

Improved estimates for nonoscillatory phase functions

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Abstract

Recently, it was observed that solutions of a large class of highly oscillatory second order linear ordinary differential equations can be approximated using nonoscillatory phase functions. In particular, under mild assumptions on the coefficients and wavenumber λ of the equation, there exists a function whose Fourier transform decays as $\exp(-\mu|\xi|)$ and which represents solutions of the differential equation with accuracy on the order of $\lambda^{-1} \exp(-\mu\lambda)$. In this article, we establish an improved existence theorem for nonoscillatory phase functions. Among other things, we show that solutions of second order linear ordinary differential equations can be represented with accuracy on the order of $\lambda^{-1} \exp(-\mu\lambda)$ using functions in the space of rapidly decaying Schwartz functions whose Fourier transforms are both exponentially decaying and compactly supported. These new observations are used in the analysis of a method for the numerical solution of second order ordinary differential equations whose running time is independent of the parameter λ . This algorithm will be reported at a later date.

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1. Introduction

Given a differential equation

$$y''(t) + \lambda^2 q(t)y(t) = 0 \quad \text{for all } a \leq t \leq b, \quad (1)$$

where λ is a real number and $q : [0, 1] \rightarrow \mathbb{R}$ is smooth and strictly positive, a sufficiently smooth $\alpha : [a, b] \rightarrow \mathbb{R}$ is a phase function for (1) if the pair of functions u, v defined by the formulas

$$u(t) = \frac{\cos(\alpha(t))}{|\alpha'(t)|^{1/2}} \quad (2)$$

and

$$v(t) = \frac{\sin(\alpha(t))}{|\alpha'(t)|^{1/2}} \quad (3)$$

form a basis in the space of solutions of (1). Phase functions have been extensively studied: they were first introduced in [1], play a key role in the theory of global transformations of ordinary differential equations [2, 3], and frequently arise in the theory of special functions [4, 5, 6, 7].

It was observed by E.E. Kummer in [1] that α is a phase function for (1) if and only if it satisfies the

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third order nonlinear differential equation

$$(\alpha'(t))^2 = \lambda^2 q(t) - \frac{1}{2} \frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4} \left(\frac{\alpha''(t)}{\alpha'(t)} \right)^2 \quad (4)$$

on the interval $[a, b]$. The presence of quotients in (4) is often inconvenient, and we prefer the more tractable equation

$$r''(t) - \frac{1}{4} (r'(t))^2 + 4\lambda^2 (\exp(r(t)) - q(t)) = 0 \quad (5)$$

obtained from (4) by letting

$$\alpha'(t) = \lambda \exp\left(\frac{r(t)}{2}\right). \quad (6)$$

Of course, if r is a solution of (5) then the function α defined by the formula

$$\alpha(t) = \lambda \int_a^t \exp\left(\frac{r(u)}{2}\right) du \quad (7)$$

is a solution of (4). We will refer to (4) as Kummer's equation and (5) as the logarithm form of Kummer's equation. The form of these equations and the appearance of λ in them suggests that their solutions will be oscillatory — and most of them are. However, there are several well-known examples of second order ordinary differential equations which admit nonoscillatory phase functions. For example, the function

$$\alpha(t) = \lambda \arccos(t), \quad (8)$$

is a phase function for Chebyshev's equation

$$y''(t) + \left(\frac{2 + t^2 + 4\lambda^2(1 - t^2)}{4(1 - t^2)^2} \right) y(t) = 0 \quad \text{for all } -1 \leq t \leq 1. \quad (9)$$

Its existence is the basis of many numerical algorithms, including the fast Chebyshev transform (see, for instance, [8]). Bessel's equation

$$y''(t) + \left(1 - \frac{\lambda^2 - 1/4}{t^2} \right) y(t) = 0 \quad \text{for all } 0 < t < \infty \quad (10)$$

also admits a nonoscillatory phase function, although it cannot be expressed in terms of elementary functions (see, for instance, [9]).

Exact solutions of (4) which are nonoscillatory need not exist in the general case. However, as we show in this article, when the coefficient q appearing in (4) is nonoscillatory, there exists a nonoscillatory function α such that (2), (3) approximate solutions of (1) in the space $L^\infty([a, b])$ with accuracy on the order of $\lambda^{-1} \exp(-\mu\lambda)$. In order to make this statement rigorous, we will use the Fourier transform to quantify the notion of “nonoscillatory function.” Accordingly, we assume that the coefficient q in (1) extends to a strictly positive function on the entire real line. Moreover, we define the function x via the formula

$$x(t) = \int_a^t \sqrt{q(u)} du, \quad (11)$$

and let $p(x)$ be twice the Schwarzian derivative of the variable t with respect to the variable x (see Section 2.8). We suppose that there exist constants Γ and μ such that

$$|\widehat{p}(\xi)| \leq \Gamma \exp(-\mu|\xi|) \quad \text{for all } \xi \in \mathbb{R}, \quad (12)$$

that $\lambda > \frac{2}{\mu}$, and that $\lambda > 2\Gamma$. Then there exists a function δ in the space $S(\mathbb{R})$ of rapidly decaying Schwartz functions (see Section 2.1) such that

$$r(t) = \log(q(t)) + \delta(x(t)) \quad (13)$$

closely approximates a solution of (5), the support of the Fourier transform $\widehat{\delta}$ of δ is contained in the interval $(-\sqrt{2\lambda}, \sqrt{2\lambda})$, and

$$\left| \widehat{\delta}(\xi) \right| \leq \frac{\Gamma}{\lambda} \exp(-\mu |\xi|) \quad \text{for all } |\xi| < \sqrt{2\lambda}. \quad (14)$$

Moreover the function α defined via the formula

$$\alpha(t) = \lambda \int_a^t \exp\left(\frac{r(u)}{2}\right) du \quad (15)$$

is a phase function for a second order differential equation of the form

$$y''(t) + \lambda^2 \left(1 + \frac{\nu(t)}{4\lambda^2}\right) q(t)y(t) = 0 \quad \text{for all } a \leq t \leq b, \quad (16)$$

where ν is an element of $S(\mathbb{R})$ whose $L^\infty(\mathbb{R})$ norm is on the order of $\exp(-\mu\lambda)$. While α is not a phase function for the original equation (1), the functions u , v obtained by inserting (15) into the formulas (2) and (3) approximate solutions of (1) with accuracy on the order of $\lambda^{-1} \exp(-\mu\lambda)$.

The bound (14) implies that when δ is approximated using various series expansions, the number of terms required to represent it to a specified precision is independent of λ . For instance, the minimum number of terms in the Legendre expansion of the restriction of δ to the interval $[a, b]$ required to achieve a specified precision is independent of λ . Assuming that q is nonoscillatory, it follows that r and α can be represented likewise; that is, using finite series expansions whose number of terms is independent of λ (it is in this sense that they are nonoscillatory). In other words: $O(1)$ terms are required to represent the function δ which enables us to approximate solutions of (1) with $O(\lambda^{-1} \exp(-\mu\lambda))$ accuracy. This is in contrast to superasymptotic and hyperasymptotic expansions (see, for instance, [10, 11]) which approximate solutions of (1) to accuracy on the order of $\exp(-\rho\lambda)$, but which require $O(\lambda)$ terms in order to do so. Note that we avoid discussing the Fourier transforms of r and α because they need not decay at infinity and so there is no assurance that their Fourier transforms are functions (as opposed to tempered distributions).

The results presented in this article improve upon those of [12], which observed that solutions of (1) can be represented with accuracy on the order of $\exp(-\mu\lambda)$ using functions which are in $L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and whose Fourier transforms decay exponentially. Here we show that the function δ which represents the solutions of (1) is an element of the space $S(\mathbb{R})$ of rapidly decaying Schwartz functions (see Section 2.1) and that its Fourier transform is both exponentially decaying and compactly supported (so that δ is entire). Moreover, we substantially reduce the constants appearing in the bounds on the decay of the Fourier transform of δ and in the error of the associated approximations of the solutions of (1). Among other things, these new observations are useful for analyzing an algorithm for constructing a nonoscillatory solution α of (4) whose running time is independent of λ . This algorithm allows for the numerical evaluation of solutions of second order linear ordinary differential equations of the form (1) using a number of operations which is independent of λ . It will be reported at a later date.

The remainder of this paper is organized as follows. Section 2 summarizes a number of well-known mathematical facts and establishes the notation which is used throughout this article. In Section 3, we reformulate Kummer's equation as a nonlinear integral equation. The statement of the main result of this paper, Theorem 12, is given in Section 4 and its proof is divided among Sections 5, 6 and 7.

2. Preliminaries

2.1. Function spaces

We denote by $C(\mathbb{R})$ the set of continuous functions $\mathbb{R} \rightarrow \mathbb{C}$. If $f \in C(\mathbb{R})$ and, for each $\epsilon > 0$, there exists a compact set K such that $|f(x)| < \epsilon$ for all $x \notin K$, then we say that f vanishes at infinity. We denote the set of continuous functions which vanish at infinity by $C_0(\mathbb{R})$. By $C^\infty(\mathbb{R})$ we mean the set of infinitely differentiable functions $\mathbb{R} \rightarrow \mathbb{C}$, and $C_c^\infty(\mathbb{R})$ is the set of compact supported functions in $C^\infty(\mathbb{R})$.

We say that $\varphi \in C^\infty(\mathbb{R})$ is a Schwartz function if φ and all of its derivatives decay faster than any polynomial. That is, if

$$\sup_{t \in \mathbb{R}} |t^i \varphi^{(j)}(t)| < \infty \quad (17)$$

for all pairs i, j of nonnegative integers. The set of all Schwartz functions is denoted by $S(\mathbb{R})$; clearly, it contains the set $C_c^\infty(\mathbb{R})$. We endow $S(\mathbb{R})$ with the topology generated by the family of seminorms

$$\|\varphi\|_k = \sum_{j=0}^k \sup_{t \in \mathbb{R}} |t^k \varphi^{(j)}(x)| \quad k = 0, 1, 2, \dots \quad (18)$$

so that a sequence $\{\varphi_n\}$ of functions in $S(\mathbb{R})$ converges to φ in $S(\mathbb{R})$ if and only if

$$\lim_{n \rightarrow \infty} \|\varphi_n - \varphi\|_k = 0 \quad \text{for all } k = 0, 1, 2, \dots \quad (19)$$

We denote the space of continuous linear functionals on $S(\mathbb{R})$, which are known as tempered distributions, by $S'(\mathbb{R})$. We endow $S'(\mathbb{R})$ with the weak-* topology so that a sequence $\{\omega_n\}$ in $S'(\mathbb{R})$ converges to $\omega \in S'(\mathbb{R})$ if and only if

$$\lim_{n \rightarrow \infty} \omega_n(\varphi) = \omega(\varphi) \quad (20)$$

for all $\varphi \in S(\mathbb{R})$. We refer the reader to [13] for a thorough discussion of the properties of Schwartz functions and tempered distributions.

2.2. The Fourier transform

We define the Fourier transform of a function $f \in S(\mathbb{R})$ via the formula

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} \exp(-ix\xi) f(x) dx. \quad (21)$$

The Fourier transform is an isomorphism $S(\mathbb{R}) \rightarrow S(\mathbb{R})$ (meaning that it is a continuous, invertible mapping $S(\mathbb{R}) \rightarrow S(\mathbb{R})$ whose inverse is also continuous). The formula

$$\langle \widehat{\omega}, \varphi \rangle = \langle \omega, \widehat{\varphi} \rangle \quad (22)$$

extends the Fourier transform to an isomorphism $S'(\mathbb{R}) \rightarrow S'(\mathbb{R})$. The definition (22) coincides with (21) when $f \in L^1(\mathbb{R})$. Moreover, when $f \in L^2(\mathbb{R})$,

$$\widehat{f}(\xi) = \lim_{R \rightarrow \infty} \int_{-R}^R \exp(-ix\xi) f(x) dx. \quad (23)$$

Owing to our choice of convention for the Fourier transform,

$$\widehat{f * g}(\xi) = \widehat{f}(\xi) \widehat{g}(\xi) \quad (24)$$

and

$$\widehat{f \cdot g}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi - \eta) \widehat{g}(\eta) d\eta \quad (25)$$

whenever f and g are elements of $L^1(\mathbb{R})$. Moreover,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ix\xi) \widehat{f}(\xi) d\xi \quad (26)$$

whenever f and \widehat{f} are elements of $L^1(\mathbb{R})$. The observation that f is an entire function when \widehat{f} is a compactly supported distribution is one consequence of the well-known Paley-Wiener theorem. See [14, 15] for a thorough treatment of the Fourier transform.

2.3. Convolution exponentials

Formally, the Fourier transform of

$$\exp(f(x)) \quad (27)$$

is the sum

$$2\pi\delta(\xi) + \psi(\xi) + \frac{\psi * \psi(\xi)}{2!(2\pi)} + \frac{\psi * \psi * \psi(\xi)}{3!(2\pi)^2} + \dots, \quad (28)$$

where ψ denotes the Fourier transform of f and δ is the delta distribution. The expression (28), which is referred to as the convolution exponential of ψ , is typically denoted by $\exp^*[\psi]$.

We will not encounter the expression (28) in this article; however, we will consider the Fourier transforms of functions of the form

$$\exp(f(x)) - 1 \quad (29)$$

and

$$\exp(f(x)) - f(x) - 1. \quad (30)$$

So, in analogy with the definition of \exp_* , we define $\exp_1^*[\psi]$ and $\exp_2^*[\psi]$ for $\psi \in L^1(\mathbb{R})$ via the formulas

$$\exp_1^*[\psi](\xi) = \psi(\xi) + \frac{\psi * \psi(\xi)}{2!(2\pi)} + \frac{\psi * \psi * \psi(\xi)}{3!(2\pi)^2} + \dots \quad (31)$$

and

$$\exp_2^*[\psi](\xi) = \frac{\psi * \psi(\xi)}{2!(2\pi)} + \frac{\psi * \psi * \psi(\xi)}{3!(2\pi)^2} + \dots. \quad (32)$$

That is, $\exp_1^*[\psi]$ is obtained by truncating the leading term of $\exp^*[\psi]$ and $\exp_2^*[\psi]$ is obtained by truncated the first two leading terms of $\exp^*[\psi]$. By repeatedly applying the inequality

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1, \quad (33)$$

which can be found in [16] (for instance), we obtain

$$\|\psi\|_1 + \left\| \frac{\psi * \psi}{2!(2\pi)} \right\|_1 + \left\| \frac{\psi * \psi * \psi}{3!(2\pi)^2} \right\|_1 + \dots \leq \sum_{n=1}^{\infty} \frac{\|\psi\|_1^n}{n!(2\pi)^{n-1}} \leq \|\psi\|_1 \exp\left(\frac{\|\psi\|_1}{2\pi}\right) \quad (34)$$

and

$$\left\| \frac{\psi * \psi}{2!(2\pi)} \right\|_1 + \left\| \frac{\psi * \psi * \psi}{3!(2\pi)^2} \right\|_1 + \dots \leq \sum_{n=2}^{\infty} \frac{\|\psi\|_1^n}{n!(2\pi)^{n-1}} \leq \frac{\|\psi\|_1^2}{4\pi} \exp\left(\frac{\|\psi\|_1}{2\pi}\right), \quad (35)$$

from which we see that the series (31) and (32) converge absolutely in $L^1(\mathbb{R})$ when $\psi \in L^1$ and therefore define $L^1(\mathbb{R})$ functions. Suppose that f is the inverse Fourier transform of $\psi \in L^1(\mathbb{R})$. For each nonnegative integer n , we define f_n by

$$f_n(x) = \sum_{k=1}^n \frac{(f(x))^k}{k!}. \quad (36)$$

We observe that $\{f_n\}$ converges in $L^\infty(\mathbb{R})$ to $\exp(f(x)) - 1$. Since

$$\widehat{f}_n(\xi) = \psi(\xi) + \frac{\psi * \psi(\xi)}{2!(2\pi)} + \cdots + \frac{\psi * \psi * \cdots * \psi(\xi)}{n!(2\pi)^{n-1}} \quad (37)$$

for each nonnegative integer n , it follows from (34) that $\{\widehat{f}_n\}$ converges in $L^1(\mathbb{R})$ to $\exp_1^*[\psi]$. We conclude that the Fourier transform of $\exp(f(x)) - 1$ is $\exp_1^*[\psi]$. A nearly identical argument shows that the Fourier transform of $\exp(f(x)) - f(x) - 1$ is $\exp_2^*[\psi]$.

Theorem 1. *If $f \in S(\mathbb{R})$, then $\exp(f(x)) - 1$ and $\exp(f(x)) - f(x) - 1$ are elements of $S(\mathbb{R})$.*

Proof. Since $f \in S(\mathbb{R})$, $\exp(|f(x)|)$ is bounded and

$$\sup_{x \in \mathbb{R}} |x^k f(x)| < \infty \quad (38)$$

for any nonnegative integer k . Consequently,

$$\sup_{x \in \mathbb{R}} |x^k (\exp(f(x)) - 1)| \leq \sup_{x \in \mathbb{R}} |x^k f(x)| \exp(|f(x)|) < \infty \quad (39)$$

for any nonnegative integer k . We conclude that $\exp(f(x)) - 1$ decays faster than any polynomial. We observe that the n^{th} derivative of $\exp(f(x)) - 1$ is of the form

$$P(f(x), f'(x), f''(x), \dots, f^{(n)}(x)) \exp(f(x)), \quad (40)$$

where P is a polynomial in n variables of the form

$$P(x_1, x_2, \dots, x_n) = \sum_{1 \leq k_1 + k_2 + \dots + k_n \leq n} C_{k_1, k_2, \dots, k_n} x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}. \quad (41)$$

Since $f, f', f'', \dots, f^{(n)}$ are elements of $S(\mathbb{R})$,

$$(f(x))^{k_1} (f'(x))^{k_2} \dots (f^{(n)}(x))^{k_n} \quad (42)$$

decays faster than any polynomial whenever k_1, k_2, \dots, k_n are nonnegative integers not all of which are 0. We combine this observation with the fact that $\exp(f(x))$ is bounded in order to conclude that the function (40) decays faster than any polynomial; i.e., the n^{th} derivative of $\exp(f(x)) - 1$ decays faster than any polynomial. Therefore $\exp(f(x)) - 1$ is in $S(\mathbb{R})$. Since $\exp(f(x)) - f(x) - 1$ is obtained from $\exp(f(x)) - 1$ by subtracting the Schwartz function f , it is also an element of $S(\mathbb{R})$. \square

We combine Theorem 1 with the observation that the Fourier transform is a continuous mapping $S(\mathbb{R}) \rightarrow S(\mathbb{R})$ in order to obtain the following theorem.

Theorem 2. *If $\psi \in S(\mathbb{R})$, then $\exp_1^*[\psi]$ and $\exp_2^*[\psi]$ are elements of $S(\mathbb{R})$.*

2.4. The constant coefficient Helmholtz equation

Under certain conditions on the function f , a solution of the inhomogeneous Helmholtz equation

$$y''(x) + \lambda^2 y(x) = f(x) \quad \text{for all } x \in \mathbb{R} \quad (43)$$

can be obtained via the Fourier transform. For instance, the following theorem is a special case of a more general one which can be found in [17].

Theorem 3. *Suppose that $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, and that λ is a positive real number. Then the function g defined by the formula*

$$g(x) = \frac{1}{2\lambda} \int_{-\infty}^{\infty} \sin(\lambda|x-y|) f(y) dy \quad (44)$$

is twice continuously differentiable,

$$g''(x) + \lambda^2 g(x) = f(x) \quad \text{for all } x \in \mathbb{R}, \quad (45)$$

and

$$\widehat{g}(\xi) = \frac{\widehat{f}(\xi)}{\lambda^2 - \xi^2}. \quad (46)$$

We interpret the Fourier transform (46) of g as a tempered distribution defined via principal value integrals; that is to say that for all $\varphi \in S(\mathbb{R})$,

$$\left\langle \frac{\widehat{f}(\xi)}{\lambda^2 - \xi^2}, \varphi \right\rangle = \frac{1}{2\lambda} \left(\lim_{\epsilon \rightarrow 0} \int_{|\xi - \lambda| > \epsilon} \frac{\widehat{f}(\xi)\varphi(\xi)}{\lambda - \xi} d\xi - \lim_{\epsilon \rightarrow 0} \int_{|\xi + \lambda| > \epsilon} \frac{\widehat{f}(\xi)\varphi(\xi)}{\lambda + \xi} d\xi \right). \quad (47)$$

We also note that the requirement that f is continuous ensures that (45) holds for all $x \in \mathbb{R}$; without such an assumption on f , we are only guaranteed that (45) holds for almost all $x \in \mathbb{R}$.

When $f \in L^2(\mathbb{R})$ and λ is real-valued, the integral (44) defining the function g is not necessarily absolutely convergent and the expression

$$\frac{\widehat{f}(\xi)}{\lambda^2 - \xi^2}, \quad (48)$$

which is formally the Fourier transform of g , need not define a tempered distribution. If, however, the support of \widehat{f} is contained in $(-\lambda, \lambda)$, then (48) is a compactly supported element of $L^1(\mathbb{R})$. In this case, we define g through the formula

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ix\xi) \frac{\widehat{f}(\xi)}{\lambda^2 - \xi^2} d\xi. \quad (49)$$

Since g is the inverse Fourier transform of a compactly supported function, it is entire. Moreover, the Fourier transform of

$$g''(x) + \lambda^2 g(x) \quad (50)$$

is

$$-\xi^2 \frac{\widehat{f}(\xi)}{\lambda^2 - \xi^2} + \lambda^2 \frac{\widehat{f}(\xi)}{\lambda^2 - \xi^2} = \widehat{f}(\xi), \quad (51)$$

from which we conclude that

$$g''(x) + \lambda^2 g(x) = f(x) \quad (52)$$

almost everywhere. Since both g and f are both entire, (52) in fact holds for all $x \in \mathbb{R}$. We record these observations as follows.

Theorem 4. *Suppose that $f \in L^2(\mathbb{R})$, that λ is a positive real number, and that the support of \widehat{f} is contained in the interval $(-\lambda, \lambda)$. Then the function g defined via the formula*

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ix\xi) \frac{\widehat{f}(\xi)}{\lambda^2 - \xi^2} d\xi \quad (53)$$

is entire,

$$g''(x) + \lambda^2 g(x) = f(x) \quad \text{for all } x \in \mathbb{R}, \quad (54)$$

and

$$\widehat{g}(\xi) = \frac{\widehat{f}(\xi)}{\lambda^2 - \xi^2}. \quad (55)$$

The following variant of Theorem 3 can be found in [18].

Theorem 5. *Suppose that f is continuous on the interval $[a, b]$, and that λ is a positive real number. Suppose also that $y : [a, b] \rightarrow \mathbb{C}$ is twice continuously differentiable, and that*

$$y''(x) + \lambda^2 y(x) = f(x) \quad \text{for all } a \leq x \leq b. \quad (56)$$

Then

$$y(x) = y(a) + y'(a)(x - a) + \frac{1}{\lambda} \int_a^x \sin(\lambda(x - u)) f(u) du \quad \text{for all } a \leq x \leq b. \quad (57)$$

2.5. Modified Bessel functions

The modified Bessel function $K_\nu(t)$ of the first kind of order ν is defined for $t \in \mathbb{R}$ and $\nu \in \mathbb{C}$ by the formula

$$K_\nu(t) = \int_0^\infty \exp(-t \cosh(t)) \cosh(\nu t) dt. \quad (58)$$

The following bound on the ratio of $K_{\nu+1}$ to K_ν can be found in [19].

Theorem 6. *Suppose that $t > 0$ and $\nu > 0$ are real numbers. Then*

$$\frac{K_{\nu+1}(t)}{K_\nu(t)} < \frac{\nu + \sqrt{\nu^2 + t^2}}{t} \leq \frac{2\nu}{t} + 1. \quad (59)$$

2.6. The binomial theorem

A proof of the following can be found in [20], as well as many other sources.

Theorem 7. *Suppose that r is a real number, and that y is a real number such that $|y| < 1$. Then*

$$(1 + y)^r = \sum_{k=0}^{\infty} \frac{\Gamma(r + 1)}{\Gamma(k + 1)\Gamma(r - k + 1)} y^k. \quad (60)$$

2.7. Fréchet derivatives and the contraction mapping principle

Given Banach spaces X, Y and a mapping $f : X \rightarrow Y$ between them, we say that f is Fréchet differentiable at $x \in X$ if there exists a bounded linear operator $X \rightarrow Y$, denoted by f'_x , such that

$$\lim_{h \rightarrow 0} \frac{\|f(x + h) - f(x) - f'_x[h]\|}{\|h\|} = 0. \quad (61)$$

Theorem 8. *Suppose that X and Y are a Banach spaces and that $f : X \rightarrow Y$ is Fréchet differentiable at every point of X . Suppose also that D is a convex subset of X , and that there exists a real number $M > 0$ such that*

$$\|f'_x\| \leq M \quad (62)$$

for all $x \in D$. Then

$$\|f(x) - f(y)\| \leq M\|x - y\| \quad (63)$$

for all x and y in D .

Suppose that $f : X \rightarrow X$ is a mapping of the Banach space X into itself. We say that f is contractive on a subset D of X if there exists a real number $0 < \alpha < 1$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| \quad (64)$$

for all $x, y \in D$. Moreover, we say that $\{x_n\}_{n=0}^{\infty}$ is a sequence of fixed point iterates for f if $x_{n+1} = f(x_n)$ for all $n \geq 0$.

Theorem 8 is often used to show that a mapping is contractive so that the following result can be applied.

Theorem 9. (*The Contraction Mapping Principle*) *Suppose that D is a closed subset of a Banach space X . Suppose also that $f : X \rightarrow X$ is contractive on D and $f(D) \subset D$. Then the equation*

$$x = f(x) \quad (65)$$

has a unique solution $\sigma^ \in D$. Moreover, any sequence of fixed point iterates for the function f which contains an element in D converges to σ^* .*

A discussion of Fréchet derivatives and proofs of Theorems 8 and 9 can be found in, for instance, [21].

2.8. Schwarzian derivatives

The Schwarzian derivative of a smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ is

$$\{f, t\} = \frac{f'''(t)}{f'(t)} - \frac{3}{2} \left(\frac{f''(t)}{f'(t)} \right)^2. \quad (66)$$

If the function $x(t)$ is a diffeomorphism of the real line (that is, a smooth, invertible mapping $\mathbb{R} \rightarrow \mathbb{R}$), then the Schwarzian derivative of $x(t)$ can be related to the Schwarzian derivative of its inverse $t(x)$; in particular,

$$\{x, t\} = - \left(\frac{dx}{dt} \right)^2 \{t, x\}. \quad (67)$$

The identity (67) can be found, for instance, in Section 1.13 of [6].

2.9. A bump function

It is well known that the function φ defined by the formula

$$\varphi(\xi) = \left(\int_{-1}^1 \exp\left(\frac{1}{u^2 - 1}\right) du \right)^{-1} \int_{-\infty}^{\xi} \exp\left(\frac{1}{u^2 - 1}\right) \chi_{(-1,1)}(u) du \quad (68)$$

is an element of $C^\infty(\mathbb{R})$ such that $\varphi(\xi) = 0$ for all $\xi < -1$, $\varphi(\xi) = 1$ for all $\xi > 1$ and $0 \leq \varphi(\xi) \leq 1$ for all $\xi \in \mathbb{R}$ (see, for instance, [16]). We suppose that λ is a positive real number and define the bump function \hat{b} via the formula

$$\hat{b}(\xi) = \frac{1}{2} \left(\varphi\left(\frac{\xi + c}{\alpha}\right) - \varphi\left(\frac{\xi - c}{\alpha}\right) \right), \quad (69)$$

where

$$c = \frac{\sqrt{2}\lambda + \lambda}{2} \quad (70)$$

and

$$\alpha = \frac{c - \lambda}{4} = \frac{\sqrt{2}\lambda - \lambda}{4}. \quad (71)$$

We observe that \widehat{b} is an element of $C_c^\infty(\mathbb{R})$, the support of \widehat{b} is contained in $(-\sqrt{2}\lambda, \sqrt{2}\lambda)$, $0 \leq \widehat{b}(\xi) \leq 1$ for all $\xi \in \mathbb{R}$, and $\widehat{b}(\xi) = 1$ for all $|\xi| \leq \lambda$. Since \widehat{b} is an element of $C_c^\infty(\mathbb{R})$, its inverse Fourier transform b is an entire function and an element of $S(\mathbb{R})$.

2.10. The Liouville-Green transform

The Liouville-Green transform is a well-known tool for analyzing the variable coefficient Helmholtz equation

$$y''(t) + \lambda^2 q(t)y(t) = f(t). \quad (72)$$

The following can be found in Chapter 2 of [22] (for instance).

Theorem 10. *Suppose that $q : [a, b] \rightarrow \mathbb{R}$ is twice continuously differentiable and strictly positive, that $f : [a, b] \rightarrow \mathbb{C}$ is continuous, that the function x is defined by the formula*

$$x(t) = \int_a^t \sqrt{q(u)} \, du, \quad (73)$$

and that the function p is defined by the formula

$$p(t) = \frac{1}{q(t)} \left(\frac{5}{4} \left(\frac{q'(t)}{q(t)} \right)^2 - \frac{q''(t)}{q(t)} \right). \quad (74)$$

Suppose also that $y : [a, b] \rightarrow \mathbb{R}$ is twice continuously differentiable, and that

$$y''(t) + \lambda^2 q(t)y(t) = f(t) \quad \text{for all } a \leq t \leq b. \quad (75)$$

Then the inverse $t(x)$ of the function $x(t)$ is continuously differentiable, and the function $\varphi : [0, x(b)] \rightarrow \mathbb{R}$ defined by the formula

$$\varphi(x) = (q(t(x)))^{1/4} y(t(x)) \quad (76)$$

is the unique solution of the initial value problem

$$\begin{cases} \varphi''(x) + \lambda^2 \varphi(x) = (q(x))^{-3/4} f(x) - \frac{1}{4} p(x) \varphi(x) & \text{for all } 0 \leq x \leq x(b) \\ \varphi(0) = (q(a))^{1/4} y(a) \\ \varphi'(0) = q'(a)(q(a))^{-5/4} y(a) + (q(a))^{-1/4} y'(a). \end{cases} \quad (77)$$

Remark 1. *We observe that due to Formula (67), the function $p(x)$ appearing in (77) is twice the Schwarzian derivative of the inverse $t(x)$ of the function $x(t)$ defined in (73). That is, $p(x) = 2\{t, x\}$.*

2.11. Gronwall's inequality

The following well-known inequality can be found in, for instance, [23].

Theorem 11. *Suppose that f and g are continuous functions on the interval $[a, b]$ such that*

$$f(t) \geq 0 \quad \text{and} \quad g(t) \geq 0 \quad \text{for all } a \leq t \leq b. \quad (78)$$

Suppose further that there exists a real number $C > 0$ such that

$$f(t) \leq C + \int_a^t f(s)g(s) \, ds \quad \text{for all } a \leq t \leq b. \quad (79)$$

Then

$$f(t) \leq C \exp \left(\int_a^t g(s) \, ds \right) \quad \text{for all } a \leq t \leq b. \quad (80)$$

3. Integral equation formulation

In this section, we reformulate Kummer's equation

$$(\alpha'(t))^2 = \lambda^2 q(t) - \frac{1}{2} \frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4} \left(\frac{\alpha''(t)}{\alpha'(t)} \right)^2 \quad (81)$$

as a nonlinear integral equation. We assume that the function q has been extended to the real line and we seek a function α which satisfies (81) there.

By letting

$$(\alpha'(t))^2 = \lambda^2 \exp(r(t)) \quad (82)$$

in (81), we obtain the equation

$$r''(t) - \frac{1}{4} (r'(t))^2 + 4\lambda^2 (\exp(r(t)) - q(t)) = 0 \quad \text{for all } t \in \mathbb{R}. \quad (83)$$

We next take r to be of the form

$$r(t) = \log(q(t)) + \delta(t), \quad (84)$$

which results in

$$\delta''(t) - \frac{1}{2} \frac{q'(t)}{q(t)} \delta'(t) - \frac{1}{4} (\delta'(t))^2 + 4\lambda^2 q(t) (\exp(\delta(t)) - 1) = q(t)p(t), \quad \text{for all } t \in \mathbb{R}, \quad (85)$$

where p is defined by the formula

$$p(t) = \frac{1}{q(t)} \left(\frac{5}{4} \left(\frac{q'(t)}{q(t)} \right)^2 - \frac{q''(t)}{q(t)} \right). \quad (86)$$

Expanding the exponential in a power series and rearranging terms yields the equation

$$\delta''(t) - \frac{1}{2} \frac{q'(t)}{q(t)} \delta'(t) + 4\lambda^2 q(t) \delta(t) - \frac{1}{4} (\delta'(t))^2 + 4\lambda^2 q(t) \left(\frac{(\delta(t))^2}{2} + \frac{(\delta(t))^3}{3!} + \dots \right) = q(t)p(t). \quad (87)$$

Applying the change of variables

$$x(t) = \int_a^t \sqrt{q(u)} \, du \quad (88)$$

transforms (87) into

$$\delta''(x) + 4\lambda^2 \delta(x) - \frac{1}{4} (\delta'(x))^2 + 4\lambda^2 \left(\frac{(\delta(x))^2}{2} + \frac{(\delta(x))^3}{3!} + \dots \right) = p(x) \quad \text{for all } x \in \mathbb{R}. \quad (89)$$

At first glance, the relationship between the function $p(x)$ appearing in (89) and the coefficient $q(t)$ in the ordinary differential equation (1) is complex. However, the function $p(t)$ defined via (86) is related to the Schwarzian derivative (see Section 2.8) of the function $x(t)$ defined in (88) via the formula

$$p(t) = -\frac{2}{q(t)} \{x, t\} = -2 \left(\frac{dt}{dx} \right)^2 \{x, t\}. \quad (90)$$

It follows from (90) and Formula (67) in Section 2.8 that

$$p(x) = 2 \{t, x\}. \quad (91)$$

That is to say: p , when viewed as a function of x , is simply twice the Schwarzian derivative of t with respect to x .

It is also notable that the part of (89) which is linear in δ is the constant coefficient Helmholtz equation. This suggests that we form an integral equation for (89) using a Green's function for the Helmholtz

equation. To that end, we define the linear integral operator T for functions $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ via the formula

$$T[f](x) = \frac{1}{4\lambda} \int_{-\infty}^{\infty} \sin(2\lambda|x-y|) f(y) dy. \quad (92)$$

We extend the domain of T to include functions $f \in L^2(\mathbb{R})$ whose Fourier transforms have support in the interval $(-2\lambda, 2\lambda)$ through the formula

$$T[f](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ix\xi) \frac{\widehat{f}(\xi)}{4\lambda^2 - \xi^2} d\xi. \quad (93)$$

Introducing the representation

$$\delta(x) = T[\sigma](x) \quad (94)$$

into (89) yields the nonlinear integral equation

$$\sigma(x) = S[T[\sigma]](x) + p(x) \quad \text{for all } x \in \mathbb{R}, \quad (95)$$

where S is the nonlinear differential operator defined by the formula

$$S[f](x) = \frac{(f'(x))^2}{4} - 4\lambda^2 \left(\frac{(f(x))^2}{2!} + \frac{(f(x))^3}{3!} + \frac{(f(x))^4}{4!} + \dots \right). \quad (96)$$

According to Theorems 3 and 4, if σ is a solution of the integral equation (95) and either $\sigma \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ or $\sigma \in L^2(\mathbb{R})$ and the support of $\widehat{\sigma}$ is contained in $(-2\lambda, 2\lambda)$, then the function δ defined via formula (94) is a solution of (89). Moreover, the function r defined via the formula

$$r(t) = \log(q(t)) + \delta(x(t)) \quad (97)$$

is a solution of (83), and

$$\alpha(t) = \lambda \int_a^t \exp\left(\frac{r(u)}{2}\right) du \quad (98)$$

is a phase function for (1).

4. Existence of nonoscillatory phase functions

The nonlinear integral equation (95) is not solvable for arbitrary p . However, when the Fourier transform of the function p decays exponentially, there exists a function σ whose Fourier transform is compactly supported and a function ν of magnitude on the order of $\exp(-\rho\lambda)$, where ρ is a real constant, such that

$$\sigma(x) = S[T[\sigma]](x) + p(x) + \nu(x) \quad \text{for all } x \in \mathbb{R}. \quad (99)$$

The following theorem, which is the principal result of this article, makes these statements precise. Its proof is given in Sections 5, 6 and 7.

Theorem 12. *Suppose that $q \in C^\infty(\mathbb{R})$ is strictly positive, that $x(t)$ is defined by the formula*

$$x(t) = \int_0^t \sqrt{q(u)} du, \quad (100)$$

and that the function p defined via the formula

$$p(x) = 2\{t, x\} \quad (101)$$

is an element of $S(\mathbb{R})$. Suppose furthermore that there exist positive real numbers λ , Γ and μ such that

$$\lambda > 2 \max \left\{ \frac{1}{\mu}, \Gamma \right\} \quad (102)$$

and

$$|\hat{p}(\xi)| \leq \Gamma \exp(-\mu|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \quad (103)$$

Then there exist functions σ and ν in $S(\mathbb{R})$ such that σ is a solution of the nonlinear integral equation

$$\sigma(x) = S[T[\sigma]](x) + p(x) + \nu(x), \quad \text{for all } x \in \mathbb{R}, \quad (104)$$

$$|\hat{\sigma}(\xi)| \leq \left(1 + \frac{2\Gamma}{\lambda}\right) \Gamma \exp(-\mu|\xi|) \quad \text{for all } |\xi| \leq \sqrt{2}\lambda, \quad (105)$$

$$\hat{\sigma}(\xi) = 0 \quad \text{for all } |\xi| > \sqrt{2}\lambda, \quad (106)$$

and

$$\|\nu\|_{\infty} \leq \frac{\Gamma}{2\mu} \left(1 + \frac{4\Gamma}{\lambda}\right) \exp(-\mu\lambda). \quad (107)$$

Remark 2. By combining (102) with (105), we see that

$$|\hat{\sigma}(\xi)| \leq 2\Gamma \exp(-\mu|\xi|) \quad (108)$$

for all $|\xi| \leq \sqrt{2}\lambda$. Similarly, we conclude from (102) and (107) that

$$\|\nu\|_{\infty} \leq \frac{3\Gamma}{2\mu} \exp(-\mu\lambda). \quad (109)$$

Suppose that σ and ν are the functions obtained by invoking Theorem 12, and that $x(t)$ is the function defined by the formula

$$x(t) = \int_a^t \sqrt{q(u)} \, du. \quad (110)$$

We define δ by the formula

$$\delta(x) = T[\sigma](x), \quad (111)$$

r by the formula

$$r(t) = \log(q(t)) + \delta(x(t)), \quad (112)$$

and α by the formula

$$\alpha(t) = \lambda \int_a^t \exp\left(\frac{r(u)}{2}\right) \, du. \quad (113)$$

From the discussion in Section 3, we conclude that $\delta(x)$ is a solution of the nonlinear differential equation

$$\delta''(x) + 4\lambda^2\delta(x) = S[\delta](x) + p(x) + \nu(x) \quad \text{for all } x \in \mathbb{R}, \quad (114)$$

that $r(t)$ is a solution of the nonlinear differential equation

$$r''(t) - \frac{1}{4}(r'(t))^2 + 4\lambda^2(\exp(r(t)) - q(t)) = q(t)\nu(t) \quad \text{for all } t \in \mathbb{R}, \quad (115)$$

and that α is a solution of the nonlinear differential equation

$$(\alpha'(t))^2 = \lambda^2 \left(\frac{\nu(t)}{4\lambda^2} + 1\right) q(t) - \frac{1}{2} \frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4} \left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 \quad \text{for all } t \in \mathbb{R}. \quad (116)$$

From (116), we see that α is a phase function for the second order linear ordinary differential equation

$$y''(t) + \lambda^2 \left(1 + \frac{\nu(t)}{4\lambda^2}\right) q(t)y(t) = 0 \quad \text{for all } a \leq t \leq b. \quad (117)$$

Since the magnitude of ν is small, we expect that the difference between solutions of (117) and those of (1) will be small as well. Indeed, the following theorem follows easily by applying the Liouville-Green

transform and invoking Gronwall's inequality.

Theorem 13. *Suppose that q is continuous and strictly positive on the interval $[a, b]$, that f is a continuous function $[a, b] \rightarrow \mathbb{C}$, and that λ is a positive real number. Suppose also that $z : [a, b] \rightarrow \mathbb{C}$ is a twice continuously differentiable function such that*

$$z''(t) + \lambda^2 q(t)z(t) = 0 \quad \text{for all } a \leq t \leq b, \quad (118)$$

and that $y : [a, b] \rightarrow \mathbb{C}$ is a twice continuously differentiable function such that

$$y''(t) + \lambda^2 q(t)y(t) = f(t) \quad \text{for all } a \leq t \leq b, \quad (119)$$

$y(a) = z(a)$, and $y'(a) = z'(a)$. Then there exists a positive real number C such that

$$|y(t) - z(t)| \leq \frac{C}{\lambda} \sup_{a \leq t \leq b} |f(t)| \quad \text{for all } a \leq t \leq b. \quad (120)$$

The constant C depends on q but not on λ or f .

Proof. We define the function Δ by the formula

$$\Delta(t) = y(t) - z(t) \quad (121)$$

and observe that Δ is the unique solution of the boundary value problem

$$\begin{cases} \Delta''(t) + \lambda^2 q(t)\Delta(t) = f(t) & \text{for all } a \leq t \leq b \\ \Delta(a) = \Delta'(a) = 0. \end{cases} \quad (122)$$

We define the function $x(t)$ via the formula

$$x(t) = \int_a^t \sqrt{q(u)} \, du \quad (123)$$

and use $t(x)$ to denote its inverse; moreover, we define the function $p(t)$ via the formula

$$p(t) = \frac{1}{q(t)} \left(\frac{5}{4} \left(\frac{q'(t)}{q(t)} \right)^2 - \frac{q''(t)}{q(t)} \right). \quad (124)$$

According to Theorem 10, the function φ defined via

$$\varphi(x) = (q(t(x)))^{1/4} \Delta(t(x)), \quad (125)$$

is the unique solution of the initial value problem

$$\begin{cases} \varphi''(x) + \lambda^2 \varphi(x) = (q(x))^{-3/4} f(x) - \frac{1}{4} p(x) \varphi(x) & \text{for all } 0 \leq t \leq x(b) \\ \varphi(0) = \varphi'(0) = 0. \end{cases} \quad (126)$$

We conclude from (126) and Theorem 5 that

$$\varphi(x) = \frac{1}{\lambda} \int_0^x \sin(\lambda(x-u)) \left((q(u))^{-3/4} f(u) - \frac{1}{4} p(u) \varphi(u) \right) du \quad (127)$$

for all $0 \leq x \leq x(b)$. We let

$$C_1 = x(b) \cdot \sup_{a \leq u \leq b} (q(u))^{-3/4} \quad (128)$$

and

$$\|f\|_\infty = \sup_{a \leq u \leq b} |f(u)|. \quad (129)$$

We observe that

$$\left| \frac{1}{\lambda} \int_0^x \sin(\lambda(x-u)) (q(u))^{-3/4} f(u) \, du \right| \leq \frac{C_1}{\lambda} \|f\|_\infty \quad (130)$$

for all $0 \leq x \leq x(b)$. Now we let

$$C_2 = \sup_{a \leq u \leq b} p(u) \quad (131)$$

and observe that

$$\left| \frac{1}{4\lambda} \int_0^x \sin(\lambda(x-u)) p(u) \phi(u) du \right| \leq \frac{C_2}{4\lambda} \int_0^x |\phi(u)| du \quad (132)$$

for all $0 \leq x \leq x(b)$. We combine (127), (130) and (132) in order to conclude that

$$|\varphi(x)| \leq \frac{C_1}{\lambda} \|f\|_\infty + \frac{C_2}{4\lambda} \int_0^x |\phi(u)| du \quad (133)$$

for all $0 \leq x \leq x(b)$. By invoking Gronwall's inequality (which is Theorem 11 in Section 2.11) we conclude that

$$|\varphi(x)| \leq \frac{C_1}{\lambda} \|f\|_\infty \exp\left(\frac{C_2 x}{4\lambda}\right) \quad (134)$$

for all $0 \leq x \leq x(b)$. We conclude from (121), (125) and (134) that

$$|y(t) - z(t)| \leq \frac{C}{\lambda} \|f\|_\infty \quad (135)$$

for all $a \leq t \leq b$, where C is defined via the formula

$$C = C_1 \exp\left(\frac{C_2 x(b)}{4\lambda}\right) \cdot \sup_{a \leq u \leq b} (q(u))^{-1/4}. \quad (136)$$

□

By applying Theorem 13 to (117) and (1) we obtain the following.

Theorem 14. *Suppose that the hypotheses of Theorem 12 are satisfied, that σ and ν are the functions obtained by invoking it. Suppose also that α is defined as in (113), and that u, v are the functions defined via the formulas*

$$u(t) = \frac{\cos(\alpha(t))}{\sqrt{\alpha'(t)}} \quad (137)$$

and

$$v(t) = \frac{\sin(\alpha(t))}{\sqrt{\alpha'(t)}}. \quad (138)$$

Then there exist a constant C and a basis $\{\tilde{u}, \tilde{v}\}$ in the space of solutions of (1) such that

$$|u(t) - \tilde{u}(t)| \leq \frac{C}{\lambda} \exp(-\mu\lambda) \quad \text{for all } a \leq t \leq b \quad (139)$$

and

$$|v(t) - \tilde{v}(t)| \leq \frac{C}{\lambda} \exp(-\mu\lambda) \quad \text{for all } a \leq t \leq b. \quad (140)$$

The constant C depends on the coefficient q appearing in (1), but not on the parameter λ .

The rest of this article is devoted to the proof of Theorem 12. It is divided among Sections 5, 6 and 7. The principal difficulty lies in constructing a function ν such that (99) admits a solution. We accomplish this by introducing a modified integral equation

$$\sigma_b(x) = S[T_b[\sigma_b]](x) + p(x), \quad (141)$$

where T_b is a "band-limited" version of T . That is, $T_b[f]$ is defined via the formula

$$\widehat{T_b[f]}(\xi) = \widehat{T[f]}(\xi) \widehat{b}(\xi), \quad (142)$$

where $\widehat{b}(\xi)$ is the $C_c^\infty(\mathbb{R})$ bump function given by Formula (69) of Section 2.9. In Section 5, we apply the Fourier transform to (141) and use the contraction mapping principle to show that under mild conditions on p and λ the resulting equation admits a solution. This gives rise to a solution σ_b of (141). In Section 6, we show that if the Fourier transform of the function p is exponentially decaying, then the Fourier transform of σ_b is as well. In Section 7 we use the solution σ_b of (141) in order to construct functions σ and ν which satisfy (99). Moreover, we show that σ can be taken to be an element of the space $S(\mathbb{R})$ of rapidly decaying Schwartz functions (see Section 2.1), that the Fourier transform of σ is compactly supported and exponentially decaying, and that ν is an element of $S(\mathbb{R})$ whose $L^\infty(\mathbb{R})$ norm decays exponentially with λ .

5. Band-limited integral equation

In this section, we introduce a “band-limited” version of the operator T , use it to form an alternative to the integral equation (95), and apply the contraction mapping principle in order to show that this alternate equation admits a solution under mild conditions on the function p and the parameter λ .

Let $\widehat{b}(\xi)$ be the bump function defined via formula (69) so that

1. $\widehat{b}(\xi) = 1$ for all $|\xi| \leq \lambda$,
2. $0 \leq \widehat{b}(\xi) \leq 1$ for all $\xi \in \mathbb{R}$, and
3. \widehat{b} is supported on a proper subset of $(-\sqrt{2}\lambda, \sqrt{2}\lambda)$.

We define the operator $T_b[f]$ for functions $f \in C_0(\mathbb{R})$ such that $\widehat{f} \in L^1(\mathbb{R})$ via the formula

$$\widehat{T_b[f]}(\xi) = \widehat{f}(\xi) \frac{\widehat{b}(\xi)}{4\lambda^2 - \xi^2}. \quad (143)$$

We will refer to T_b as the band-limited version of the operator T and we call the nonlinear integral equation

$$\sigma_b(x) = S[T_b[\sigma_b]](x) + p(x) \quad \text{for all } x \in \mathbb{R} \quad (144)$$

obtained by replacing T with T_b in (95) the “band-limited” version of (95).

Remark 3. *The function*

$$\frac{\widehat{b}(\xi)}{4\lambda^2 - \xi^2} \quad (145)$$

is an element of $C_c^\infty(\mathbb{R})$ since the support of \widehat{b} is bounded away from the points $\pm 2\lambda$ at which the denominator vanishes. Consequently, (143) is a compactly supported tempered distribution and $T_b[f]$ is an entire function whenever f is a tempered distribution. For our purposes, it suffices to know that $T_b[f]$ is defined for $\widehat{f} \in L^1(\mathbb{R})$.

It is convenient to analyze (144) in the Fourier domain rather than the space domain. We denote by W_b and \widetilde{W}_b the linear operators defined for $f \in L^1(\mathbb{R})$ via the formulas

$$W_b[f](\xi) = f(\xi) \frac{\widehat{b}(\xi)}{4\lambda^2 - \xi^2} \quad (146)$$

and

$$\widetilde{W}_b[f](\xi) = f(\xi) \frac{-i\xi\widehat{b}(\xi)}{4\lambda^2 - \xi^2}, \quad (147)$$

where $\widehat{b}(\xi)$ is the function used to define the operator T_b . We define functions $\psi(\xi)$ and $w(\xi)$ using the formulas

$$\psi(\xi) = \widehat{\sigma}_b(\xi) \quad (148)$$

and

$$w(\xi) = \widehat{p}(\xi). \quad (149)$$

Finally, we define $R[f]$ for functions $f \in L^1(\mathbb{R})$ via

$$R[f](\xi) = \frac{1}{8\pi} \widetilde{W}_b[f] * \widetilde{W}_b[f](\xi) - 4\lambda^2 \exp_2^*[W_b[f]](\xi) + w(\xi), \quad (150)$$

where \exp_2^* is the operator defined by Formula (32) of Section 2.3. Applying Fourier transform to both sides of (144) results in the nonlinear equation

$$\psi(\xi) = R[\psi](\xi). \quad (151)$$

The following theorem gives conditions under which the sequence $\{\psi_n\}_{n=0}^\infty$ of fixed point iterates for (151) obtained by using the function w defined by (149) as an initial approximation converges. More explicitly, ψ_0 is defined by the formula

$$\psi_0(\xi) = w(\xi), \quad (152)$$

and for each integer $n \geq 0$, ψ_{n+1} is obtained from ψ_n via

$$\psi_{n+1}(\xi) = R[\psi_n](\xi). \quad (153)$$

Theorem 15. *Suppose that $\lambda > 0$ is a real number, and that w is an element of $L^1(\mathbb{R})$ such that*

$$\|w\|_1 \leq \frac{\pi}{2} \lambda^2. \quad (154)$$

Then the sequence $\{\psi_n\}$ defined by (152) and (153) converges in $L^1(\mathbb{R})$ norm to a function $\psi \in L^1(\mathbb{R})$ such that

$$\psi(\xi) = R[\psi](\xi) \quad \text{for all } \xi \in \mathbb{R}. \quad (155)$$

Proof. We observe that the Fréchet derivative (see Section 2.7) of R at f is the linear operator $R'_f : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ given by the formula

$$R'_f[h](\xi) = \frac{\widetilde{W}_b[f] * \widetilde{W}_b[h](\xi)}{4\pi} - 4\lambda^2 \exp_1^* \left[\frac{W_b[f]}{2\pi} \right] * W_b[h](\xi). \quad (156)$$

From formulas (146) and (147) and the definition of $\widehat{b}(\xi)$ we see that

$$\|W_b[f]\|_1 \leq \frac{\|f\|_1}{2\lambda^2}, \quad (157)$$

and

$$\|\widetilde{W}_b[f]\|_1 \leq \frac{\|f\|_1}{\sqrt{2}\lambda} \quad (158)$$

for all $f \in L^1(\mathbb{R})$. We combine (156) with (34) in order to conclude that

$$\|R'_f[h]\|_1 \leq \frac{1}{4\pi} \|\widetilde{W}_b[f]\|_1 \|\widetilde{W}_b[h]\|_1 + \frac{2\lambda^2}{\pi} \|W_b[f]\|_1 \exp\left(\frac{\|W_b[f]\|_1}{2\pi}\right) \|W_b[h]\|_1 \quad (159)$$

for all f and h in $L^1(\mathbb{R})$. By inserting (157) and (158) into (159) we see that

$$\begin{aligned} \|R'_f[h]\|_1 &\leq \frac{\|f\|_1 \|h\|_1}{8\pi\lambda^2} + \frac{2\lambda^2}{\pi} \frac{\|f\|_1}{2\lambda^2} \frac{\|h\|_1}{2\lambda^2} \exp\left(\frac{\|f\|_1}{4\pi\lambda^2}\right) \\ &\leq \left(\frac{\|f\|_1}{8\pi\lambda^2} + \frac{\|f\|_1}{2\pi\lambda^2} \exp\left(\frac{\|f\|_1}{4\pi\lambda^2}\right)\right) \|h\|_1 \end{aligned} \quad (160)$$

for all f and h in $L^1(\mathbb{R})$. Similarly, by combining (150), (35), (157) and (158) we conclude that

$$\begin{aligned} \|R[f]\|_1 &\leq \frac{1}{8\pi} \|\widetilde{W}_b[f]\|_1^2 + \frac{\lambda^2}{\pi} \|W_b[f]\|_1^2 \exp\left(\frac{\|W_b[f]\|_1}{2\pi}\right) + \|w\|_1 \\ &\leq \frac{\|f\|_1^2}{16\pi\lambda^2} + \frac{\|f\|_1^2}{4\pi\lambda^2} \exp\left(\frac{\|f\|_1}{4\pi\lambda^2}\right) + \|w\|_1 \end{aligned} \quad (161)$$

whenever $f \in L^1(\mathbb{R})$. We now let $r = \pi\lambda^2$ and denote by Ω the closed ball of radius r centered at 0 in $L^1(\mathbb{R})$. Suppose that $f \in L^1(\mathbb{R})$ such that

$$\|f\|_1 \leq r = \pi\lambda^2, \quad (162)$$

and that

$$\|w\|_1 \leq \frac{r}{2} = \frac{\pi\lambda^2}{2}. \quad (163)$$

We insert (162) and (163) into (161) in order to obtain

$$\begin{aligned} \|R[f]\|_1 &\leq \frac{r^2}{16\pi\lambda^2} + \frac{r^2}{4\pi\lambda^2} \exp\left(\frac{\|f\|_1}{4\pi\lambda^2}\right) + \frac{r}{2} \\ &= \left(\frac{r}{16\pi\lambda^2} + \frac{r}{4\pi\lambda^2} \exp\left(\frac{\|f\|_1}{4\pi\lambda^2}\right) + \frac{1}{2}\right) r \\ &= \left(\frac{1}{16} + \frac{1}{4} \exp\left(\frac{1}{4}\right) + \frac{1}{2}\right) r \\ &\leq \frac{9}{10} r, \end{aligned} \quad (164)$$

from which we conclude that R maps Ω into itself. Next, we insert (162) into (160) in order to obtain

$$\begin{aligned} \|R'_f[h]\|_1 &\leq \left(\frac{r}{8\pi\lambda^2} + \frac{r}{2\pi\lambda^2} \exp\left(\frac{r}{4\pi\lambda^2}\right)\right) \|h\|_1 \\ &\leq \left(\frac{1}{8} + \frac{1}{2} \exp\left(\frac{1}{4}\right)\right) \|h\|_1 \\ &\leq \frac{8}{10} \|h\|_1, \end{aligned} \quad (165)$$

which shows that R is a contraction on Ω . We now invoke the contraction mapping theorem (Theorem 9 in Section 2.7) in order to conclude that any sequence of fixed point iterates for (151) which originates in Ω will converge in $L^1(\mathbb{R})$ to a solution of (151). Since $\{\psi_n\}$ is such a sequence, it converges in $L^1(\mathbb{R})$ to a function ψ such that

$$\psi(\xi) = R[\psi](\xi) \quad \text{for almost all } \xi \in \mathbb{R}. \quad (166)$$

We change the values of ψ on a set of measure zero (which does not affect $R[\psi]$) in order to ensure that (166) holds for all $\xi \in \mathbb{R}$. \square

6. Fourier estimate

In this section, we derive a pointwise estimate on the solution ψ of Equation (151) under additional assumptions on the function w .

Lemma 1. *Suppose that μ and C are real numbers such that*

$$0 \leq C < \mu. \quad (167)$$

Suppose also that $f \in L^1(\mathbb{R})$, and that

$$|f(\xi)| \leq C \exp(-\mu|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \quad (168)$$

Then

$$|\exp_2^*[f](\xi)| \leq \frac{C^2}{2\pi} \exp(-\mu|\xi|) \frac{1 + \mu|\xi|}{\mu} \exp\left(\frac{C}{2\pi\mu}\right) \exp\left(\frac{C}{2\pi}|\xi|\right) \quad \text{for all } \xi \in \mathbb{R}, \quad (169)$$

where \exp_2^ is the operator defined in (32).*

Proof. We let

$$g_1(\xi) = C \exp(-\mu|\xi|) \quad (170)$$

and for each integer $m > 0$, we define g_{m+1} in terms of g_m through the formula

$$g_{m+1}(\xi) = \frac{1}{2\pi} g_m * g_1(\xi). \quad (171)$$

We observe that for each integer $m > 0$ and all $\xi \in \mathbb{R}$,

$$g_m(\xi) = 2\sqrt{\mu C} \left(\frac{C|\xi|}{2\pi}\right)^{m-1/2} \frac{K_{m-1/2}(\mu|\xi|)}{\Gamma(m)}, \quad (172)$$

where K_ν denotes the modified Bessel function of the second kind of order ν (see Section 2.5). By repeatedly applying Theorem 6 of Section 2.5, we conclude that for all integers $m > 0$ and all real t ,

$$\begin{aligned} K_{m-1/2}(t) &\leq K_{1/2}(t) \prod_{j=1}^{m-1} \left(\frac{2(j-\frac{1}{2})}{t} + 1 \right) \\ &= K_{1/2}(t) \left(\frac{2}{t}\right)^{m-1} \frac{\Gamma(\frac{1+t}{2} + m - 1)}{\Gamma(\frac{1+t}{2})}. \end{aligned} \quad (173)$$

We insert the identity

$$K_{1/2}(t) = \sqrt{\frac{\pi}{2t}} \exp(-t) \quad (174)$$

into (173) in order to conclude that for all integers $m > 0$ and all real numbers $t > 0$,

$$K_{m-1/2}(t) \leq \frac{\sqrt{\pi}}{2} \left(\frac{t}{2}\right)^{1/2-m} \exp(-t) \frac{\Gamma(\frac{1+t}{2} + m - 1)}{\Gamma(\frac{1+t}{2})}. \quad (175)$$

By combining (175) and (172) we conclude that

$$g_m(\xi) \leq C \exp(-\mu|\xi|) \left(\frac{C}{\pi\mu}\right)^{m-1} \frac{\Gamma\left(\frac{1+\mu|\xi|}{2} + m - 1\right)}{\Gamma(m)\Gamma\left(\frac{1+\mu|\xi|}{2}\right)} \quad (176)$$

for all integers $m > 0$ and all $\xi \neq 0$. Moreover, the limit as $\xi \rightarrow 0$ of each side of (176) is finite and the two limits are equal, so (176) in fact holds for all $\xi \in \mathbb{R}$. We sum (176) over $m = 2, 3, \dots$ in order to conclude that

$$\begin{aligned} \exp_2^*[g](\xi) &\leq C \exp(-\mu|\xi|) \sum_{m=2}^{\infty} \left(\frac{C}{\pi\mu}\right)^{m-1} \frac{\Gamma\left(\frac{1+\mu|\xi|}{2} + m - 1\right)}{\Gamma(m+1)\Gamma(m)\Gamma\left(\frac{1+\mu|\xi|}{2}\right)} \\ &= C \exp(-\mu|\xi|) \sum_{m=1}^{\infty} \left(\frac{C}{\pi\mu}\right)^m \frac{\Gamma\left(\frac{1+\mu|\xi|}{2} + m\right)}{\Gamma(m+2)\Gamma(m+1)\Gamma\left(\frac{1+\mu|\xi|}{2}\right)} \end{aligned} \quad (177)$$

for all $\xi \in \mathbb{R}$. Now we observe that

$$\frac{1}{\Gamma(m+2)} \leq \left(\frac{1}{2}\right)^m \quad \text{for all } m = 0, 1, 2, \dots \quad (178)$$

Inserting (178) into (177) yields

$$\exp_2^*[g](\xi) \leq C \exp(-\mu|\xi|) \sum_{m=1}^{\infty} \left(\frac{C}{2\pi\mu}\right)^m \frac{\Gamma\left(\frac{1+\mu|\xi|}{2} + m\right)}{\Gamma(m+1)\Gamma\left(\frac{1+\mu|\xi|}{2}\right)} \quad (179)$$

for all $\xi \in \mathbb{R}$. Now we apply the binomial theorem (Theorem 7 of Section 2.6), which is justified since $C < a < 2\pi\mu$, to conclude that

$$\begin{aligned} \exp_2^*[g](\xi) &\leq C \exp(-\mu|\xi|) \left(\left(1 - \frac{C}{2\pi\mu}\right)^{-\frac{1+\mu|\xi|}{2}} - 1 \right) \\ &= C \exp(-\mu|\xi|) \left(\exp\left(\frac{1+\mu|\xi|}{2} \log\left(\frac{1}{1 - \frac{C}{2\pi\mu}}\right)\right) - 1 \right) \end{aligned} \quad (180)$$

for all $\xi \in \mathbb{R}$. We observe that

$$\exp(x) - 1 \leq x \exp(x) \quad \text{for all } x \geq 0, \quad (181)$$

and

$$0 \leq \log\left(\frac{1}{1-x}\right) \leq 2x \quad \text{for all } 0 \leq x \leq \frac{1}{2\pi}. \quad (182)$$

By combining (181) and (182) with (180) we conclude that

$$\begin{aligned} \exp_2^*[g](\xi) &\leq C \exp(-\mu|\xi|) \left(\frac{1+\mu|\xi|}{2}\right) \log\left(\frac{1}{1 - \frac{C}{2\pi\mu}}\right) \exp\left(\frac{1+\mu|\xi|}{2} \log\left(\frac{1}{1 - \frac{C}{2\pi\mu}}\right)\right) \\ &\leq \frac{C^2}{2\pi} \exp(-\mu|\xi|) \left(\frac{1+\mu|\xi|}{\mu}\right) \exp\left(\frac{C}{2\pi\mu}\right) \exp\left(\frac{C}{2\pi}|\xi|\right) \end{aligned} \quad (183)$$

for all $\xi \in \mathbb{R}$. Note that in (183), we used the assumption that $C < a$ in order to apply the inequality (182). Owing to (168),

$$|\exp_2^*[f](\xi)| \leq \exp_2^*[g](\xi) \quad \text{for all } \xi \in \mathbb{R}. \quad (184)$$

By combining this observation with (183), we obtain (169), which completes the proof. \square

Remark 4. *Kummer's confluent hypergeometric function $M(a, b, z)$ is defined by the series*

$$M(a, b, z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \frac{(a)_3 z^3}{(b)_3 3!} + \dots, \quad (185)$$

where $(a)_n$ is the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\dots(a+n-1). \quad (186)$$

By comparing the definition of $M(a, b, z)$ with (177), we conclude that

$$|\exp_2^*[f](\xi)| \leq C \exp(-\mu|\xi|) \left(M\left(\frac{1+\mu|\xi|}{2}, 2, \frac{C}{\pi\mu}\right) - 1 \right) \quad \text{for all } \xi \in \mathbb{R} \quad (187)$$

provided

$$|f(\xi)| \leq C \exp(-\mu|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \quad (188)$$

The weaker bound (169) is sufficient for our immediate purposes, but formula (187) might serve as a basis for improved estimates on solutions of Kummer's equation.

The following lemma is a special case of Formula (172).

Lemma 2. *Suppose that $C \geq 0$ and $\mu > 0$ are real numbers, and that $f \in L^1(\mathbb{R})$ such that*

$$|f(\xi)| \leq C \exp(-\mu|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \quad (189)$$

Then

$$|f * f(\xi)| \leq C^2 \exp(-\mu|\xi|) \left(\frac{1 + \mu|\xi|}{\mu} \right) \quad \text{for all } \xi \in \mathbb{R}. \quad (190)$$

We will also make use of the following elementary observation.

Lemma 3. *Suppose that $\mu > 0$ is a real number. Then*

$$\exp(-\mu|\xi|)|\xi| \leq \frac{1}{\mu \exp(1)} \quad \text{for all } \xi \in \mathbb{R}. \quad (191)$$

We combine Lemmas 1 and 2 with (157) and (158) in order to obtain the following key estimate.

Theorem 16. *Suppose that $\Gamma > 0$, $\lambda > 0$, $\mu > 0$ and $C > 0$ are real numbers such that*

$$0 \leq C < 2\mu\lambda^2. \quad (192)$$

Suppose also that $f \in L^1(\mathbb{R})$ such that

$$|f(\xi)| \leq C \exp(-\mu|\xi|) \quad \text{for all } |\xi| \leq \sqrt{2}\lambda, \quad (193)$$

and that $w \in L^1(\mathbb{R})$ such that

$$|w(\xi)| \leq \Gamma \exp(-\mu|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \quad (194)$$

Suppose further that R is the operator defined via (150). Then

$$|R[f](\xi)| \leq \exp(-\mu|\xi|) \left(\frac{C^2}{\lambda^2} \left(\frac{1 + \mu|\xi|}{\mu} \right) \left(\frac{1}{16\pi} + \frac{1}{2\pi} \exp\left(\frac{C}{4\pi\lambda^2\mu} \right) \exp\left(\frac{C}{4\pi\lambda^2} |\xi| \right) \right) + \Gamma \right) \quad (195)$$

for all $\xi \in \mathbb{R}$.

Proof. We define the operator R_1 via the formula

$$R_1[f](\xi) = \frac{1}{8\pi} \widetilde{W}_b[f] * \widetilde{W}_b[f](\xi) \quad (196)$$

and R_2 by the formula

$$R_2[f](\xi) = -4\lambda^2 \exp_2^*[W_b[f]](\xi), \quad (197)$$

where W_b and \widetilde{W}_b are defined as in Section 5. Then

$$R[f](\xi) = R_1[f](\xi) + R_2[f](\xi) + w(\xi) \quad (198)$$

for all $\xi \in \mathbb{R}$. We observe that

$$\left| \widetilde{W}_b[f](\xi) \right| \leq \frac{C}{\sqrt{2}\lambda} \exp(-\mu|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \quad (199)$$

By combining Lemma 2 with (199) we obtain

$$|R_1[f](\xi)| \leq \frac{C^2}{16\pi\lambda^2} \exp(-\mu|\xi|) \left(\frac{1 + \mu|\xi|}{\mu} \right) \quad \text{for all } \xi \in \mathbb{R}. \quad (200)$$

Now we observe that

$$|W_b[f](\xi)| \leq \frac{C}{2\lambda^2} \exp(-\mu|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \quad (201)$$

Combining Lemma 1 with (201) yields

$$|R_2[f](\xi)| \leq \frac{C^2}{2\pi\lambda^2} \exp(-\mu|\xi|) \left(\frac{1 + \mu|\xi|}{\mu} \right) \exp\left(\frac{C}{4\pi\lambda^2\mu}\right) \exp\left(\frac{C}{4\pi\lambda^2}|\xi|\right) \quad (202)$$

for all $\xi \in \mathbb{R}$. Note that (192) ensures that the hypothesis (167) in Lemma 1 is satisfied. We combine (200) with (202) and (194) in order to obtain (195), and by so doing we complete the proof. \square

Remark 5. *Note that Theorem 16 only requires that $f(\xi)$ satisfy a bound on the interval $[-\sqrt{2}\lambda, \sqrt{2}\lambda]$ and not on the entire real line.*

In the following theorem, we use Theorem 16 to bound the solution of (151) under an assumption on the decay of w .

Theorem 17. *Suppose that $\lambda > 0$, $\mu > 0$ and $\Gamma > 0$ are real numbers such that*

$$\lambda > 2 \max \left\{ \Gamma, \frac{1}{\mu} \right\}. \quad (203)$$

Suppose also that $w \in L^1(\mathbb{R})$ such that

$$|w(\xi)| \leq \Gamma \exp(-\mu|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \quad (204)$$

Then there exists a solution ψ of (151) in $L^1(\mathbb{R})$ such that

$$|\psi(\xi)| \leq \left(1 + \frac{2\Gamma}{\lambda} \right) \Gamma \exp(-\mu|\xi|) \quad \text{for all } |\xi| \leq \sqrt{2}\lambda \quad (205)$$

and

$$|\psi(\xi)| \leq \left(1 + \frac{4\Gamma}{\lambda} \right) \Gamma \exp\left(-\left(\mu - \frac{1}{2\pi\lambda}\right)|\xi|\right) \quad \text{for all } \xi \in \mathbb{R}. \quad (206)$$

Proof. From (203) and (204) we obtain

$$\|w\|_1 \leq \Gamma \int_{-\infty}^{\infty} \exp(-\mu|\xi|) d\xi = \frac{\Gamma}{\mu} < \frac{\lambda^2}{4}. \quad (207)$$

It follows from Theorem 15 and (207) that a solution $\psi(\xi)$ of (151) is obtained as the limit of the sequence of fixed point iterates $\{\psi_n(\xi)\}$ defined by the formula

$$\psi_0(\xi) = w(\xi) \quad (208)$$

and the recurrence

$$\psi_{n+1}(\xi) = R[\psi_n](\xi). \quad (209)$$

We now derive pointwise estimates on the iterates $\psi_n(\xi)$ in order to establish (205) and (206).

We denote by $\{\beta_k\}$ be the sequence of real numbers generated by the recurrence relation

$$\beta_{k+1} = \frac{\beta_k^2}{2\lambda} + \Gamma \quad (210)$$

with the initial value

$$\beta_0 = \Gamma. \quad (211)$$

From mathematical induction and (203), we conclude that $\{\beta_k\}$ is a monotonically increasing sequence which converges to

$$\beta = \lambda - \sqrt{\lambda^2 - 2\Gamma\lambda}. \quad (212)$$

We also observe that

$$\beta_n \leq \beta < \left(1 + \frac{2\Gamma}{\lambda}\right) \Gamma \leq 2\Gamma. \quad (213)$$

Now suppose that $n \geq 0$ is an integer, and that

$$|\psi_n(\xi)| \leq \beta_n \exp(-\mu|\xi|) \quad \text{for all } |\xi| \leq \sqrt{2}\lambda. \quad (214)$$

When $n = 0$, this is simply the assumption (204). The function $\psi_{n+1}(\xi)$ is obtained from $\psi_n(\xi)$ via the formula

$$\psi_{n+1}(\xi) = R[\psi](\xi). \quad (215)$$

We combine Theorem 16 with (215) and (214) to conclude that

$$|\psi_{n+1}(\xi)| \leq \exp(-\mu|\xi|) \left(\frac{\beta_n^2}{\lambda^2} \left(\frac{1 + \mu|\xi|}{\mu} \right) \left(\frac{1}{16\pi} + \frac{1}{2\pi} \exp\left(\frac{\beta_n}{4\pi\lambda^2\mu}\right) \exp\left(\frac{\beta_n}{4\pi\lambda^2}|\xi|\right) \right) + \Gamma \right) \quad (216)$$

for all $\xi \in \mathbb{R}$. The hypothesis (192) of Theorem 16 is satisfied since

$$\beta_n \leq 2\Gamma \leq 2\lambda^2\mu \quad (217)$$

for all integers $n \geq 0$. We restrict ξ to the interval $[-\sqrt{2}\lambda, \sqrt{2}\lambda]$ in (216) and use the fact that

$$\frac{1}{\mu\lambda} < \frac{1}{2}, \quad (218)$$

which is a consequence of (203), in order to conclude that

$$\begin{aligned} |\psi_{n+1}(\xi)| &\leq \exp(-\mu|\xi|) \left(\frac{\beta_n^2}{\lambda^2} \left(\frac{1 + \mu\sqrt{2}\lambda}{\mu} \right) \left(\frac{1}{16\pi} + \frac{1}{2\pi} \exp\left(\frac{\beta_n}{4\pi\lambda^2\mu}\right) \exp\left(\frac{\beta_n}{4\pi\lambda^2}\sqrt{2}\lambda\right) \right) + \Gamma \right) \\ &\leq \exp(-\mu|\xi|) \left(\frac{\beta_n^2}{\lambda} \left(\frac{1}{2} + \sqrt{2} \right) \left(\frac{1}{16\pi} + \frac{1}{2\pi} \exp\left(\frac{\beta_n}{8\pi\lambda}\right) \exp\left(\frac{\beta_n}{2\sqrt{2}\pi\lambda}\right) \right) + \Gamma \right) \end{aligned} \quad (219)$$

for all $|\xi| \leq \sqrt{2}\lambda$. By combining (219) with inequality

$$\frac{\beta_n}{\lambda} \leq \frac{2\Gamma}{\lambda} \leq 1 \quad \text{for all } n \geq 0 \quad (220)$$

and the observation that

$$\left(\frac{1}{2} + \sqrt{2} \right) \left(\frac{1}{16\pi} + \frac{1}{2\pi} \exp\left(\frac{1}{8\pi}\right) \exp\left(\frac{1}{2\sqrt{2}\pi}\right) \right) < \frac{1}{2}, \quad (221)$$

we arrive at the inequality

$$\begin{aligned} |\psi_{n+1}(\xi)| &\leq \left(\frac{\beta_n^2}{2\lambda} + \Gamma \right) \exp(-\mu|\xi|) \\ &= \beta_{n+1} \exp(-\mu|\xi|), \end{aligned} \quad (222)$$

which holds for all $|\xi| \leq \sqrt{2}\lambda$. We conclude by induction that (214) holds for all integers $n \geq 0$.

The sequence $\{\psi_n\}$ converges to ψ in $L^1(\mathbb{R})$ norm and so a subsequence of ψ_n converges to ψ pointwise almost everywhere. From (213) and (214) we conclude that

$$|\psi(\xi)| \leq \beta \exp(-\mu|\xi|) \quad (223)$$

for almost all $|\xi| \leq \sqrt{2}\lambda$. By changing the values of ψ on a set of measure 0 (which does not affect the value of $R[\psi]$), we ensure that (223) holds for all $|\xi| \leq \sqrt{2}\lambda$. By inserting (213) into (223) we obtain

$$|\psi(\xi)| \leq \left(1 + \frac{2\Gamma}{\lambda}\right) \Gamma \exp(-\mu|\xi|) \quad \text{for all } |\xi| \leq \sqrt{2}\lambda, \quad (224)$$

which is the conclusion (205).

We combine (223) with Theorem 16 (the application of which is justified since $\beta < 2\Gamma < 2\lambda^2\mu$) to

conclude that

$$|\psi(\xi)| \leq \exp(-\mu|\xi|) \left(\frac{\beta^2}{\lambda^2} \left(\frac{1 + \mu|\xi|}{\mu} \right) \left(\frac{1}{16\pi} + \frac{1}{2\pi} \exp\left(\frac{\beta}{4\pi\lambda^2\mu}\right) \exp\left(\frac{\beta}{4\pi\lambda^2}|\xi|\right) \right) + \Gamma \right) \quad (225)$$

for all $\xi \in \mathbb{R}$. Note the distinction between (214) and (225) is that the former only holds for ξ in the interval $[-\sqrt{2}\lambda, \sqrt{2}\lambda]$, while the later holds for all ξ on the real line. It follows from (203) that

$$\frac{1}{\lambda\mu} < \frac{1}{2} \quad \text{and} \quad \beta < \frac{2\Gamma}{\lambda} < 1. \quad (226)$$

We insert these bounds into (225) in order to conclude that

$$|\psi(\xi)| \leq \Gamma \exp(-a|\xi|) \left(\frac{\beta^2}{\lambda} \left(\frac{1}{2} + \frac{|\xi|}{\lambda} \right) \left(\frac{1}{16\pi} + \frac{1}{2\pi} \exp\left(\frac{1}{8\pi}\right) \exp\left(\frac{1}{4\pi\lambda}|\xi|\right) \right) + \Gamma \right) \quad (227)$$

for all $\xi \in \mathbb{R}$. We conclude from Lemma 3 that

$$\exp\left(-\frac{1}{4\pi\lambda}|\xi|\right) \left(\frac{1}{2} + \frac{|\xi|}{\lambda} \right) \leq \left(\frac{1}{2} + \frac{4\pi}{\exp(1)} \right) \quad \text{for all } \xi \in \mathbb{R}. \quad (228)$$

We observe that

$$\exp\left(-\frac{1}{4\pi\lambda}|\xi|\right) \left(\frac{1}{16\pi} + \frac{1}{2\pi} \exp\left(\frac{1}{8\pi}\right) \exp\left(\frac{1}{4\pi\lambda}|\xi|\right) \right) \leq \left(\frac{1}{16\pi} + \frac{1}{2\pi} \exp\left(\frac{1}{8\pi}\right) \right) \quad (229)$$

for all $\xi \in \mathbb{R}$, and that

$$\left(\frac{1}{2} + \frac{4\pi}{\exp(1)} \right) \cdot \left(\frac{1}{16\pi} + \frac{1}{2\pi} \exp\left(\frac{1}{8\pi}\right) \right) < 1. \quad (230)$$

We combine (228), (229) and (230) in order to conclude that

$$\exp\left(-\frac{1}{2\pi\lambda}|\xi|\right) \left(\frac{\beta^2}{\lambda} \left(\frac{1}{2} + \frac{|\xi|}{\lambda} \right) \left(\frac{1}{16\pi} + \frac{1}{2\pi} \exp\left(\frac{1}{8\pi}\right) \exp\left(\frac{1}{4\pi\lambda}|\xi|\right) \right) + \Gamma \right) \leq \frac{\beta^2}{\lambda} + \Gamma \quad (231)$$

for all $\xi \in \mathbb{R}$. From (210) and (213) we obtain

$$\frac{\beta^2}{\lambda} + \Gamma = \frac{\beta^2}{\lambda} + 2\Gamma - \Gamma = 2\beta - \Gamma < \left(1 + \frac{4\Gamma}{\lambda} \right) \Gamma. \quad (232)$$

By inserting (231) into (225) we arrive at

$$|\psi(\xi)| < \left(1 + \frac{4\Gamma}{\lambda} \right) \Gamma \exp\left(-\left(\mu - \frac{1}{2\pi\lambda}\right)|\xi|\right) \quad \text{for all } \xi \in \mathbb{R}, \quad (233)$$

which is (206). \square

Suppose that p is an element of $L^1(\mathbb{R})$, and that there exist positive real numbers μ and Γ such that

$$|\hat{p}(\xi)| \leq \Gamma \exp(-\mu|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \quad (234)$$

Suppose further that $\psi \in L^1(\mathbb{R})$ is the solution of (151) obtained by invoking Theorem 15. Then the function σ_b defined by the formula

$$\sigma_b(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ix\xi) \psi(\xi) d\xi \quad (235)$$

is a solution of the integral equation (144). Moreover, according to Theorem 17, if $\lambda > 2\mu^{-1}$ and $\lambda > 2\Gamma$, then the Fourier transform of σ_b (which is, of course, ψ) decays faster than any polynomial. In this event, σ_b is infinitely differentiable, $\hat{\sigma}_b \in L^2(\mathbb{R})$, and $\sigma_b \in L^2(\mathbb{R})$. We record these observations in the following theorem.

Theorem 18. *Suppose that there exist real numbers $\lambda > 0$, $\Gamma > 0$ and $\mu > 0$ such that*

$$\lambda > 2 \max \left\{ \Gamma, \frac{1}{\mu} \right\}. \quad (236)$$

Suppose also that p is an element of $L^1(\mathbb{R})$ such that

$$|\hat{p}(\xi)| \leq \Gamma \exp(-\mu|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \quad (237)$$

Then there exists a solution $\sigma_b \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$ of the integral equation (144) such that

$$|\hat{\sigma}_b(\xi)| \leq \left(1 + \frac{2\Gamma}{\lambda}\right) \Gamma \exp(-\mu|\xi|) \quad \text{for all } |\xi| \leq \sqrt{2}\lambda, \quad (238)$$

and

$$|\hat{\sigma}_b(\xi)| \leq \left(1 + \frac{4\Gamma}{\lambda}\right) \Gamma \exp\left(-\left(\mu - \frac{1}{2\pi\lambda}\right)|\xi|\right) \quad \text{for all } \xi \in \mathbb{R}. \quad (239)$$

7. Perturbed integral equation

Suppose that σ_b is the function obtained by invoking Theorem 18 so that

$$\sigma_b(x) = S[T_b[\sigma_b]](x) + p(x). \quad (240)$$

We rearrange (240) as

$$\sigma_b(x) = S[T[\sigma_b]](x) + p(x) + (S[T_b[\sigma_b]](x) - S[T[\sigma_b]](x)) \quad (241)$$

and define ν_b via the formula

$$\nu_b(x) = S[T_b[\sigma_b]](x) - S[T[\sigma_b]](x) \quad (242)$$

so that σ_b is a solution of the perturbed integral equation

$$\sigma_b(x) = S[T[\sigma_b]](x) + p(x) + \nu_b(x). \quad (243)$$

From the discussion in Section 4, we see that the phase function arising from σ_b approximates solutions of (1) with accuracy on the order of $\lambda^{-1}\|\nu_b\|_\infty$. However, it is not immediately apparent that $T[\sigma_b]$ is defined: the integral

$$\frac{1}{2\lambda} \int_{-\infty}^{\infty} \sin(2\lambda|x-y|) \sigma_b(y) dy \quad (244)$$

need not converge absolutely, and without an estimate on the derivative of $\hat{\sigma}_b$, it is not clear that

$$\frac{\hat{\sigma}_b(\xi)}{4\lambda^2 - \xi^2}, \quad (245)$$

which is formally the Fourier transform of $T[\sigma_b]$, defines a tempered distribution (although, in fact, it does). It is possible obtain a bound on the derivative of $\hat{\sigma}_b$ by modifying the argument of Section 6. That bound can be used show that $T[\sigma_b]$ is defined and to estimate the magnitude of ν_b . We prefer the following, simpler approach to constructing an appropriate perturbation ν of the function p .

We define the function σ via the formula

$$\hat{\sigma}(\xi) = \hat{\sigma}_b(\xi)\hat{b}(\xi), \quad (246)$$

where $\hat{b}(\xi)$ is the function used to define the operator T_b . We observe that, unlike $T[\sigma_b]$, there is no difficulty in defining $T[\sigma]$ since $\sigma \in L^2(\mathbb{R})$ and the support of $\hat{\sigma}$ is contained in $(-\sqrt{2}\lambda, \sqrt{2}\lambda)$. Moreover, $T_b[\sigma_b] = T[\sigma]$ so that

$$\sigma_b(x) = S[T[\sigma]](x) + p(x) \quad \text{for all } x \in \mathbb{R}. \quad (247)$$

Rearranging (247), we obtain

$$\sigma(x) = S[T[\sigma]](x) + p(x) + \nu(x) \quad \text{for all } x \in \mathbb{R}, \quad (248)$$

where ν is defined the formula

$$\nu(x) = \sigma(x) - \sigma_b(x). \quad (249)$$

Using (239), (236) and (249), we conclude that under the hypotheses of Theorem 18,

$$\begin{aligned} \|\nu\|_\infty &\leq \frac{1}{2\pi} \|\hat{\sigma} - \hat{\sigma}_b\|_1 \\ &\leq \frac{1}{2\pi} \left(1 + \frac{4\Gamma}{\lambda}\right) \Gamma \int_{|\xi| \geq \lambda} \exp\left(-\left(\mu - \frac{1}{2\pi\lambda}\right)|\xi|\right) d\xi \\ &= \frac{1}{\pi\left(\mu - \frac{1}{2\pi\lambda}\right)} \left(1 + \frac{4\Gamma}{\lambda}\right) \Gamma \exp\left(-\left(\mu - \frac{1}{2\pi\lambda}\right)\lambda\right) \\ &< \frac{\Gamma}{2\mu} \left(1 + \frac{4\Gamma}{\lambda}\right) \exp(-\mu\lambda). \end{aligned} \quad (250)$$

By combining Theorem 18 with (250) we arrive at the following theorem.

Theorem 19. *Suppose that $q \in C^\infty(\mathbb{R})$ is strictly positive, and that $x(t)$ is defined by the formula*

$$x(t) = \int_0^t \sqrt{q(u)} du. \quad (251)$$

Suppose also that $p(x)$ is defined via the formula

$$p(x) = 2\{t, x\}; \quad (252)$$

that is, $p(x)$ is twice the Schwarzian derivative of the variable t with respect to the variable x defined via (251). Suppose furthermore that there exist positive real numbers λ , Γ and μ such that

$$\lambda > 2 \max\left\{\frac{1}{\mu}, \Gamma\right\} \quad (253)$$

and

$$|\hat{p}(\xi)| \leq \Gamma \exp(-\mu|\xi|) \quad \text{for all } \xi \in \mathbb{R}. \quad (254)$$

Then there exist functions ν and σ in $L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$ such that σ is a solution of the nonlinear integral equation

$$\sigma(x) = S[T[\sigma]](x) + p(x) + \nu(x), \quad (255)$$

$$|\hat{\sigma}(\xi)| \leq \left(1 + \frac{2\Gamma}{\lambda}\right) \Gamma \exp(-\mu|\xi|) \quad \text{for all } |\xi| < \sqrt{2}\lambda, \quad (256)$$

$$\hat{\sigma}(\xi) = 0 \quad \text{for all } |\xi| \geq \sqrt{2}\lambda, \quad (257)$$

and

$$\|\nu\|_\infty < \frac{\Gamma}{2\mu} \left(1 + \frac{4\Gamma}{\lambda}\right) \exp(-\mu\lambda). \quad (258)$$

The function σ_b is obtained as the inverse Fourier transform of the limit ψ of a sequence of fixed point iterates $\{\psi_n\}$ for the equation (151). As a consequence of Theorem 2, the functions ψ_n are elements of the space $S(\mathbb{R})$ of Schwartz functions if \hat{p} is an element of $S(\mathbb{R})$. Thus σ_b is the limit in $L^1(\mathbb{R})$ of a sequence of Schwartz functions. The following proof of the principal result of this paper, Theorem 12, proceeds by approximating σ_b using the inverse Fourier transform of an appropriately chosen ψ_n .

Proof of Theorem 12. We denote by $\{\psi_n\}$ the sequence of fixed point iterates for (151) generated by

the function \widehat{p} . That is, ψ_0 is defined by the formula

$$\psi_0(\xi) = \widehat{p}(\xi) \quad (259)$$

and, for each integer $n = 0, 1, 2, \dots$, ψ_{n+1} is defined via the formula

$$\psi_{n+1}(\xi) = R[\psi_n](\xi), \quad (260)$$

where R is as in (150). According to Theorem 15, the sequence $\{\psi_n\}$ converges in $L^1(\mathbb{R})$ to a function ψ such that

$$\psi(\xi) = R[\psi](\xi) \quad \text{for all } \xi \in \mathbb{R}. \quad (261)$$

Moreover, as a consequence of Theorem 2 and the assumption that $\widehat{p} \in S(\mathbb{R})$, each of the ψ_n is an element of $S(\mathbb{R})$. We denote the inverse Fourier transform of ψ by $\tilde{\sigma}_b$. According to Theorem 18, $\tilde{\sigma}_b$ is a solution of the nonlinear integral equation

$$\tilde{\sigma}_b(x) = S[T_b[\tilde{\sigma}_b]](x) + p(x). \quad (262)$$

We now define $\tilde{\sigma}$ via the formula

$$\widehat{\tilde{\sigma}}(\xi) = \psi(\xi)\widehat{b}(\xi), \quad (263)$$

where \widehat{b} is given in (69), so that

$$T_b[\tilde{\sigma}_b](\xi) = T[\tilde{\sigma}](\xi). \quad (264)$$

And we define $\tilde{\nu}$ via the formula

$$\tilde{\nu}(\xi) = \tilde{\sigma}(\xi) - \tilde{\sigma}_b(\xi). \quad (265)$$

From (262), (264) and (265), we conclude that

$$\tilde{\sigma}(x) = S[T[\tilde{\sigma}]](x) + p(x) + \tilde{\nu}(x) \quad (266)$$

for all $x \in \mathbb{R}$. According to Theorem 19,

$$\|\tilde{\nu}\|_\infty < \frac{\Gamma}{2\mu} \left(1 + \frac{4\Gamma}{\lambda}\right) \exp(-\mu\lambda). \quad (267)$$

For each integer $n = 0, 1, 2, \dots$, we denote by σ_n the inverse Fourier transform of the function

$$\psi_n(\xi)\widehat{b}(\xi). \quad (268)$$

The function ψ_n is in $S(\mathbb{R})$ and \widehat{b} is an element of $C_c^\infty(\mathbb{R})$, so the σ_n are contained in $S(\mathbb{R})$. We combine (263) and (268) in order to obtain

$$\|\sigma_n - \tilde{\sigma}\|_\infty \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \widehat{b}(\xi)\psi_n(\xi) - \widehat{b}(\xi)\psi(\xi) \right| d\xi \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\psi_n(\xi) - \psi(\xi)| d\xi \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (269)$$

Since \widehat{b} is supported on $(-\sqrt{2}\lambda, \sqrt{2}\lambda)$,

$$\begin{aligned} \|T[\sigma_n] - T[\tilde{\sigma}]\|_\infty &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\widehat{b}(\xi)\psi_n(\xi) - \widehat{b}(\xi)\psi(\xi)}{4\lambda^2 - \xi^2} \right| d\xi \\ &\leq \frac{1}{4\pi\lambda^2} \int_{-\infty}^{\infty} \left| \widehat{b}(\xi)\psi_n(\xi) - \widehat{b}(\xi)\psi(\xi) \right| d\xi \\ &\leq \frac{1}{4\pi\lambda^2} \int_{-\infty}^{\infty} |\psi_n(\xi) - \psi(\xi)| d\xi, \end{aligned} \quad (270)$$

from which we conclude that $T[\sigma_n]$ converges to $T[\tilde{\sigma}]$ in $L^\infty(\mathbb{R})$. Similarly, if we denote the derivative

of the function $T[f](x)$ with respect to x by $T[f]'(x)$, then

$$\begin{aligned} \|T[\sigma_n]' - T[\tilde{\sigma}']\|_\infty &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{-i\xi\widehat{b}(\xi)\psi_n(\xi) + i\xi\widehat{b}(\xi)\psi(\xi)}{4\lambda^2 - \xi^2} \right| d\xi \\ &\leq \frac{1}{2\sqrt{2}\pi\lambda} \int_{-\infty}^{\infty} |\psi_n(\xi) - \psi(\xi)| d\xi, \end{aligned} \quad (271)$$

from which we conclude that the derivative of $T[\sigma_n]$ converges to the derivative of $T[\tilde{\sigma}]$ in $L^\infty(\mathbb{R})$. We combine these observations with the definition (96) in order to conclude that

$$\|S[T[\sigma_n]] - S[T[\tilde{\sigma}]]\|_\infty \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (272)$$

We rearrange (266) as

$$\sigma_n(x) = S[T[\sigma_n]](x) + p(x) + \nu_n(x), \quad (273)$$

where ν_n is defined via the formula

$$\nu_n(x) = \tilde{\nu}(x) + (S[T[\tilde{\sigma}]](x) - S[T[\sigma_n]](x)) + (\sigma_n(x) - \tilde{\sigma}(x)). \quad (274)$$

We combine (274) with (269) and (272) in order to conclude that

$$\|\nu_n - \tilde{\nu}\|_\infty \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (275)$$

Together (275) and (267) imply

$$\|\nu_n\|_\infty \leq \frac{\Gamma}{2\mu} \left(1 + \frac{4\Gamma}{\lambda}\right) \exp(-\mu\lambda) \quad (276)$$

when n is sufficiently large. Note that the inequality (267) is strict.

We have already established that ψ_n and σ_n are elements of $S(\mathbb{R})$ for all nonnegative integers n . Now we observe that

$$\widehat{T[\sigma_n]}(\xi) = \frac{\widehat{\sigma}_n(\xi)}{4\lambda^2 - \xi^2} = \frac{\widehat{\psi}_n(\xi)\widehat{b}(\xi)}{4\lambda^2 - \xi^2}. \quad (277)$$

Since the support of \widehat{b} is contained in $(-\sqrt{2}\lambda, \sqrt{2}\lambda)$ and ψ_n is an element of $S(\mathbb{R})$, we conclude from (277) that the Fourier transform of $T[\sigma_n]$ — and hence $T[\sigma_n]$ — is an element of $S(\mathbb{R})$ for each nonnegative integer n . Next, we combine this observation with Theorem 1 in order to conclude that $S[T[\psi_n]] \in S(\mathbb{R})$ for all nonnegative integers n . We rearrange (273) as

$$\nu_n(x) = \sigma_n(x) - S[T[\sigma_n]](x) - p(x) \quad (278)$$

and observe that all of the functions appearing on the right-hand side of (278) are elements of $S(\mathbb{R})$. We conclude that $\nu_n \in S(\mathbb{R})$ for all nonnegative integers n .

It follows from Formulas (213) and (214), which appear in the proof of Theorem 18, that

$$|\psi_n(\xi)| \leq \left(1 + \frac{2\Gamma}{\lambda}\right) \Gamma \exp(-\mu|\xi|) \quad (279)$$

for all $|\xi| \leq \sqrt{2}\lambda$ and all nonnegative integers n . We conclude from (273) (276) and (279), and our observation that σ_n and ν_n are elements of $S(\mathbb{R})$ for all nonnegative integers n that we obtain Theorem 12 by letting $\sigma = \sigma_n$ and $\nu = \nu_n$ for a sufficiently large n . \square

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