Airy phase functions

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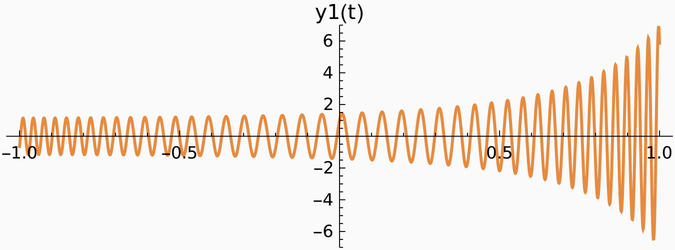
Oscillatory differential equations

$$\boldsymbol{y}'(t) = \begin{pmatrix} g(t) & -\omega f(t) \\ \omega f(t) & 0 \end{pmatrix} \boldsymbol{y}(t)$$

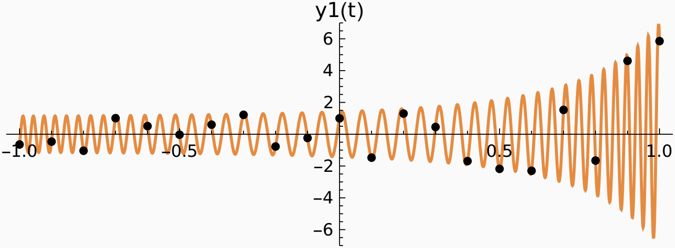
Eigenvalues are $\frac{g(t)}{2} \pm \sqrt{\frac{(g(t))^2}{2} - \omega^2 (f(t))^2}$:



Classical approaches

Step schemes which combine exponential integrators with preconditioning:

- Magnus methods: integrator is a truncated Magnus expansion — $\mathcal{O}(\omega)$
- Modified Magnus methods: integrator is truncated Magnus expansion and preconditioning is done by freezing the coefficient matrix — $\mathcal{O}\left(\omega^{\frac{3}{4}}\right)$
- "Adibiatic integrators" of Lubich, et al.: exponential integrator with first order eigenfunction preconditioner — $\mathcal{O}(\sqrt{\omega})$
- WKB marching methods: WKB preconditioners applied to scalar equations — $\mathcal{O}(\sqrt{\omega})$



Phase functions provide efficient representations

Theorem. Suppose that $q(t, \omega)$ is a smooth positive function given on $[a, b]$, and that q and its first $2M + N$ derivatives with respect to t are bounded independent of ω. Then there exist smooth functions α and $\widetilde{\alpha}$ such that

$$\left\{ \frac{\cos(\alpha(t))}{\sqrt{\alpha'(t)}},\ \frac{\sin(\alpha(t))}{\sqrt{\alpha'(t)}} \right\}$$

is a basis in the space of solutions of the equation

$$y''(t) + \omega^2 q(t, \omega) y(t) = 0, \quad a < t < b,$$

the quantities $\left| \frac{\widetilde{\alpha}^{(j)}(t)}{\widetilde{\alpha}'(t)} \right|$ are bounded independent of ω for all $j = 0, 1, \ldots, N$ and

$$\alpha(t) = \widetilde{\alpha}(t) \left(1 + \mathcal{O}\left(\frac{1}{\omega^{2(M+1)}} \right) \right) \quad \text{as } \omega \to \infty.$$

Oscillatory differential equations

$$\mathbf{y}'(t) = \begin{pmatrix} g(t) & -\omega f(t) \\ \omega f(t) & 0 \end{pmatrix} \mathbf{y}(t)$$

$$\mathbf{y}(t) = e^{\int \frac{g(t)}{2} dt} \begin{pmatrix} \sqrt{f(t)} & 0 \\ \frac{g(t)}{2\omega\sqrt{f(t)}} - \frac{f'(t)}{2\omega(f(t))^{1.5}} & -\frac{1}{\omega\sqrt{f(t)}} \end{pmatrix} \mathbf{z}(t)$$

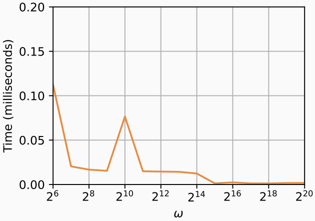
$$\mathbf{z}'(t) = \begin{pmatrix} 0 & 1 \\ -q(t) & 0 \end{pmatrix} \mathbf{z}(t)$$

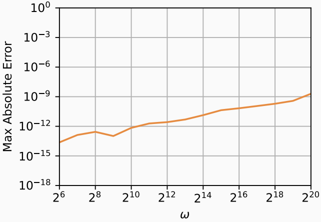
$$q(t) = \omega(f(t))^2 - \frac{3}{4}\left(\frac{f'(t)}{f(t)}\right)^2 + \frac{1}{2}\frac{f''(t)}{f(t)} - \frac{1}{4}\left((g(t))^2 + 2g'(t) - \frac{2g(t)f'(t)}{f(t)}\right)$$

Numerical experiment

$$\boldsymbol{y}'(t) = \begin{pmatrix} g(t) & -\omega f(t) \\ \omega f(t) & 0 \end{pmatrix} \boldsymbol{y}(t) \qquad \begin{aligned} f(t) &= 1 + t^2 \\ g(t) &= \exp(2t) \end{aligned}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{y}(0) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \boldsymbol{y}(1) = \begin{pmatrix} -1 \\ \frac{1}{\sqrt{\omega}} \end{pmatrix}$$

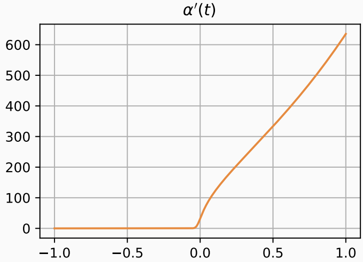




The output of the solver is an efficient representation of a fundamental matrix for the differential equation

All is not well

Trigonometric representations are expensive in the presence of turning points:



$$y''(t) + q(t)y(t) = 0$$

$$q(t) = \left(-\mu^2 + \nu(\nu+1)\,\mathrm{sech}^2\left(t - \xi_\nu^\mu\right)\right)$$

Solutions are associated Legendre functions $P_\nu^\mu\left(\tanh(t - \xi_\nu^\mu)\right)$

Trigonometric phase functions

Use a transformation of the form $y(t) = \dfrac{z(\alpha(t))}{\sqrt{\alpha'(t)}}$ to map the solutions of

$$z''(t) + z(t) = 0$$

to those of

$$y''(t) + \omega^2 q(t) y(t) = 0$$

so that

$$\left\{ \frac{\cos(\alpha(t))}{\sqrt{\alpha'(t)}}, \ \frac{\sin(\alpha(t))}{\sqrt{\alpha'(t)}} \right\}$$

is a basis in the solution space of the latter equation.

The **trigonometric phase function** α satisfies Kummer's equation

$$\omega^2 q(t) - \left(\alpha'(t)\right)^2 + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 - \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} = 0.$$

Generalized phase functions

Use a transformation of the form $y(t) = \dfrac{z(\lambda(t))}{\sqrt{\lambda'(t)}}$ to map the solutions of

$$z''(t) + \widetilde{q}(t)z(t) = 0$$

to those of

$$y''(t) + q(t)y(t) = 0, \quad q(t) \sim \widetilde{q}(t),$$

so that

$$\left\{ \frac{u\left(\lambda(t)\right)}{\sqrt{\lambda'(t)}}, \ \frac{v\left(\lambda(t)\right)}{\sqrt{\lambda'(t)}} \right\},$$

where $\{u, v\}$ is a basis in the solution space of the first equation, is a basis in the solution space of the latter equation.

The **generalized phase function** λ satisfies the generalized Kummer's equation

$$\omega^2 q(t) - \widetilde{q}\left(\lambda(t)\right)\left(\lambda'(t)\right)^2 + \frac{3}{4}\left(\frac{\lambda''(t)}{\lambda'(t)}\right)^2 - \frac{1}{2}\frac{\lambda'''(t)}{\lambda'(t)} = 0.$$

Airy phase functions

Use a transformation of the form $y(t) = \dfrac{z(\gamma(t))}{\sqrt{\gamma'(t)}}$ to map the solutions of

$$z''(t) + tz(t) = 0$$

to those of

$$y''(t) + \omega^2 q(t) y(t) = 0, \quad q(t) \sim t,$$

so that

$$\left\{ \frac{\mathrm{Ai}\,(\gamma(t))}{\sqrt{\gamma'(t)}},\ \frac{\mathrm{Bi}\,(\gamma(t))}{\sqrt{\gamma'(t)}} \right\}$$

is a basis in the solution space of the latter equation.

The **Airy phase function** γ satisfies the Airy-Kummer equation

$$\omega^2 q(t) - \gamma(t)\left(\gamma'(t)\right)^2 + \frac{3}{4}\left(\frac{\gamma''(t)}{\gamma'(t)}\right)^2 - \frac{1}{2}\frac{\gamma'''(t)}{\gamma'(t)} = 0.$$

Theorem. Suppose that $q(t,\omega) = t\,q_0(t,\omega)$ where q_0 is smooth and positive on $[-a,a]$, and that q_0 and its first $2M+N$ derivatives with respect to t are bounded independent of ω. Then there exist smooth functions γ and $\widetilde{\gamma}$ such that

$$\left\{ \frac{\mathrm{Bi}\left(\gamma(t)\right)}{\sqrt{\gamma'(t)}}, \frac{\mathrm{Ai}\left(\gamma(t)\right)}{\sqrt{\gamma'(t)}} \right\}$$

is a basis in the space of solutions of the equation

$$y''(t) + \omega^2 q(t,\omega) y(t) = 0, \quad -a < t < a,$$

the quantities $\left| \dfrac{\widetilde{\gamma}^{(j)}(t)}{\widetilde{\gamma}'(t)} \right|$ are bounded independent of ω for all $j = 0, 1, \ldots, N$ and

$$\gamma(t) = \widetilde{\gamma}(t) \left(1 + \mathcal{O}\left(\frac{1}{\omega^{2(M+1)}} \right) \right) \quad \text{as } \omega \to \infty.$$

Numerical calculation of Airy phase functions

Problem with the obvious approach: Almost all solutions of the Airy-Kummer equation

$$\omega^2 q(t) - \gamma(t)\left(\gamma'(t)\right)^2 + \frac{3}{4}\left(\frac{\gamma''(t)}{\gamma'(t)}\right)^2 - \frac{1}{2}\frac{\gamma'''(t)}{\gamma'(t)} = 0$$

are rapidly varying and we need some mechanism to select the desired slowly-varying solution.

Numerical calculation of Airy phase functions

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Solution: Exploit the fact that a sparse discretization will only see the slowly-varying solutions.

Numerical calculation of Airy phase functions

The k-point Chebyshev spectral discretization of the Airy-Kummer equation on an interval $[-a_0, a_0]$ is

$$\omega^2 \boldsymbol{q} - \boldsymbol{\gamma} \circ \left(\frac{\mathscr{D}_k}{a_0} \gamma \right)^{\circ 2} + \frac{3}{4} \left(\left(\frac{\mathscr{D}_k}{a_0} \right)^2 \gamma \right)^{\circ 2} \circ \left(\frac{\mathscr{D}_k}{a_0} \gamma \right)^{\circ -2} - \frac{1}{2} \left(\left(\frac{\mathscr{D}_k}{a_0} \right)^3 \gamma \right) \circ \left(\frac{\mathscr{D}_k}{a_0} \gamma \right)^{\circ -1} = 0.$$

Choose k so that the slowly-varying solution is discretized, but the rapidly-varying solutions are not.

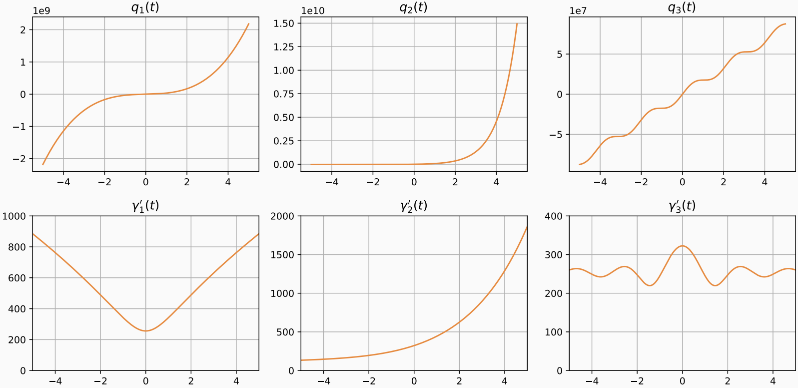
Newton-Kantorovich implies that Newton's method converges provided ω is sufficiently large relative to $\|\mathscr{D}_k\|$.

We then extend the solution to the whole interval $[-a, a]$ by solving an initial value problem.

Numerical experiments

$$y''(t) + \omega^2 q(t) y(t) = 0$$

$$q_1(t) = t + t^3 \qquad q_2(t) = -1 + (1-t)\exp(t) \qquad q_3(t) = t + \frac{\sin(3t)}{3}$$



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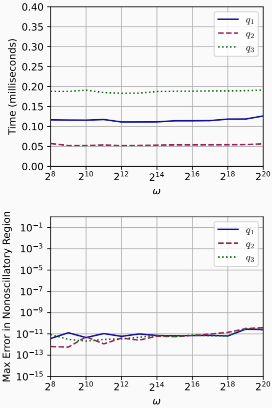
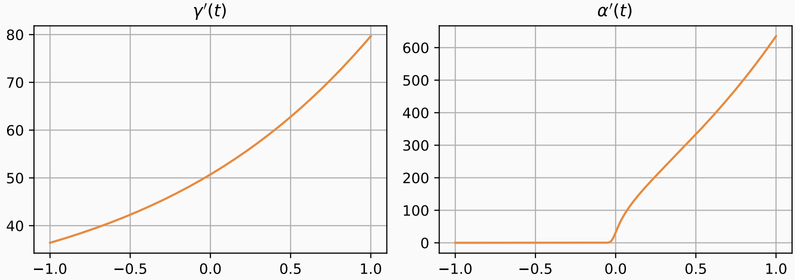


Numerical experiments

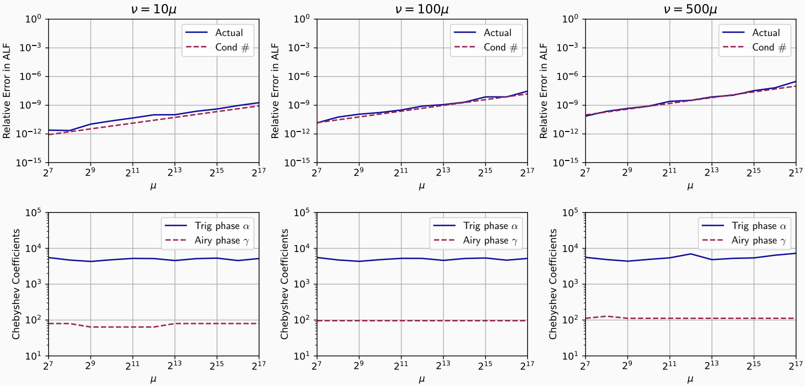
$$y''(t) + \left(-\mu^2 + \nu(\nu+1)\operatorname{sech}^2\left(t - \xi_\nu^\mu\right)\right) y(t) = 0$$



$\mu = 256$ and $\nu = 2560$

Numerical experiments

$$y''(t) + \left(-\mu^2 + \nu(\nu+1)\,\mathrm{sech}^2\left(t - \xi_\nu^\mu\right)\right) y(t) = 0$$



Numerical experiments

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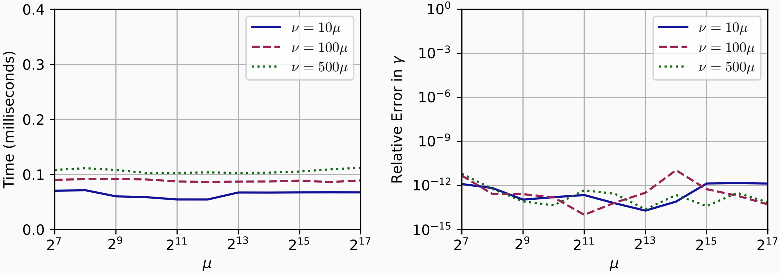

Applications to special functions

Precompute expansions of phase functions which vary with parameters to efficiently represent a family of special functions:

- $\mathcal{O}(1)$ evaluation of the special functions
- $\mathcal{O}(1)$ evaluation of their roots, as well as the weights of the associated quadrature rules, Sturm-Liouville eigenvalues, etc.
- $\mathcal{O}(n)$ application of associated Sturm-Liouville eigentransform through the evaluation of integrals such as

$$\int \exp(i\alpha(\omega, t)) f(t)\, dt \quad \text{or} \quad \int \mathrm{Ai}(\gamma(\omega, t)) f(t)\, dt$$

Spheriodal wave functions, Jacobi polynomials, ALFs, ...

Ongoing work

Map the solutions of

$$z''(t) + t^{\sigma} z(t) = 0$$

to those of

$$y''(t) + \omega^2 q(t) y(t) = 0, \quad q(t) \sim t^{\sigma},$$

by solving the generalized Kummer equation

$$\omega^2 q(t) - (\gamma(t))^{\sigma} \left(\gamma'(t)\right)^2 + \frac{3}{4}\left(\frac{\gamma''(t)}{\gamma'(t)}\right)^2 - \frac{1}{2}\frac{\gamma'''(t)}{\gamma'(t)} = 0.$$

Then we have the following basis in the solution space of the original equation:

$$\left\{ \frac{u\left(\gamma(t)\right)}{\sqrt{\gamma'(t)}}, \frac{v\left(\gamma(t)\right)}{\sqrt{\gamma'(t)}} \right\}$$

$$u(t) = \sqrt{t} J_{\frac{1}{\sigma+2}}\left(\frac{2}{2+\sigma} t^{\frac{2+\sigma}{2}}\right)$$

$$v(t) = \sqrt{t} J_{-\frac{1}{\sigma+2}}\left(\frac{2}{2+\sigma} t^{\frac{2+\sigma}{2}}\right)$$

Ongoing work

$$y''(t) + \frac{\omega^2}{\sqrt{t}}\left(1 + \frac{1}{2}t\cos(13t)\right)y(t) = 0$$