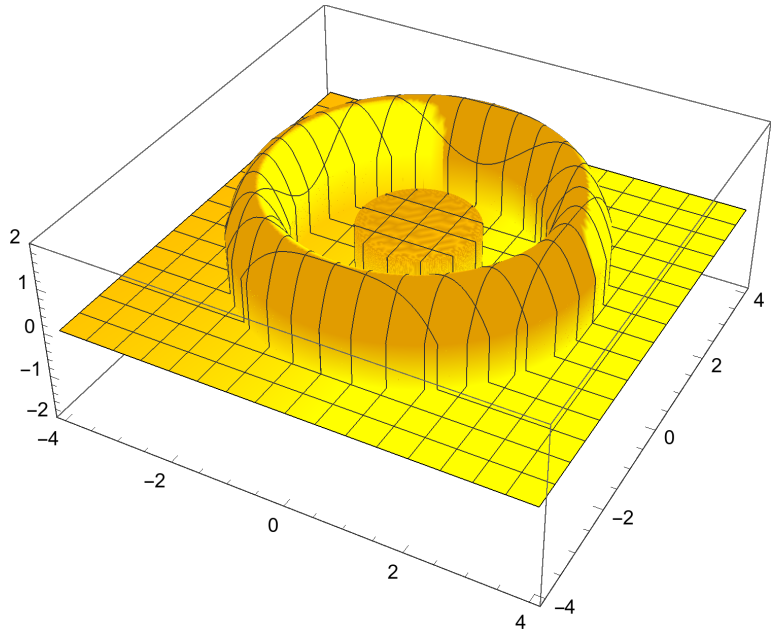


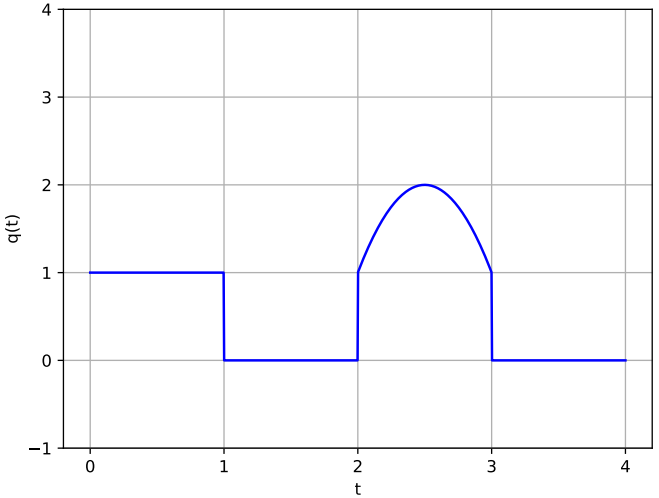
k	m	Maximum absolute error	Precomp time (in seconds)	Solve time (in seconds)
2^{8}	1608	2.44×10^{-12}	4.61×10^{-01}	2.44×10^{-02}
2^{9}	3216	5.85×10^{-12}	9.37×10^{-01}	1.03×10^{-02}
2^{10}	6433	9.34×10^{-12}	$1.98\times10^{+00}$	9.11×10^{-02}
2^{11}	12867	1.98×10^{-11}	$4.20\times10^{+00}$	2.27×10^{-01}
2^{12}	25735	4.80×10^{-11}	$9.00\times10^{+00}$	2.64×10^{-02}
2^{13}	51471	4.35×10^{-10}	$1.90\times10^{+01}$	1.49×10^{-01}
2^{14}	102943	1.91×10^{-09}	$4.09\times10^{+01}$	1.53×10^{-01}
2^{15}	205887	6.93×10^{-09}	$8.75\times10^{+01}$	2.34×10^{-01}
2^{16}	411774	3.23×10^{-08}	$1.93\times10^{+02}$	$2.53\times10^{+00}$
2^{17}	823549	1.08×10^{-07}	$4.17\times10^{+02}$	$1.13\times10^{+00}$

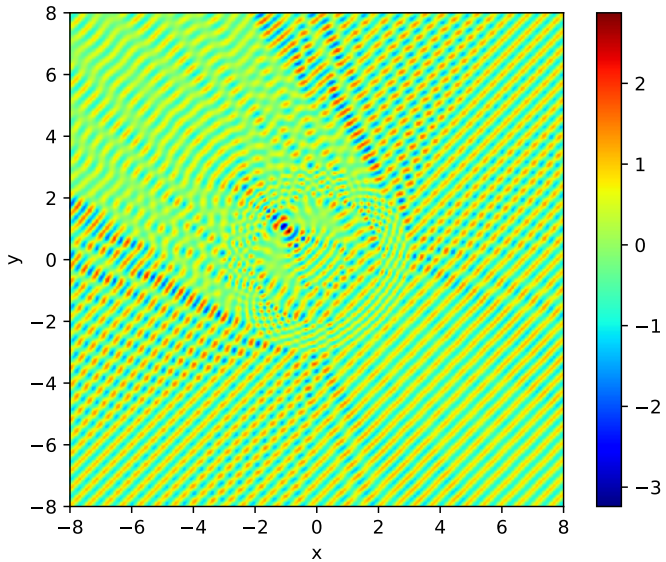


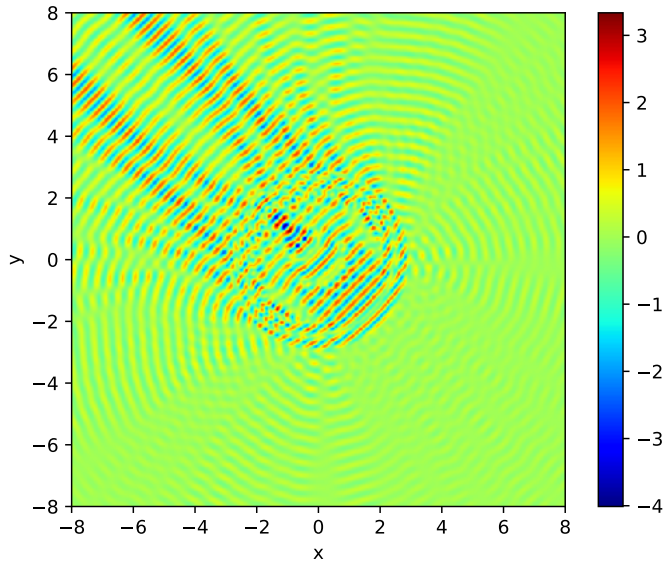


$$q(r) = \chi_{(0,1)}(r) + \chi_{(2,3)}(r)\left(2 - 4\left(r - 2.5\right)^2\right), \quad u_i(r\exp(i\theta)) = \exp\left(ikr\cos\left(\theta - \frac{\pi}{4}\right)\right)$$

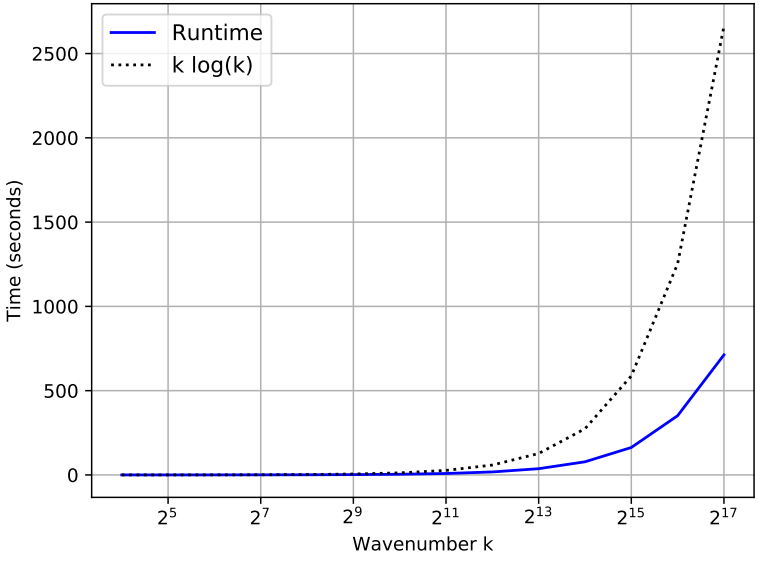


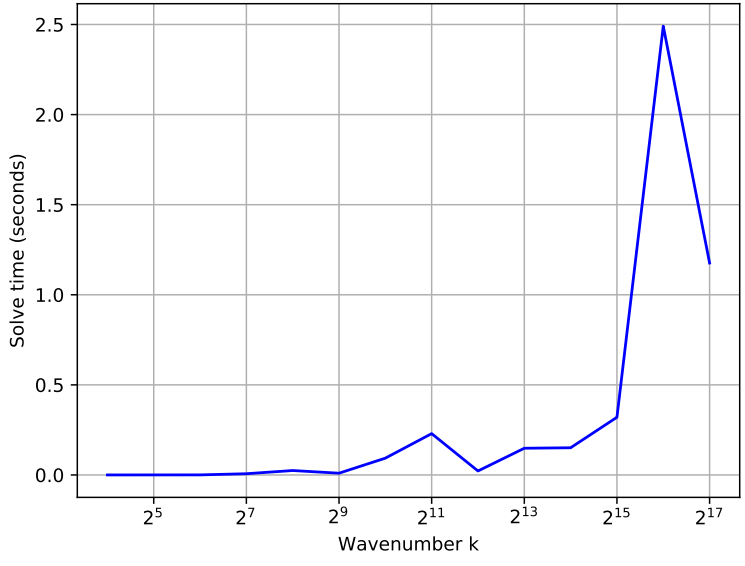






k	m	Maximum absolute error	Precomp time (in seconds)	Solve time (in seconds)
2^8	1608	2.20×10^{-12}	9.49×10^{-01}	2.44×10^{-02}
2^9	3216	5.95×10^{-12}	$1.93 \times 10^{+00}$	1.14×10^{-02}
2^{10}	6433	6.50×10^{-12}	$4.06 \times 10^{+00}$	8.99×10^{-02}
2^{11}	12867	2.04×10^{-11}	$8.58 \times 10^{+00}$	2.27×10^{-01}
2^{12}	25735	4.31×10^{-11}	$1.78 \times 10^{+01}$	2.20×10^{-02}
2^{13}	51471	2.27×10^{-10}	$3.70 \times 10^{+01}$	1.49×10^{-01}
2^{14}	102943	1.90×10^{-10}	$7.79 \times 10^{+01}$	1.55×10^{-01}
2^{15}	205887	3.48×10^{-10}	$1.62 \times 10^{+02}$	4.00×10^{-01}
2^{16}	411774	7.33×10^{-10}	$3.51 \times 10^{+02}$	$2.45 \times 10^{+00}$
2^{17}	823549	1.95×10^{-09}	$7.12 \times 10^{+02}$	$1.11 \times 10^{+00}$





Accelerating the algorithm

This algorithm has two steps which take a nontrivial amount of time:

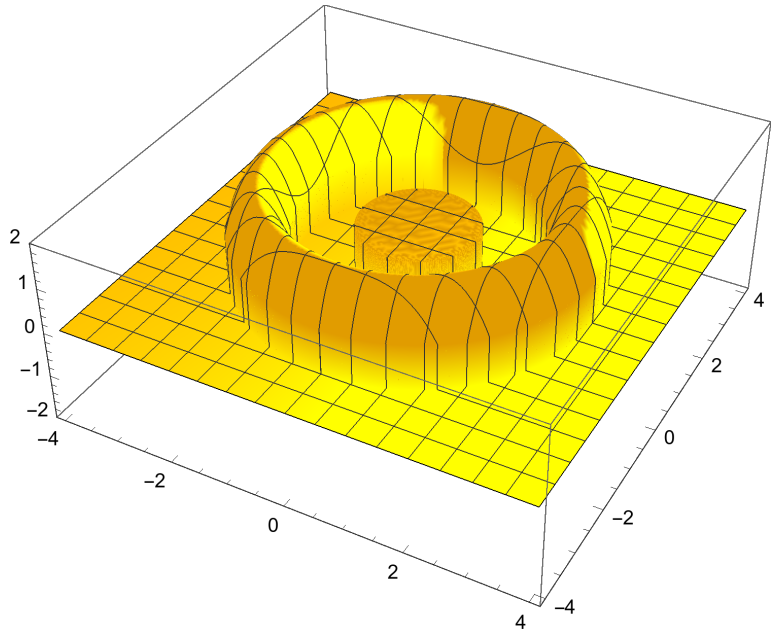
- The Fourier transform of the restriction of the incident wave to the boundary of Ω is computed
- For each Fourier mode with a nonneglible coefficient, a solution of the Riccati equation is calculated

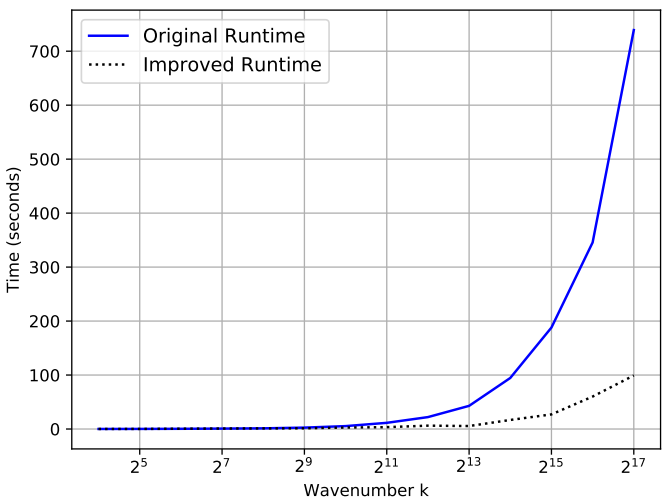
Each of these steps takes $\mathcal{O}(k \log(k))$ time. But the overwhelming majority of the computational effort is spent on computing the solutions of the Riccati equation.

The time required to compute these solutions can be dramatically reduced by observing that the solutions of the Riccati equation are smooth as functions of the wavenumber k and so can be interpolated in that variable.

$$\underbrace{k \log(k)}_{\text{FFT}} + \underbrace{\log^{\kappa}(k)??}_{\text{Compute basis}}$$

$$q(r) = \chi_{(0,1)}(r) + \chi_{(2,3)}(r)\left(2 - 4\left(r - 2.5\right)^2\right), \quad u_i(r\exp(i\theta)) = \exp\left(ikr\cos\left(\theta - \frac{\pi}{4}\right)\right)$$





k	m	Maximum absolute error	Precomp time (in seconds)
2^4	100	1.40×10^{-13}	9.06×10^{-02}
2^5	201	2.18×10^{-13}	1.54×10^{-01}
2^6	402	4.02×10^{-13}	5.57×10^{-01}
2^7	804	1.18×10^{-13}	8.87×10^{-01}
2^8	1608	1.00×10^{-12}	$1.16\times10^{+00}$
2^9	3216	9.95×10^{-12}	$1.33\times10^{+00}$
2^{10}	6433	9.03×10^{-12}	$2.74\times10^{+00}$
2^{11}	12867	2.65×10^{-11}	$3.45\times10^{+00}$
2^{12}	25735	3.42×10^{-11}	$6.23\times10^{+00}$
2^{13}	51471	1.07×10^{-10}	$5.52\times10^{+00}$
2^{14}	102943	2.88×10^{-10}	$1.67\times10^{+01}$
2^{15}	205887	6.52×10^{-10}	$2.70\times10^{+01}$
2^{16}	411774	6.78×10^{-10}	$6.01\times10^{+01}$
2^{17}	823549	2.53×10^{-09}	$9.92\times10^{+01}$

Three-dimensional problems

This has important implications for higher dimensions.

In three dimensions, without an acceleration of this type, the cost of the algorithm is expected to be on the order of

$$\underbrace{k^2 \log(k)}_{\text{SHT}} + \underbrace{k^2 \log(k)}_{\text{Compute basis}}$$

With such an acceleration, I expect the running time to be much improved, perhaps on the order of

$$\underbrace{k^2 \log(k)}_{\text{SHT}} + \underbrace{\log^{\kappa}(k)??}_{\text{Compute basis}}$$

Thank you for your attention!

You can find an implementation of the radially symmetric solver at

`github.com/jamescbremerjr/HelmRad`

You can find an implementation of the one-dimensional solver at

`github.com/jamescbremerjr/Phase-Functions`