Spherical Varieties

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1 Toric varieties

Let $\Lambda \cong \mathbb{Z}^n$ be a free abelian group and let $k[\Lambda]$ be its group algebra. Any (split) torus T is of the form $\operatorname{Spec} k[\Lambda]$. We may recover Λ as the weight lattice $\operatorname{Hom}_{\mathbf{Grp}}(T, \mathbb{G}_m)$. An action of T on an affine variety $\operatorname{Spec} R$ is equivalent to the data of a Λ-grading on the k-algebra R. A variety X is called *toric* if there is a torus acting on X with a dense open orbit.

Let's see how to construct affine toric varieties. Let M be a submonoid of Λ which is finitely generated. The monoid algebra $k[M]$ is then a finitely generated subalgebra of $k[\Lambda]$, and $\operatorname{Spec} k[M]$ is an affine toric variety. The T-action is determined by the canonical Λ-grading on M. If $\mathbb{Z}M \subseteq \Lambda$ denotes the subgroup of Λ generated by M, then $\operatorname{Spec} k[\mathbb{Z}M] \hookrightarrow \operatorname{Spec} k[M]$ is an open subvariety isomorphic to a torus, which is an orbit of the T-action and the image of the map $T \to \operatorname{Spec} k[M]$. (We can identify this map with the action of T on the image of the identity.) It turns out that every affine toric variety X is of this form.

Example 1.1.

Let $\Lambda = \mathbb{Z}$ and let M be the additive submonoid generated by 2 and 3. Then $T = \operatorname{Spec} k[\Lambda] = \operatorname{Spec} k[t, t^{-1}]$ and $\operatorname{Spec} k[M] = \operatorname{Spec} k[t^2, t^3]$. This toric variety is isomorphic the cuspidal curve $\operatorname{Spec} k[x, y]/(y^2 - x^3) \subseteq \mathbb{A}^2$.

Let $\Lambda = \mathbb{Z}^2$ and let M be the additive submonoid generated by $(1, 0)$ and $(0, 1)$. Then $T = \operatorname{Spec} k[x^{\pm 1}, y^{\pm 1}] = \mathbb{G}_m^2$ and $\operatorname{Spec} k[M] = \operatorname{Spec} k[x, y] = \mathbb{A}^2$.

Of particular importance for us are the affine toric varieties which are normal varieties. These may be characterized as the $\operatorname{Spec} k[M]$ such that M is a *saturated* submonoid of $\mathbb{Z}M$: if $am \in M$ for some $a \in \mathbb{N}$ and $m \in \mathbb{Z}M$, then $m \in M$. We care about these because normal toric varieties can all be glued together from normal affine toric varieties in a nice combinatorial way. The idea will be to note that saturated submonoids of Λ are determined by their *cones*: the convex subset of $\mathbb{Q} \otimes \Lambda$ consisting of non-negative linear combinations of the elements of M. If $\sigma = \operatorname{cone}(M)$, then $\sigma \cap \Lambda$ is the minimal saturated submonoid of $\mathbb{Z}M$ containing M. We will find it more convenient to work with the *dual cone*: if σ is a cone in $\mathbb{Q} \otimes \Lambda$ then its dual cone is

$$\sigma^\vee := \{f \in \operatorname{Hom}(\Lambda, \mathbb{Q}) \mid \forall \lambda \in \sigma,\ \langle f, \lambda \rangle \geq 0\} \subseteq \operatorname{Hom}(\Lambda, \mathbb{Q}).$$

Similarly, if we have a cone $\sigma \subseteq \operatorname{Hom}(\Lambda, \mathbb{Q})$, we can define $\sigma^\vee \subseteq \Lambda \otimes \mathbb{Q}$. We write $X_\sigma = \operatorname{Spec} k[\Lambda \cap \sigma^\vee]$ for the associated normal affine toric variety. If a cone σ' is contained in σ, then there is an induced map $X_{\sigma'} \to X_\sigma$. If σ' is a face of σ, then this map is an open immersion. In other words, we can think of the faces of σ as open affine toric subvarieties of X_σ. The analogy extends further: if we have two cones $\sigma_1, \sigma_2 \subseteq \operatorname{Hom}(\Lambda, \mathbb{Q})$, and $\sigma' = \sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2, then there is a toric variety $X_{\{\sigma_1,\sigma_2\}}$ which is glued together from a copy of X_{σ_1} and X_{σ_2}, glued along $X_{\sigma'}$. More generally, if Σ is any fan in $\operatorname{Hom}(\Lambda, \mathbb{Q})$ (a collection of cones intersecting along their faces), then there is a toric variety X_Σ. Remarkably, every normal toric variety arises from this construction, by the following result.

Theorem 1.2: (Sumihiro).

If X is a normal variety with an action of a torus T, then X has an open cover by T-stable affine subvarieties.

Example 1.3.

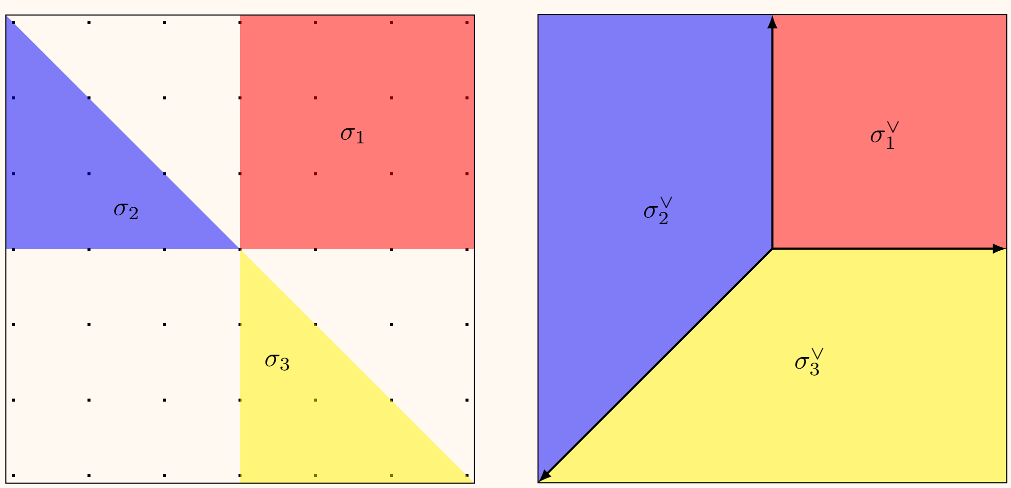
Set $\Lambda = \mathbb{Z}$ and let $\sigma_1 = \mathbb{Q}_{\geq 0}$ and $\sigma_2 = \mathbb{Q}_{\leq 0}$ be cones in $\operatorname{Hom}(\Lambda, \mathbb{Q}) \cong \mathbb{Q}$. Then $\sigma_1^\vee = \mathbb{Q}_{\geq 0}$ and $\sigma_2^\vee = \mathbb{Q}_{\leq 0}$ give rise to toric varieties $X_{\sigma_1} = \operatorname{Spec} k[t]$ and $X_{\sigma_2} = \operatorname{Spec} k[t^{-1}]$, respectively. The intersection $\sigma_1 \cap \sigma_2 = \{0\}$ has dual cone $(\sigma_1 \cap \sigma_2)^\vee = \mathbb{Q}$. Hence the toric variety $X_{\{\sigma_1,\sigma_2\}}$ is given by gluing X_{σ_1} and X_{σ_2} along their mutual open subvariety $X_{\sigma_1 \cap \sigma_2} = \operatorname{Spec} k[t, t^{-1}]$. We conclude that $X_{\{\sigma_1,\sigma_2\}} \cong \mathbb{P}^1$.

Example 1.4.

Consider $\mathbb{P}^2$, with homogeneous coordinates $[x_0 : x_1 : x_2]$ and field of rational functions $k(x, y)$, where $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}$. We can describe $\mathbb{P}^2$ as a toric variety, with open torus $T = \operatorname{Spec} k[x^{\pm 1}, y^{\pm 1}]$. There are three maximal open affine T-stable subsets:

$$U_1 = \operatorname{Spec} k[x, y],\ U_2 = \operatorname{Spec} k[x^{-1}, x^{-1}y],\ U_3 = \operatorname{Spec} k[xy^{-1}, y^{-1}].$$

We depict the lattice $\Lambda = \mathbb{Z}^2$ on the left below, with the subsets associated to the three open subvarieties. On the right we show the fan Σ in the dual space $\operatorname{Hom}(\Lambda, \mathbb{Q})$.



We note that each of U_1, U_2, and U_3 contain a unique point fixed by T. More generally, for each $\sigma \in \Sigma$, the open affine toric variety X_σ contains a unique closed T-orbit.

We can deduce geometric properties of X_Σ from the combinatorial properties of the fan Σ. For instance, X_Σ is smooth if and only if every X_σ is smooth, and X_σ is smooth if and only if the cone σ is simplicial and unimodular. Furthermore, the variety X_Σ is proper over $\operatorname{Spec} k$ if and only if Σ is a complete fan (meaning the union of its cones is all of $\operatorname{Hom}(\Lambda, \mathbb{Q})$). We can also describe (toric) line bundles in a combinatorial way, and hence detect the existence of ample line bundles combinatorially, which tells us when X_Σ is quasi-projective. If X_Σ is proper, then toric line bundles correspond to polytopes which are the convex hulls of points in Λ, with normal fan coarsening Σ, up to translation. More generally, on any normal toric variety X_Σ, the Cartier divisors whose underlying Weil divisor is T-invariant correspond to piecewise-linear functions on Σ (linear on each cone), such that each linear subfunction is determined by an element of Λ. Two such Cartier divisors are linearly equivalent iff they differ by a global linear function.

2 Spherical varieties

Spherical varieties generalize (normal) toric varieties. Fix a reductive group G and a Borel subgroup B.

Definition 2.1.

A *spherical variety* is a normal (irreducible) variety X with an action of G such that B has an orbit which is an open subvariety of X.

Example 2.2.

Some familiar spaces are spherical varieties:

- Toric varieties
- Flag varieties/Grassmannians: spaces of the form G/P for some parabolic subgroup P
- Symmetric spaces: spaces of the form G/H where H is the fixed points of an involution on G
- Horospherical varieties: spaces of the form G/H where H contains the unipotent radical of a Borel subgroup

As the examples indicate, it is already interesting to examine the spherical varieties which are *homogeneous*, that is, of the form G/H for some subgroup H of G. If X is any spherical variety with open Borel orbit $\mathring{X}$, then $G\mathring{X}$ will be a homogeneous spherical variety. For tori (for which $B = G$), any homogeneous spherical varieties are also tori. For arbitrary G, we may give a similar explicit description of $\mathring{X}$ as the product of a torus and an affine space. Specifically, there is a maximal subgroup stabilizing $\mathring{X}$ (necessarily parabolic, since it will contain B) denoted $P(X)$, the *parabolic stabilizer* of X. Then the unipotent radical $U(X)$ of $P(X)$ acts freely on $\mathring{X}$, and given a point in $\mathring{X}$ and a Levi splitting of $P(X)$, there is an isomorphism $\mathring{X} \cong A_X \times U(X)$, where A_X is a torus quotient of $P(X)$ (determined by X).

Example 2.3.

Let us examine the homogeneous spherical varieties for $G = \mathrm{PGL}_2$. A subgroup H of G defines a spherical variety if and only if HB is open in PGL_2 if and only if H has an open orbit in $\mathrm{PGL}_2/B = \mathbb{P}^1$. When H is a finite subgroup this will certainly not be the case, but if H has positive dimension then this turns out to be true (since otherwise the connected component of H is contained in all conjugates of B, but this intersection is trivial).

Up to conjugation, the positive-dimensional subgroups of PGL_2 are the following:

(G) PGL_2, in which case G/H is a point,

(T) $\mathbb{G}_m$, the pointwise stabilizer of two points in $\mathbb{P}^1$, so $G/H \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$,

(N) The normalizer $N_G(\mathbb{G}_m) \cong PO_2 \cong \mathbb{Z}/2 \ltimes \mathbb{G}_m$, the stabilizer of a set of two points in $\mathbb{P}^1$, so that $G/H \cong (\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta)/(\mathbb{Z}/2)$,

(U) $S \cdot U$, where S is a subgroup of the group of diagonal matrices $\mathbb{G}_m$, and $U \cong \mathbb{G}_a$ is the elements of the form $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$. When $H = U$, the quotient G/H is a $\mathbb{G}_m$-bundle over $\mathbb{P}^1$.

We say that G/H is *horospherical* if H contains the unipotent radical of some Borel; types (G) and (U) shown here are horospherical.

The parabolic stabilizers and descriptions of $\mathring{X}$ in these four cases are

(G) $P(X) = \mathrm{PGL}_2$; $\mathring{X} = *$,

(T) $P(X) = B$; $\mathring{X} = B \cong \mathbb{G}_m \times \mathbb{A}^1$,

(N) $P(X) = B$; $\mathring{X} = B \cong \mathbb{G}_m \times \mathbb{A}^1$,

(U) $P(X) = B$; $\mathring{X} = B/S \cong \mathbb{G}_m/S \times \mathbb{A}^1$.

As a non-example, consider GL_n/T. Since T does not have an open orbit on the variety GL_n/B of complete flags in n-space for $n > 2$ (consider dimensions), there is no open B-orbit on GL_n/T in this case, so this is not a spherical variety.

If X has a G action, then the rings of regular functions and of rational functions both admit an action of G by pre-composition. If G is reductive, then these rings decompose as G-modules into a direct sum of irreducible representations. Note that for a torus $T = \operatorname{Spec} k[\Lambda]$, the irreducible T-modules are parametrized by weights in Λ, and $k[\Lambda]$ decomposes into one-dimensional weight spaces, with exactly one irreducible appearing for each weight. We say a G-module is *multiplicity-free* if each irrep appears at most once in the decomposition.

Theorem 2.4: Equivalent formulations of spherical varieties; [Gandini, Theorem 2.8].

The following are equivalent, for a normal irreducible variety X acted on by G:

- X is spherical

- Any B-invariant rational function on X is constant
- X has finitely many B-orbits
- If X is quasi-projective: for every G-linearized line bundle $\mathcal{L}$ on X, the global sections of $\mathcal{L}$ is a multiplicity-free G-module
- If X is quasi-affine: the coordinate ring $k[X]$ is a multiplicity-free G-module

If $X = G/H$ is homogeneous, then these are further equivalent to:

- Every equivariant completion of G/H contains only finitely many orbits
- The H-action on any irreducible G-module is multiplicity-free for characters of H.

The smooth affine spherical varieties have a particularly nice description:

Theorem 2.5: [Knop–Steirteghem, Cor 2.2].

If X is a smooth affine spherical variety, then $X = G \times^H V$ for some reductive subgroup H of G with G/H spherical and V is a module for H which is itself a spherical variety. In particular, if G/H is smooth and affine, then H is reductive.

Furthermore a smooth affine homogeneous spherical variety is determined by its weight monoid $\Lambda^+_{G/H}$ inside $\Lambda_{G/H}$.

The analog of the decomposition of normal toric varieties into open affine toric subvarieties is the following:

Theorem 2.6.

Let X be a spherical variety and $x \in X(k)$. Then there is an open affine B-stable subset intersecting the G-orbit through x.

3 Colored fans

Here we give a combinatorial description of spherical varieties containing G/H. If X is a spherical variety, then we fix $x_0 \in X(k)$ generating the open B-orbit $\mathring{X}$. If H is the stabilizer of x_0, then $Gx_0 \cong G/H$ is spherical.

For a representation V of a group K, we write $V^{(K)}$ for the subset of nonzero vectors on which K acts via a character. Let $T = \operatorname{Spec} k[\Lambda]$ be a maximal torus of B. Recall that the choice of $T \subseteq B \subseteq G$ determines a set of *dominant weights* $\Lambda^+ \subseteq \Lambda$.

Definition 3.1.

The *weight lattice* of X is

$$\Lambda_X = \{\lambda \in \Lambda \mid \lambda \text{ is the weight of a function in } k(X)^{(B)}\}.$$

This coincides with the weights of the torus A_X. The *weight monoid* of X is

$$\Lambda_X^+ = \{\lambda \in \Lambda^+ \mid \lambda \text{ is the weight of a function in } k[X]^{(B)}\}.$$

We write $\mathcal{Q}_X = \mathrm{Hom}(\Lambda_X, \mathbb{Q})$.

There is an exact sequence

$$0 \to k^\times \to k(X)^{(B)} \to \Lambda_X \to 0.$$

Normality of X implies that Λ_X^+ is a saturated submonoid of Λ_X, hence is determined by its dual cone. Notice that Λ_X is a birational invariant, so if the G-orbit of the open Borel orbit is G/H, then $\Lambda_X = \Lambda_{G/H}$.

Definition 3.2.

We write

$$\mathcal{D}(X) = \{B\text{-stable prime divisors of } X\}$$

$$\Delta(X) = \{D \in \mathcal{D}(X) \mid D \text{ is not } G\text{-stable}\}.$$

The elements of $\Delta(X)$ are called the *colors* of X.

The elements of $\mathcal{D}(X)$ are exactly the irreducible components of $X \setminus \mathring{X}$. The elements of $\Delta(X)$ are exactly those $D \in \mathcal{D}(X)$ such that $D \cap G/H \neq \varnothing$. As a result, the restriction map $\Delta(X) \to \Delta(G/H)$ is a bijection. For each $D \in \mathcal{D}(X)$ we get an element $\rho_X(D) \in \mathrm{Hom}(\Lambda_X, \mathbb{Z})$ via

$$\langle \rho_X(D), \lambda \rangle = \nu_D(f_\lambda),$$

where $f_\lambda \in k(X)^{(B)}$ has weight λ and where $\nu_D(f_\lambda)$ is the order of vanishing of f along D (well-defined by normality). More generally, we can define $\rho_X(\nu) \in \mathcal{Q}_X$ for any valuation $\nu : k(X) \to \mathbb{Q} \cup \{\infty\}$ invariant under the action of G. Let $\mathcal{V}(X)$ denote the set of all G-invariant valuations on $k(X)$.

Theorem 3.3.

The map $\rho_X : \mathcal{V}(X) \to \mathcal{Q}_X$ is injective; its image is a cone in $\mathcal{Q}_X$.

Remark 3.4. A homogeneous spherical variety is horospherical iff $\mathcal{V}(X) = \mathcal{Q}_X$.

Example 3.5.

Consider $G = \mathrm{PGL}_2$ and $H = T = \mathbb{G}_m = \operatorname{Spec} k[\Lambda]$ the maximal torus. Then G/H is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$. Since the associated parabolic $P(G/H) = B$, and the dimension of G/H is two, we find that $\mathring{X} \cong \mathbb{G}_m \times \mathbb{A}^1$. The torus $\mathbb{G}_m$ acts freely on $\mathring{X}$, so $A_X = T$ and $\Lambda_X = \Lambda$. We compute

$$k[\mathrm{PGL}_2/T] = k[\mathrm{PGL}_2]^T = \bigoplus_{m \geq 0} V_{m,-m} \otimes (V_{m,-m}^*)^T = \bigoplus_{m \geq 0} V_{m,-m},$$

so $\Lambda_X^+ = \Lambda^+$, the set of dominant weights for PGL_2. (Here $V_{m,-m}$ denotes the irreducible

GL_2 representation of highest weight $(m, -m)$; these exhaust the irreps of PGL_2, are self-dual, and each have a one dimensional subspace of T-invariants.) It will be convenient to have an explicit highest weight vector for one of the nontrivial representations. We can take the function sending a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $\frac{cd}{ad-bc}$; this gives a regular function on PGL_2/T which is a B-eigenvector of weight $(1, -1)$.

The colors of G/H are its B-stable prime divisors. There are just three B-orbits on X. If $p \in \mathbb{P}^1$ is the unique point fixed by B, then the orbits are

$$D_1 \coloneqq (\mathbb{P}^1 - \{p\}) \times \{p\}, \quad D_2 \coloneqq \{p\} \times (\mathbb{P}^1 - \{p\}), \quad (\mathbb{P}^1 - \{p\}) \times (\mathbb{P}^1 - \{p\}).$$

The first two are divisors and the last is the open orbit. Neither divisor is G-stable. So our colors are

$$\Delta(G/H) = \{D_1, D_2\}.$$

Since X is not horospherical, $\mathcal{V}(X)$ is not all of $\mathcal{Q}_X$. Furthermore it is nonzero, since there is a G-invariant valuation ν from the divisor $\mathbb{P}^1 \overset{\Delta}{\hookrightarrow} \mathbb{P}^1 \times \mathbb{P}^1$ (noting that $k(\mathbb{P}^1 \times \mathbb{P}^1) = k(X)$). Let us pair ν with our highest weight function above. We want to find the order of the pole of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{cd}{ad-bc}$ as the homogeneous coordinate $[a : c]$ approaches $[b : d]$, which we find to be 1. Hence $\nu(f) = -1$. Since $\mathcal{V}(X)$ must be a ray in this case, ν must generate $\mathcal{V}(X)$, so we conclude that

$$\mathcal{V}(X) = \{f : \Lambda \to \mathbb{Q} \mid f(\Lambda^+) \leq 0\}.$$

Finally, we compute the vectors $\rho_X(D_1)$ and $\rho_X(D_2)$. In this case we want to compute the order of vanishing of $f = \frac{cd}{ad-bc}$ on the divisors D_1 and D_2. We compute $\nu_{D_1}(f) = 1$ and $\nu_{D_2}(f) = 1$.

Call a spherical embedding $G/H \hookrightarrow X$ a *simple embedding* if it has a unique closed orbit Y. Write $\mathcal{D}_Y(X)$ for the B-stable divisors $D \in \mathcal{D}(X)$ such that $Y \subseteq X$. We can associate a set of colors to $Y \subseteq X$ by

$$\Delta_Y(X) = \{D \cap G/H \in \mathcal{D}(G/H) \mid D \in \mathcal{D}_Y(X) \text{ which is not } G\text{-stable}\}.$$

We can also associate a cone $\mathcal{C}_Y(X)$ in $\mathcal{Q}(G/H)$, which is generated by

$$\{\rho_X(D) \mid D \in \mathcal{D}_Y(X) \text{ is } G\text{-stable}\} \cup \rho_X(\Delta_Y(X)).$$

Theorem 3.6.

Simple embeddings up to isomorphism are in bijection with (strictly convex) *colored cones*: a cone $\mathcal{C} \subseteq \mathcal{Q}(G/H)$ and a subset $\mathcal{F} \subseteq \Delta(G/H)$ satisfying:

- $\mathcal{C}$ is a convex cone generated by $\rho(\mathcal{F})$ and finitely many elements of $\mathcal{V}(G/H)$,
- The interior $\mathcal{C}^\circ$ has nonempty intersection with $\mathcal{V}(G/H)$, and
- $\mathcal{C}$ is strictly convex, and $0 \notin \rho(\mathcal{F})$.

See Gandini's survey for many more examples. We can also classify the embeddings with multiple closed orbits.

Theorem 3.7.

Embeddings of $G/H \hookrightarrow X$ up to isomorphism are in bijection with (strictly convex) *colored fans*: a nonempty set $\mathfrak{F}$ of strictly convex colored cones such that:

- Every face of a colored cone in $\mathfrak{F}$ belongs to $\mathfrak{F}$, and
- For all $\nu \in \mathcal{V}(G/H)$ there is at most one colored cone $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$ such that $\nu \in \mathcal{C}^\circ$.

We can write $X_{\mathfrak{F}}$ for the corresponding spherical variety. Then $X_{\mathfrak{F}}$ is complete if and only if the union of all of the cones contains $\mathcal{V}(G/H)$. The embedding is called *toroidal* if the set of colors is empty. Every spherical homogeneous space admits a complete toroidal embedding.

Example 3.8.

Return to $G = \mathrm{PGL}_2$ acting on $G/H \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$.

The spherical variety $X = \mathbb{P}^1 \times \mathbb{P}^1$ gives a simple embedding of G/H, with unique closed orbit $Y = \mathbb{P}^1 \overset{\Delta}{\hookrightarrow} \mathbb{P}^1 \times \mathbb{P}^1$. Hence there should be a colored cone associated to X. Its colors should be the B-invariant prime divisors containing Y and intersecting G/H, of which there are none. So $\Delta_Y(X) = \varnothing$. The cone $\mathcal{C}_Y(X)$ is generated by ρ_X applied to the G-stable prime divisors containing Y; the unique such is Y itself. Hence the cone is generated by $\rho_X(Y)$, which we have previously computed to be the unique generator of $\mathcal{V}(G/H)$.

Note that this is the only possible (strictly convex) colored cone for G/H! Since both possible colors for G/H generate the cone which is the negation of $\mathcal{V}(G/H)$, there is no way to build a strictly convex cone containing any points of $\mathcal{V}(G/H)$ in its interior, unless we use no colors. If we do use no colors, the unique cone containing points of $\mathcal{V}(G/H)$ in its interior is $\mathcal{V}(G/H)$ itself. Hence there is exactly one nonzero (strictly convex) colored cone, and therefore exactly one nontrivial colored fan for G/H. So the only spherical embeddings of G/H are the trivial one, and $G/H \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

Write $\Lambda = \Lambda_{G/H}$ and $\mathcal{Q} = \mathcal{Q}_{G/H}$.

Definition 3.9.

The *spherical roots* of G/H, denoted $\Sigma_{G/H}$ are the primitive elements in Λ generating an extremal ray of $-\mathcal{V}^\vee$.

The *little Weyl group* of G/H is the Weyl group $W_{G/H}$ of the root system generated by $\Sigma_{G/H}$; equivalently, the subgroup of $\mathrm{GL}(\mathcal{Q})$ generated by reflections about the codimension 1 faces of $\mathcal{V}$.

Theorem 3.10.

The cone $\mathcal{V}(G/H)$ is a fundamental domain for the action of the little Weyl group on $\mathcal{Q}$

(except in characteristic 2).

Definition 3.11.

If it exists, then the *wonderful embedding* of G/H is the unique complete, simple, toroidal, smooth embedding $G/H \hookrightarrow X$.

Theorem 3.12.

G/H admits a wonderful embedding iff Σ is a basis of Λ.

Example 3.13.

The embedding $\mathrm{PGL}_2/\mathbb{G}_m \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ studied in earlier examples is a wonderful embedding. The spherical roots are the unique basis for Λ.

4 Gaitsgory–Nadler theorem on $G(\mathcal{O})$ orbits

Gaitsgory and Nadler give a description of $\mathcal{V}(G/H)$ generalizing the description of fundamental coweights via orbits on the affine Grassmannian (a key input to the geometric Satake equivalence). Let $\mathcal{O} = \mathbb{C}[[x]]$ and $\mathcal{K} = \mathbb{C}((x))$. If $T = \operatorname{Spec} k[\Lambda]$ is a torus, then the $T(\mathcal{K})$ points include those coming from coweights $\mathbb{G}_m \to T$. Hence we can identify the coweight lattice $\Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z})$ as a subset of $T(\mathcal{K})$. In fact each $T(\mathcal{O})$ orbit in $T(\mathcal{K})$ contains a unique such coweight. More generally, the coweights of the associated Cartan $A_{G/H}$ are a subset of $(G/H)(\mathcal{K})$ and we have:

Theorem 4.1.

If G is reductive and H is a subgroup, then the following are equivalent:

- G/H is a spherical variety
- $H(\mathcal{K})$ acts on the affine Grassmannian Gr_G with countably many orbits
- $G(\mathcal{O})$ acts on $(G/H)(\mathcal{K})$ with countably many orbits.

Theorem 4.2: [Gaitsgory–Nadler, Thm 3.3.1].

Each element of $(G/H)(\mathcal{K})$ contains a unique element of $\mathcal{V}(G/H) \cap \Lambda^\vee$ in the $G(\mathcal{O})$ orbit through it, and every such element is realized.

In other words, we can identify the $G(\mathcal{O})$ orbits in $(G/H)(\mathcal{K})$ with (the integral points of) the cone $\mathcal{V}(G/H) \subseteq \mathcal{Q}$.

Example 4.3.

Consider the group $G \times G$ with H the diagonal embedding of G. This is a symmetric space, since G is the fixed points of the evident $\mathbb{Z}/2$ action. In this case the Gaitsgory–Nadler theorem says that

$$(G(\mathcal{O}) \times G(\mathcal{O}))\backslash(G(K) \times G(K))/G(K) = G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}) \cong \Lambda_G^+,$$

which is a key input to the geometric (and classical) Satake equivalence.