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Relative Langlands

Geometric Langlands: Duality of 4d field theories

automorphic $\curvearrowright$ $\mathcal{A}_G \simeq \mathcal{B}_{G^\vee}$ $\curvearrowleft$ spectral

(Equiv. of 3-cats?)

G red gp w/ dual $G^\vee$

<u>Simplify</u>: Compactify to lower-dimensional theories

(To get to arithmetic, compactify on one further circle by taking Frob. trace)

Take Frob trace: get something like

Category of Sh on space of G-isocrystals $=$ IndCoh (Langlands param space) (?)

<u>Local Langlands</u>: $\mathcal{A}_G[S^1] \underset{\text{2-categories}}{\simeq} \mathcal{B}_{G^\vee}[S^1]$

$\mathcal{A}_G[S^1]$	$\mathcal{B}_{G^\vee}[S^1]$
$=$ cats w/ $\mathcal{L}G$-action	$=$ cats linear over $(\mathfrak{g}^\vee)^*[2]/G^\vee$
<u>Ex</u>: $Sh(\mathcal{L}X)$ when $X \curvearrowleft G$ (constructible sh w/ ss condition, or D-modules, or ...)	<u>Ex</u>: $QCoh(M^\vee/G^\vee)$ with $M^\vee/G^\vee \to \mathfrak{g}^{\vee *}[2]/G^\vee$

<u>Global Langlands</u>: $\mathcal{A}_G[\Sigma] \simeq \mathcal{B}_{G^\vee}[\Sigma]$

$= Sh(Bun_G(\Sigma))$ $\qquad$ $= Coh_N(Loc_{G^\vee}(\Sigma))$

local systems in Betti case, flat conns in de Rham

<u>Boundary condition</u> for the theory gives an object in these categories / 2-categories

2) Langlands duality $\Rightarrow$ b.c. M for A_G corresponds to dual b.c. $M^\vee$ for $B_{G^\vee}$.

Goal: Understand consequences of this duality

Observation. $A_G[S^1] \simeq B_{G^\vee}[S^1]$ has distinguished b.c.

$$\begin{array}{ccc} A_G[S^1] & \simeq & B_{G^\vee}[S^1] \\ \Downarrow & & \Downarrow \\ A_G[D^2] & & B_{G^\vee}[D^2] \end{array}$$

concretely, this is $\quad Sh(\mathcal{L}G/\mathcal{L}^+G)$, $QCoh(\mathfrak{g}^{\vee *}[2]/G^\vee)$

$\quad Sh(\mathcal{L}G/\mathcal{L}^+G) \curvearrowleft Sh(\mathcal{L}G)$-module category $\qquad QCoh(\mathfrak{g}^{\vee *}[2]/G^\vee)$ categorical linear over $\mathfrak{g}^{\vee *}[2]/G^\vee$.

The endomorphisms of this object is

$$A_G[\underbrace{D^2 \cup_{S^1} D^2}_{=S^2 \text{ (raviolo } D \cup_{D^\times} D)}] \simeq B_{G^\vee}[S^2]$$

$$\begin{array}{ccc} \shortparallel & & \shortparallel \\ Sh(\mathcal{L}^+G\backslash\mathcal{L}G/\mathcal{L}^+G) & \underset{\text{geom. Satake}}{\simeq} & QCoh(\mathfrak{g}^{\vee *}[2]/G^\vee) \end{array}$$

as ($\mathbb{E}_3$-) monoidal categories.

Now consider $X \curvearrowleft G \rightsquigarrow LG \curvearrowright LX$ b.c. for $A_G[S^1]$.

Then $Sh(L^+G\backslash\mathcal{L}X)$ is a module category for

$Sh(L^+G\backslash LG/\mathcal{L}^+G) \simeq QCoh(\mathfrak{g}^{\vee *}[2]/G^\vee) \implies$ expect this category is of the form $Coh(M^\vee/G^\vee)$, module for $QCoh(\mathfrak{g}^{\vee *}/G^\vee)$ via

Ham. $G^\vee$-space

Hamiltonian moment map $\quad \begin{array}{c} M^\vee/G^\vee \\ \mu\downarrow \\ \mathfrak{g}^{\vee *}/G^\vee \end{array} \qquad (\text{Ex}: M^\vee = T^*X^\vee,\ G^\vee \curvearrowright X^\vee.)$

Warning: To make this work, we had to drop the shift [2] somehow.

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In fact, $Sh(\mathcal{L}^+G \backslash \mathcal{L}X)$ is an algebra object in $Sh(\mathcal{L}^+G\backslash \mathcal{L}G/\mathcal{L}^+G)$-mod, with unit object $\mathbb{1}$ given by $\delta_{\mathcal{L}^+X}$. $End(\mathbb{1})$ is the Plancherel algebra.

Dually, the unit in $QCoh(M^\vee/G^\vee)$ is $\mathcal{O}_{M^\vee}$, and $End(\mathbb{1}) \simeq \mu_* \mathcal{O}_{M^\vee}$.
(A special interpretation of the Plancherel algebra is a geometric version of the problem of determining the Plancherel measure.)

~~Dually, the~~ Similarly for the global case
$$A_G[\Sigma] \simeq B_{G^\vee}[\Sigma]$$
$$\begin{array}{cc} \| & \| \\ Sh(Bun_G \Sigma) & Coh_N(Loc_{G^\vee}(\Sigma)) \\ \Psi & \Psi \\ \mathcal{P}_X \ \text{"period sheaf"} & \mathbb{L}_X \ \text{"L-sheaf"} \end{array}$$

obtained by pushfwd along $Bun_G^X \to Bun_G$, $\quad Loc_G^X \to Loc_G$

$Bun_G^X = \{$G-bundle + section of associated X-bundle$\}$

skyscraper at point of $Loc_{G^\vee}(\Sigma)$ i.e., at Galois rep ρ

(Claim: $Hom(\mathbb{L}_X, \delta_\rho)$ Frob. trace ~~mod~~ L-value)

Ex. $G = \mathbb{G}_m$, $G^\vee = \mathbb{G}_m$. $\quad X = \mathbb{A}^1 \longleftrightarrow X^\vee = \mathbb{A}^1$.
Full (not just unramified) duality studied by Hilburn-Raskin (de Rham case) ; G.-Hilburn (Betti case)

In general, we need $X, X^\vee$ to be spherical: they are normal with dense open B-orbit.
This is the analogue of toric varieties for non-abelian G.

④ Theorem. Let $H \subseteq G$. Then ~~iff~~

$$X = G/H \text{ is spherical} \iff L^+G \curvearrowright LX \text{ has countably many orbits.}$$

Ex. $Gl_n/Gl_{n-1} \sim S^{2n-1}$ is a spherical Gl_n-variety.

(Sanath's perspective: many questions here can be expressed purely homotopically

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Questions (to be addressed in future talks):

① Given a spherical variety, how to construct the dual?

② What if M is not a cotangent bundle? spherical variety

So far we have considered $M = T^*X \curvearrowleft G$, but (Ham)
there are Hamiltonian G-varieties similar to this which are not cotangent bundles.

New ingredient. Whittaker reduction. Instead of cotangent bundle $T^*(G/H) = T^*G/\!/H$, we take twisted cotangent bundle $T^*(G/_\chi N) = T^*G/\!/_\chi N$.

This will be essential, since $M = T^*(G/_\chi N)$ will be dual to $M^\vee = pt$.

Thus we have $Sh(L^+G \backslash LG/_\chi LN) \simeq QCoh(pt/G^\vee)$ as modules for the Satake category. (Casselman-Shalika)