Dual Group of Spherical Variety

Setup

M = hyperspherical variety $\circlearrowleft G$ red. gp $/\mathbb{F}$ = alg. closed char = 0

Assume that M admits a distinguished polarization.

Then $\exists$ $H \times SL_2 \longrightarrow G$, $H \circlearrowright S$ symplectic rep'n
$LU_P = P \supset B^-$; $S = S^+ \oplus S^-$ H-stable polarization

s.t. $X_L := S^+ \overset{H}{\times} L$ spherical L-variety

$$X = X_L \overset{P}{\times} G = S^+ \overset{HU_P}{\times} G$$

$SL_2 \to G \rightsquigarrow f : U_P \longrightarrow \mathbb{G}_a$, $U_P' := \mathrm{Ker}(f)$

$$\begin{array}{c} S^+ \overset{HU_P}{\times} G \\ \uparrow {\scriptstyle \mathbb{G}_a\text{-torsor}} \\ \Psi = S^+ \overset{HU_P'}{\times} G \end{array}$$

$$M = T^*(X, \Psi)$$

Goal M as above $\rightsquigarrow \check{G}_X \times SL_2 \longrightarrow \check{G}$

Later $\rightsquigarrow S_X \in \mathrm{Rep}(\check{G}_X)$ conjecturally symplectic

$$\check{M} := \omega\text{-}\mathrm{ind}_{\check{G}_X}^{\check{G}} S_X$$

Table of content

§1. $\check{G}_X$ via root datum

§1.1. $M = T^*X$

<u>Goal</u> $\quad X$ spherical var. $\quad$ up to birational equiv. $\rightsquigarrow \hat{G}_X \times SL_2 \longrightarrow \hat{G}$

§1.1.1. $\check{G}_X$ via combinatorics

WLOG $X = G/H$ for $H \subset G$

maximal torus

$A \subset B \subset G$, Δ = simple roots , Φ = roots , W = Weyl gp

Borel

$G \circlearrowright X$ spherical , $k(X)$ = function field of X , X° = open B-orbit $\quad$ Or equivalently $\cdot$ $X^\circ = B_X \backslash B \rightsquigarrow A_X = B_X \cdot U \backslash B$

$$\begin{tikzcd} & \bigcup & \\ f_\chi \in \arrow[d, mapsto] & k(X)^{(B)} \arrow[d] \arrow[dr] & = \text{subset of } B\text{-eigenfunctions} \\ \chi \in & X^*(A_X) \arrow[r, hook] & X^*(A) \end{tikzcd}$$

This gives a definition of A_X

$\mathcal{V} = \{G\text{-invariant discrete } \mathbb{Q}\text{-valued valuations on } k(X)\}$

There is a map $\quad \mathcal{V} \xrightarrow{\rho_{\mathcal{V}}} X_*(A_X)_{\mathbb{Q}} = \mathrm{Hom}(X^*(A_X), \mathbb{Q})$, denote $X_*(A_X)^+_{\mathbb{Q}} = -\overline{\mathrm{Im}(\rho_{\mathcal{V}})}$

$\qquad v \longmapsto (\chi \longmapsto v(f_\chi))$

<u>Fact</u> [Brion] $\cdot$ $\rho_{\mathcal{V}}$ is injective

$\cdot$ Under $\quad X_*(A) \xrightarrow{p} X_*(A_X)$

$\qquad p(X_*(A)^+) \subset X_*(A_X)^+$

When $p(X_*(A)^+_{\mathbb{Q}}) = \mathcal{V}$, $\quad X$ is called <u>wavefront</u>.

satisfied in many examples

Denote $\mathcal{V}^\perp = \{\chi \in X^*(A_X)_{\mathbb{Q}} \mid \chi|_{\mathcal{V}} \leq 0\}$

Denote $\{l_i\}$ to be set of extremal rays of $\mathcal{V}^\perp$

Pick $\alpha_i \in l_i$ s.t. $l_i \cap X^*(A_X) = \mathbb{Z}_{\geq 0} \cdot \alpha_i$ $\quad$ (Then $X_*(A_X)^+_{\mathbb{Q}} = \{\check{\lambda} \in X_*(A_X)_{\mathbb{Q}} \mid \langle \check{\lambda}, \alpha_i \rangle \geq 0, i = 1, \cdots\}$)

associated root

Then either $\cdot$ α_i is proportional to $\tilde{\alpha}_i \in \Phi$ $\quad$ (in fact, $\alpha_i = \tilde{\alpha}_i$ or $\alpha_i = 2\tilde{\alpha}_i$) $\rightsquigarrow \{\alpha^\vee\}$

$\cdot$ α_i is proportional to $\tilde{\alpha}_i = \beta + \gamma$ for $\beta, \gamma \in \Phi$ strongly orthogonal $\rightsquigarrow \{\beta^\vee, \gamma^\vee\}$

(In fact, can choose β, γ uniquely s.t.

Denote $\Delta_X = \{\tilde{\alpha}_i\}_i$, called (normalized) (simple) spherical roots.

- β, γ are positive
- $(\mathbb{Q}\beta + \mathbb{Q}\gamma) \cap \Phi = \{\pm\beta, \pm\gamma\}$ (strongly orthogonal)
- $\beta^\vee - \gamma^\vee = \delta_1^\vee - \delta_2^\vee$ for $\delta_1, \delta_2 \in \Delta$)

Fact [Brion] $\exists\ W_X \subset W$ subgp s.t. • W_X stablizes $X^*(A_X) \subset X^*(A)$

• $\mathcal{V} \subset X_*(A_X)_{\mathbb{Q}}$ is a fundamental domain of W_X

$\Xi := X^*(A_X) + \mathbb{Z}\cdot\Delta_X$ Rmk When T^*X hyperspherical, $X^*(A_X) = X^*(A_X)_{\mathbb{Q}} \cap X^*(A)$ so $\Xi = X^*(A_X)$

Fact $(\Delta_X \subset \Xi,\ W_X)$ defines a root datum

Take $\check{G}_X$ to be associated to the dual root datum (This def'n is given by Knop-Schalke)

Denote $X^\circ \subset X$ open B-orbit, $P(X) = \{g \in G \mid gX^\circ = X^\circ\} = L(X)\cdot U(X)$ Levi decomp.

• Thm [Knop-Schalke] There is a unique distinguished map $\check{G}_X \times SL_2 \longrightarrow \check{G}$ up to conjugation.

(Distinguished : • $\begin{array}{ccc} \check{G}_X & \longrightarrow & \check{G} \\ \cup & & \cup \\ \check{A}_X & \longrightarrow & \check{A} \end{array}$ commutes

when $\gamma \in \Delta$, $Asoc_\gamma = \{\gamma^\vee\}$
$\gamma = \alpha + \beta$, $Asoc_\gamma = \{\alpha^\vee, \beta^\vee\}$

• For every $\gamma \in \Delta_X$ with associated root $Asoc_\gamma \subset \Phi^\vee$

$\check{\mathfrak{g}}_{X,\gamma^\vee}$ is mapped into $\bigoplus_{\check{\alpha} \in Asoc_\gamma} \check{\mathfrak{g}}_{\check{\alpha}}$

• SL_2 is a principal-SL_2 of $\check{L}(X)$)

Note When $B \curvearrowright X$ has generically connected stablizer, $\check{G}_X \subset \check{G}$ (as $X^*(A_X) = X^*(A_X)_{sat} \subset X^*(A)$)

Next More computable def'n of W_X, Φ_X under additional assumptions.

§1.1.2. W_X via moment map & GIT

Assume $X = G/H$ is quasi-affine spherical

Thm [Knop]
$$\begin{tikzcd} T^*X \arrow[r, "\mu"] \arrow[rr, bend left, "\bar{\mu}"] \arrow[dr] & \mathfrak{g}^* \arrow[r] & \mathfrak{a}^*/\!/W \\ & T^*X/\!/G \simeq \mathfrak{a}_X^*/\!/W_X \arrow[ur] & \end{tikzcd}$$
factors as

Sketch of $\mathrm{Im}(\bar{\mu}) = \mathrm{Im}(\rho)$ · $\mathfrak{a}_X^* \times X^\circ \xrightarrow{s} T^*X$

$(\sum a_i x_i, x) \longmapsto \sum a_i \frac{df_{x_i}}{f_{x_i}}\Big|_x$

· $\overline{G \cdot \mathrm{Im}(s)} = T^*X$ by tangent space computation

where $\mathfrak{a}_X = \mathrm{Lie}\, A_X$, $\mathfrak{a} = \mathrm{Lie}\, A$, $\mathfrak{a}_X^* \subset \mathfrak{a}^*$, $W_X \subset W$ stablizes $\mathfrak{a}_X^*$ and faithfully acts on $\mathfrak{a}_X^*$.

· $T^*X/\!/G \simeq \mathfrak{a}_X^*/\!/W_X$ i.e. $k[T^*X]^G \simeq k[\mathfrak{a}_X^*]^{W_X}$ (need quasi-affine to make sense of this)

This determines $W_X \subset N_W(\mathfrak{a}_X^*)/\underbrace{C_W(\mathfrak{a}_X^*)}_{\substack{\shortparallel \\ W_{L(X)}}}$

§. 1.1.3. W_X via Knop's action

Let $\mathcal{B}(X) = \{B\text{-orbit on } X\}$. There is an action $W \curvearrowright \mathcal{B}(X)$ defined as follows

· For $\alpha \in \Delta$, $Y \in \mathcal{B}(X)$
$$\begin{array}{c} Y \\ \cap \\ X \end{array}$$

$\overline{Y}_\alpha := Y \cdot P_\alpha / \mathrm{rad}(P_\alpha) \circlearrowleft L_\alpha^{ad}$ is a spherical $L_\alpha^{ad} \simeq PGL_2$-variety

minimal parabolic containing B, $U_{-\alpha}$

Only 4 cases can occur :

(Type G) $\overline{Y}_\alpha = pt$ i.e. $YP_\alpha = Y$, define $s_\alpha Y = Y$ (One say Y is of type ⋯)

B-orbit decomp.

(Type T) $\overline{Y}_\alpha = \mathbb{G}_m \backslash PGL_2 = \underbrace{\overline{Y}_\alpha^\circ}_{\text{open}} \sqcup \underbrace{\overline{Z}_+ \sqcup \overline{Z}_-}_{\text{closed}}$

preimage

$Y \cdot P_\alpha = Y_\alpha^\circ \sqcup Z_+ \sqcup Z_-$

define $s_\alpha Y_\alpha^\circ = Y_\alpha^\circ$

$s_\alpha Z_\pm = Z_\mp$

· $\dim \overline{Z}_1 = \dim \overline{Z}_2 = \dim \overline{Y}_\alpha^\circ - 1$

(Type U) $\overline{Y_\alpha} = U\cdot S\backslash PGL_2 = \overline{Y_\alpha^\circ} \sqcup \overline{Z}$ for $S \subset \mathbb{G}_m \subset PGL_2$

$$\wr$$

$Y\cdot P_\alpha = Y_\alpha^\circ \sqcup Z$

define $s_\alpha \cdot Y_\alpha^\circ = Z$, $s_\alpha \cdot Z = Y_\alpha^\circ$

· $\dim Z = \dim Y_\alpha^\circ - 1$

(Type N) $\overline{Y_\alpha} = N_{PGL_2}(\mathbb{G}_m)\backslash PGL_2 = \overline{Y_\alpha^\circ} \sqcup \overline{Z}$

$$\wr$$

$Y_\alpha \cdot P_\alpha = Y_\alpha^\circ \sqcup Z$

$s_\alpha Y_\alpha^\circ = Y_\alpha^\circ$, $s_\alpha Z = Z$

· $\dim Z = \dim Y_\alpha^\circ - 1$

<u>Rmk</u> $X^*(A_{wy}) = {}^{w}X^*(A_y)$ for $w \in W$, $y \in \mathcal{B}(X)$

<u>Thm</u> [Knop]

Above actions extend to $W \circlearrowright \mathcal{B}(X)$

<u>Rmk</u> · <u>Fact</u> [Knop] $\{Y \in \mathcal{B}(X) \mid \operatorname{rank} X^*(A_Y) = \operatorname{rank} X^*(A_X)\} = W \cdot X^\circ$

· Above action can be defined for non-spherical X, with $\mathcal{B}(X)$ modified.

Denote $W_{(X)}$ = stablizer of $\overset{\circ}{X}$, easy to see $W_{L(X)} \subset W_{(X)}$

<u>Thm</u> [Knop] $\exists$ canonical injection $W_X \subset W_{(X)}$

s.t. $W_{(X)} = W_X \ltimes W_{L(X)}$

§1.1.4. <u>Φ_X, W_X of symmetric variety</u>

When $X = G/G_0$ for $G_0 = G^\theta$ where $\theta \in \operatorname{Aut}(G)\backslash\{e\}$, $\theta^2 = \mathrm{id}$

say X is a symmetric variety

<u>Fact</u> X is spherical (b/c $G_1 \longrightarrow {}_H\backslash G/_H$ has dense image (by tangent space computation) + Gelfand's trick)

Denote $G_1 := \{g \in G \mid \theta(g) = g^{-1}\}$, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$

<u>Fact</u> [Richardson] $\exists$ $\widetilde{A}_X \subset A$ s.t.

· $\widetilde{A}_X$ is a maximal subtorus of G_1

· A is a maximal torus of G s.t. $\theta(A) = A$

Then $\widetilde{A}_X \subset A \twoheadrightarrow A_X \simeq A/A^\theta$ is an isogeny

Define $\Phi_X' \subset X^*(\widetilde{A}_X) \subset X^*(A_X)_{\mathbb{Q}}$ to be the roots associated to $\widetilde{A}_X \overset{Ad}{\curvearrowright} \mathfrak{g}$

i.e. $\mathfrak{g} = \bigoplus_{\alpha \in \Phi_X'} \mathfrak{g}_\alpha$

<u>Fact</u> [Knop] · Φ_X is proportional to Φ_X'

· $W_X = N_{G_0^\circ}(\widetilde{A}_X) / C_{G_0^\circ}(\widetilde{A}_X)$

· $\Delta_{L(X)} = \{ \alpha \in \Delta \mid \alpha^\vee |_{X^*(\widetilde{A}_X)} = 0 \}$

§1.2. $M = T^*(X, \Psi)$

Assume we are in hyperspherical case.

In this case, $\exists\ H \times SL_2 \longrightarrow G$, $H \curvearrowright S^+ \oplus S^-$ s.t.

$$X_L = S^+ \overset{H}{\times} L$$

$$X = X_L \overset{P}{\times} G = S^+ \overset{HU}{\times} G$$

$$\big\uparrow \mathbb{G}_a\text{-torsor}$$

$$\Psi = S^+ \overset{HU'}{\times} G$$

<u>Fact</u> For $\alpha \in \Delta \setminus \Delta_L$, $\alpha \in X^*(A_X) \subset X^*(A)$

<u>Proof</u> Choose $x \in X_L^\circ$, denote $B_x =$ stablizer of x in $B \cap L$

For $b \in B_x$, write $b = tu$ for $t \in A \subset B$, $u \in U \cap L$, wts $\alpha(t) = 1$

Consider $X_L \longrightarrow H \backslash L$, then $l^{-1} b l \in H$
$x \longmapsto l$

<u>Fact</u> ${}^{l^{-1}}f|_{U_{-\alpha}}$ is non-trivial (f is generic)

Recall that we have $H \subset C_L(f)$

$\Rightarrow {}^{l^{-1}bl}f = f \Rightarrow {}^{l}f = {}^{b^{-1}l}f$

$\Rightarrow ({}^{l}f)|_{U_{-\alpha}} = {}^{u^{-1}t^{-1}}({}^{l}f))|_{U_{-\alpha}} = {}^{t^{-1}}({}^{l}f)|_{U_{-\alpha}} = \alpha(t)^{-1}({}^{l}f)|_{U_{-\alpha}}$

b/c $[U \cap L, U_{-\alpha}] = \{1\}$

$\Rightarrow \alpha(t) = 1$

To construct $\check{G}_{X,\Psi} \times SL_2 \longrightarrow \check{G}$

· Start from $(X^*(A_{X_L}) \supset \Delta_{X_L}, W_{X_L})$

$$\begin{array}{ccc} \| & & \wr \\ (X^*(A_X) \supset \Delta_{X,\Psi} := \Delta_{X_L} \cup (\Delta \setminus \Delta_L) & , & W_{X,\Psi} := \langle W_{X_L}, s_\alpha, \alpha \in \Delta \setminus \Delta_L \rangle) \end{array}$$

$\Big\downarrow$ dual datum

$\check{G}_{X,\Psi}$

<u>Rmk</u> When X_L is <u>wavefront</u>, [SV] proves that

$\check{G}_{X,\Psi} = \check{G}_\Psi$ where $\check{G}_\Psi$ is dual gp for non-spherical variety constructed by Knop-Schalke.

When X_L not wavefront, $\check{G}_{X,\Psi} \overset{?}{=} \check{G}_\Psi$ is not known.

Fact $\check{G}_{X,\Psi}$ = subgp of $\check{G}$ generated by $\check{G}_{X_L}$ and $U_{\pm\alpha^\vee}$ for $\alpha \in \Delta \setminus \Delta_L$

$SL_2 \longrightarrow \check{G}$ is given by $SL_2 \longrightarrow \check{G}_{X_L} \longrightarrow \check{G}$

$\rightsquigarrow \check{G}_{X,\Psi} \times SL_2 \longrightarrow \check{G}$

SL_2 commutes with $\check{G}_{X_L}$ by construction

For $\alpha \in \Delta_{L(X)}$, $\beta \in \Delta \setminus \Delta_L$, $\langle \alpha^\vee, \beta \rangle = 0$ as $\beta \in X^*(A_X) \subset X^*(L(X)^{ab})$

$\Rightarrow SL_2$ commutes with $\check{G}_{X,\Psi}$

§2. Examples

§2.1. Reality check

- $X \circlearrowleft G = T$ toric

$A_X = A = T$, $W_X = \text{trivial}$.
$$\begin{tikzcd} \check{G}_X \times SL_2 \arrow[r] & \check{G} \\ \check{T} \times SL_2 \arrow[u, equal] \arrow[r, "pr_1"] & \check{T} \arrow[u, equal] \end{tikzcd}$$

- $X = pt \circlearrowleft G$

$$\underset{\substack{\| \\ W}}{W_{(X)}} = \underset{\substack{\| \\ W}}{W_X} \ltimes W_{L(X)}$$

$A_X = \text{trivial}$, $P(X) = G \Rightarrow W_X = \{e\}$

$$\begin{tikzcd} T^*X/\!/G \arrow[d] & & pt \arrow[d] \\ \mathfrak{g}^*/\!/G & = & \mathfrak{t}^*/\!/W \end{tikzcd} \Rightarrow W_X = \{e\}$$

$$\Rightarrow \begin{tikzcd} \check{G}_X \times SL_2 \arrow[r] & \check{G} \\ \{e\} \times SL_2 \arrow[u, equal] \arrow[ur, "\text{principal } SL_2"'] & \end{tikzcd}$$

$S_X = \text{triv} \in \mathrm{Rep}(\{e\})$

- $X = G \circlearrowleft G \times G$

$$\begin{tikzcd} A \arrow[r] & A_X \\ A_G^2 \arrow[u, equal] \arrow[r] & A_G \arrow[u, equal] \end{tikzcd}, \quad P(X) = B^- \times B, \quad \underset{\substack{\| \\ W_G}}{W_{(X)}} = W_X \ltimes \underset{\substack{\| \\ \{e\}}}{W_{P(X)}}$$

B-orbits in open G-orbits $\simeq W_G \circlearrowleft W_G \times W_G$

$\Rightarrow W_X = W_G$

$$\begin{tikzcd} T^*X/\!/G\times G \arrow[d] & & \mathfrak{t}^*/\!/W_G \arrow[d, "\Delta"] \\ \mathfrak{g}^* \times \mathfrak{g}^*/\!/G \times G & = & \mathfrak{t}^*/\!/W_G \times \mathfrak{t}^*/\!/W_G \end{tikzcd}$$

$$\Rightarrow \begin{tikzcd} \check{G}_X \times SL_2 \arrow[r] & \check{G} \times \check{G} \\ \check{G} \times SL_2 \arrow[u, equal] \arrow[ur, "(\Delta, \text{triv})"'] & \end{tikzcd}$$

$S_X = \text{triv} \in \mathrm{Rep}(\check{G})$

- $X = U \backslash G \circlearrowleft G$ (not affine, so T^*X not hyperspherical)

$A \xrightarrow{\sim} A_X$, $P(X) = B$, $\underset{\substack{\| \\ \{e\}}}{W_{(X)}} = W_X \ltimes W_{P(X)}$

B-orbits $\simeq W \circlearrowleft W$

$\Rightarrow W_X = \{e\}$

$$\begin{tikzcd} T^*X/\!/G \arrow[d] & & \mathfrak{b}/\!/U = \mathfrak{t} \arrow[d] \\ \mathfrak{g}^*/\!/W & = & \mathfrak{t}/\!/W \end{tikzcd}$$

$$\Rightarrow \begin{tikzcd} \check{G}_X \times SL_2 \arrow[r] & \check{G} \\ \check{A} \times SL_2 \arrow[u, equal] \arrow[ur, "pr_1"'] & \end{tikzcd}$$

• $(X, \Psi) = (U\backslash G,\ U^{\prime}\backslash G) \circlearrowleft G$

$X = \underset{\overset{\shortparallel}{T}}{X_T} \overset{T}{\times} B$ for spherical $T \circlearrowleft T$

$$\begin{tikzcd} \check{G}_{X_T} \times SL_2 \arrow[r] & \check{T} \subset \check{G} \\ \check{T} \times SL_2 \arrow[u, equal] \arrow[ur, "pr_1"'] & \end{tikzcd}$$

$\check{G}_{X,\Psi}$ satisfies $\Delta_{X,\Psi} = \Delta_{X_T} \sqcup (\Delta \backslash \Delta_T) = \Delta$

$$\rightsquigarrow \begin{tikzcd} \check{G}_{X,\Psi} \times SL_2 \arrow[r] & \check{G} \\ \check{G} \times SL_2 \arrow[u, equal] \arrow[ur, "pr_1"'] & \end{tikzcd}$$

$S_X = \mathrm{triv} \in \mathrm{Rep}(\check{G})$

§2.2. Rank one cases

• $X = \mathbb{G}_m \backslash PGL_2$

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[d, equal] & A_X \arrow[d, equal] \\ \mathbb{G}_m \arrow[r, "\sim"] & \mathbb{G}_m \end{tikzcd}$$

, $P(X) = B$, $W_{(X)} = \underset{\mathbb{Z}/2\mathbb{Z}}{W_X} \ltimes \underset{\{e\}}{W_{L(X)}}$ $\Leftarrow$

$\Rightarrow W_X = W = \mathbb{Z}/2\mathbb{Z}$

$\Rightarrow \Delta_X = \Delta \Rightarrow \check{G}_X = \check{G} = SL_2$

Knop's action

$T^*X/\!/G \xrightarrow{\sim} \mathfrak{g}^*/\!/G$

$\{(y\ \ x) \sim (\lambda^{-1}y\ \ \lambda x)\} \xrightarrow{\sim} xy$

B-orbits: X°

$Z_+ \overset{s}{\longleftrightarrow} Z_-$ $s \in W$

$$\begin{tikzcd} \check{G}_X \times SL_2 \arrow[r] & \check{G} \\ SL_2 \times SL_2 \arrow[u, equal] \arrow[r, "pr_1"] & SL_2 \arrow[u, equal] \end{tikzcd}$$

$S_X = \mathrm{Std} \otimes \mathrm{Std}^{\vee} \in \mathrm{Rep}(SL_2)$

Compare • $X = \mathbb{G}_m \backslash SL_2$

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & A_X \arrow[d, equal] \\ \mathbb{G}_m \arrow[r, "(\cdot)^2"] & \mathbb{G}_m \end{tikzcd}$$

, $P(X) = B$, $W_X = \mathbb{Z}/2\mathbb{Z}$

$\Rightarrow \Delta_X = \underset{\{2\}}{\Delta} \subset \underset{2\mathbb{Z}}{X^*(A_X)} \subset \underset{\mathbb{Z}}{X^*(A)}$

$$\rightsquigarrow \begin{tikzcd} \check{G}_X \times SL_2 \arrow[r] & PGL_2 \\ SL_2 \times SL_2 \arrow[u, equal] \arrow[ur, "pr_1"'] & \end{tikzcd}$$

$\check{G}_X \not\subset \check{G}$ as generic B-stablizer $= \mu_2$

not connected.

• $X = N(\mathbb{G}_m) \backslash PGL_2 \circlearrowleft PGL_2$ (We use Knop-Schalke's def'n)

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d, equal] & A_X \arrow[d, equal] \\ \mathbb{G}_m \arrow[r, "(\cdot)^2"] & \mathbb{G}_m \end{tikzcd}$$

, $P(X) = B$, $W_X = W = \mathbb{Z}/2\mathbb{Z}$

$\Delta_X = \Delta \not\subset X^*(A_X) \subset X^*(A) = \Xi = X^*(A_X) + \mathbb{Z}\Delta$

$\{1\} \not\subset \underset{}{2\mathbb{Z}} \subset \mathbb{Z}$

$$\rightsquigarrow \begin{tikzcd} \check{G}_X \times SL_2 \arrow[r] & SL_2 \\ SL_2 \times SL_2 \arrow[u, equal] \arrow[ur, "pr_1"'] & \end{tikzcd}$$

- $X = SO_{2n}\backslash SO_{2n+1} \circlearrowleft G = SO_{2n+1}$

$Q = x_1x_{2n+1} + x_2x_{2n} + \cdots + x_nx_{n+2} + x_{n+1}^2$ quadratic form on $\mathbb{F}^{2n+1} = V$

$X = \{ \underline{x} \in \mathbb{F}^{2n+1} \mid Q(\underline{x}) = 1 \} \subset V$

Take $B \subset SO_{2n+1}$ preserving $\{0\} \overset{1}{\subset} F_1 \overset{1}{\subset} F_2 \overset{1}{\subset} \cdots \overset{1}{\subset} F_{2n} \overset{1}{\subset} V$

$$\begin{array}{ccc} A & \longrightarrow & A_X \\ \| & & \| \\ \mathbb{G}_m^n & \xrightarrow[pr_1]{} & \mathbb{G}_m \end{array}, \quad P(X) = \text{preserving } \{0\} \overset{1}{\subset} F_1 \subset F_{2n} \overset{1}{\subset} V$$

$$\begin{array}{ccccc} & & S_{n-1} \ltimes (\mathbb{Z}/2\mathbb{Z})^n & & S_{n-1} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1} \\ & & \| & & \| \\ W(X) & = & W_X & \ltimes & W_{L(X)} \end{array}$$

$\Rightarrow W_X = \mathbb{Z}/2\mathbb{Z}$

x_{2n+1} has eigencharacter $e_1 \in X^*(A)$

$$\Rightarrow \begin{array}{ccc} X^*(A_X) & \hookrightarrow & X^*(A) \\ \| & & \| \\ \mathbb{Z}e_1 & \subset & \bigoplus_{i=1}^{n} \mathbb{Z}e_i \end{array}$$

$\Delta_X = \{e_1\}$

$$\Rightarrow \begin{array}{ccc} \check{G}_X \times SL_2 & \longrightarrow & \check{G} \\ \| & & \| \\ SL_2 \times SL_2 & & Sp_{2n} \\ \downarrow (id,\, principal) & \nearrow & \\ Sp_2 \times Sp_{2n-2} & & \end{array}$$

$S_X = Std \oplus Std^\vee \in Rep(SL_2)$

Knop's action

B-orbits : $X \cap (V\backslash F_{2n})$, $X \cap (F_{2n} \backslash F_{2n-1})$, $\vdots$, $X \cap (F_{n+2}\backslash F_{n+1})$ — $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})$

S_n - acts via permutation

$$(X \cap F_{n+1})^+ \longleftrightarrow (X \cap F_{n+1})^-$$

$(\mathbb{Z}/2\mathbb{Z})^n \xrightarrow[pr_n]{} \mathbb{Z}/2\mathbb{Z}$

- $X = \mathbb{A}^n \circlearrowleft G = GL_n$

$$\begin{array}{ccc} A & \longrightarrow & A_X \\ \| & & \| \\ \mathbb{G}_m^n & \xrightarrow{pr_1} & \mathbb{G}_m \end{array}, \quad P(X) = \begin{array}{cc} \begin{array}{cc} 1 & n-1 \end{array} & \\ \begin{pmatrix} * & * \\ & * \end{pmatrix} & \begin{array}{c} 1 \\ n-1 \end{array} \end{array}, \quad \begin{array}{ccccc} S_{n-1} & & & & S_{n-1} \\ \| & & & & \| \\ W(X) & = & W_X & \ltimes & W_{P(X)} \end{array}$$

$\Rightarrow W_X = \text{trivial}$

$$\Rightarrow \begin{array}{ccc} \check{G}_X \times SL_2 & \longrightarrow & \check{G} \\ \| & & \| \\ \mathbb{G}_m \times SL_2 & \longrightarrow & GL_n \\ \downarrow (id,\, principal) & \nearrow & \\ GL_1 \times GL_{n-1} & & \end{array}$$

$S_X = triv \in Rep(\mathbb{G}_m)$

Knop's action

B-orbits $=$ $\mathbb{A}^n \backslash \mathbb{A}^{n-1}$ (where $\mathbb{A}^{n-1} = \{0\} \times \mathbb{A}^{n-1}$), $\mathbb{A}^{n-1}\backslash \mathbb{A}^{n-2}$, $\vdots$, $\mathbb{A}^1 \backslash \{0\}$ — $W = S_n$ permutation

$\{0\}$

§2.3. Symmetric cases

- $X = \underbrace{GL_n \times GL_n}_{\overset{\shortparallel}{G_0}} \backslash GL_{2n} \circlearrowleft GL_{2n} = G$

$\theta = \mathrm{Ad}_{\begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}}$, then $GL_n \times GL_n = G^\theta$

Take $\widetilde{A}_X := (SO_2)^n \hookrightarrow G$ s.t. $\mathbb{F}^{\oplus n} \simeq \widetilde{\mathfrak{a}}_X \hookrightarrow \mathfrak{gl}_{2n}$

$e_j \longmapsto i(E_{j,j+n} - E_{j+n,j})$ $\rightsquigarrow \begin{pmatrix} & i \\ -i & \end{pmatrix}$

$A_X^\theta := \mathbb{G}_m^n \hookrightarrow G$, $\quad A := \widetilde{A}_X \cdot A_X^\theta$

$(\lambda_i) \longmapsto \mathrm{diag}(\lambda_1, \cdots, \lambda_n, \lambda_1, \cdots, \lambda_n)$

Set $X^*(\widetilde{A}_X) = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$, $\quad X^*(A^\theta) = \bigoplus_{i=1}^n \mathbb{Z}\beta_i$

$$1 \to \mu_2^n \to \widetilde{A}_X \times A^\theta \longrightarrow A \longrightarrow 1$$

$$1 \to \mu_2^n \to \widetilde{A}_X \longrightarrow A_X \longrightarrow 1$$

$\rightsquigarrow$

$$\begin{array}{ccc} X^*(A_X) & \longrightarrow & X^*(A) \\ \shortparallel & & \shortparallel \\ \bigoplus_{i=1}^n 2\mathbb{Z}\alpha_i & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z}(\alpha_i \pm \beta_i) \end{array}$$

$$\rightsquigarrow \Phi_G = \{\pm\alpha_i,\ \alpha_i - \alpha_j \pm \beta_i \pm \beta_j\ (i \neq j)\} \subset X^*(A)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\Phi_X' = \{\pm\alpha_i,\ \alpha_i - \alpha_j\ (i \neq j)\} \subset X^*(\widetilde{A}_X)$$

$\Rightarrow \Phi_X = \{\pm 2\alpha_i,\ 2(\alpha_i - \alpha_j)\ (i \neq j)\} \subset X^*(A_X)$

$W_X = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$

$P(X) = B$

$\Rightarrow \check{G}_X \times SL_2 \longrightarrow \check{G}$

$\quad\ \ \shortparallel$

$Sp_{2n} \times SL_2 \xrightarrow{1 \times \text{trivial}} GL_{2n}$

$S_X = \mathrm{Std} \oplus \mathrm{Std}^\vee \in \mathrm{Rep}(Sp_{2n})$

- $X = \underset{\overset{\shortparallel}{G_0}}{Sp_{2n}} \backslash GL_{2n} \circlearrowleft GL_{2n} = G$

$\theta = \mathrm{Ad}_J \circ {}^t(\cdot)^{-1}$ where $J = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$

$Sp_{2n} = G^\theta$

Take $\widetilde{A}_X = \mathbb{G}_m^n \hookrightarrow G$, $\quad A^\theta = \mathbb{G}_m^n \hookrightarrow G$, $\quad A = \widetilde{A}_X \cdot A^\theta$

$(\lambda_i) \longmapsto \mathrm{diag}(\lambda_1, \cdots, \lambda_n, \lambda_1, \cdots, \lambda_n)$, $\quad (\lambda_i) \longmapsto \mathrm{diag}(\lambda_1, \cdots, \lambda_n, \lambda_1^{-1}, \cdots, \lambda_n^{-1})$

$X^*(\widetilde{A}_X) = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$, $\quad X^*(A^\theta) = \bigoplus_{i=1}^n \mathbb{Z}\beta_i$

Then

$$\begin{array}{ccc} X^*(A_X) & \longrightarrow & X^*(A) \\ \shortparallel & & \shortparallel \\ \bigoplus_{i=1}^n 2\mathbb{Z}\alpha_i & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z}(\alpha_i \pm \beta_i) \end{array}$$

$$\Phi_G = \{\pm 2\beta_i,\ \alpha_i - \alpha_j \pm \beta_i \pm \beta_j\} \subset X^*(A)$$
$$\downarrow \qquad\qquad \downarrow$$
$$\Phi_X'' = \{\alpha_i - \alpha_j\} \subset X^*(\widetilde{A_X})$$

$$\left.\begin{array}{l} \Rightarrow \Phi_X = \{2(\alpha_i - \alpha_j)\} \subset X^*(A_X) \\ W_X = S_n \\ \Phi_{L(X)} = \{\pm 2\beta_i\} \end{array}\right\} \Rightarrow \begin{array}{ccc} \check{G}_X \times SL_2 & \longrightarrow & \check{G} \\ \shortparallel & & \shortparallel \\ GL_n \times SL_2 & \longrightarrow & GL_{2n} \\ (g, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) & \longmapsto & (\begin{pmatrix} g & \\ & g \end{pmatrix}, \begin{pmatrix} aI_n & bI_n \\ cI_n & dI_n \end{pmatrix}) \end{array}$$

$S_X = triv \in Rep(GL_n)$

§2.4. Horospherical cases

Suppose X horospherical. WLOG $X = G/S$ where $U \subset S$

$P := N_G(S)$

Fact [Knop] $\quad P^{der} \subset S \subset P$

Then $T^*(G/S) = G \overset{S}{\times} (\mathfrak{g}/\mathfrak{s})^\perp = G \overset{S}{\times} (\mathfrak{u}_P + \mathfrak{a}_0)$ for some $\mathfrak{a}_0 \subset \mathfrak{z}(L)$

$\mathbb{F}[T^*(G/S)]^G = \mathbb{F}[\mathfrak{u}_P + \mathfrak{a}_0]^S = \mathbb{F}[\mathfrak{a}_0] \quad \Rightarrow W_X = \{1\}$

$\Rightarrow \check{G}_X = \check{A}_X$

Rmk Actually, X horospherical $\Leftrightarrow W_X = \{1\}$ [Knop]

§2.5. Strongly tempered cases

Call cases that $\check{G}_X = \check{G}$ strongly tempered, examples include

- $X = GL_n \backslash GL_n \times GL_{n+1} \circlearrowleft G = GL_n \times GL_{n+1}$ $\quad S_X = Std_n \boxtimes Std_{n+1} \oplus Std_n^\vee \boxtimes Std_{n+1} \in Rep(GL_n \times GL_{n+1})$
- $X = SO_{2n} \backslash SO_{2n} \times SO_{2n+1} \circlearrowleft G = SO_{2n} \times SO_{2n+1}$ $\quad S_X = Std_{2n} \boxtimes Std_{2n} \in Rep(SO_{2n} \times Sp_{2n})$
  $\wr\ n = 3$
- $X = PGL_2 \backslash (PGL_2)^3 \circlearrowleft G = (PGL_2)^3$ $\quad S_X = Std^{\boxtimes 3} \in Rep((SL_2)^3)$

One can check $\check{G}_X = \check{G}$ by showing $T^*X \longrightarrow \mathfrak{a}^*/\!/W$ has generically connected fibers.

§3. $\check{G}_{X,GN}$ via Tannakian duality

When spherical X is affine smooth, Gaitsgory-Nadler constructs $\check{G}_{X,GN} \subset \check{G}$ via Tannakian duality

Conj $\check{G}_{X,GN} \simeq \mathrm{Im}(\check{G}_X \longrightarrow \check{G})$

Rmk When $B \curvearrowright X$ has generically connected stablizer, conjecture $\Leftrightarrow$ $\check{G}_{X,GN} = \check{G}_X$

Goal Construct

$$\begin{tikzcd} \mathrm{Rep}(\check{G}) \arrow[r] \arrow[dr] & \mathrm{Rep}(\check{G}_{X,GN}) \arrow[d] \\ & \mathrm{Rep}(\check{A}_0) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} P_{G(\mathcal{O})}(Gr_G) \arrow[r, "\mathrm{conv}", "①"'] \arrow[dr, "\mathrm{conv.}", "③"'] & Q(\mathcal{Z}) \arrow[d, "\Psi", "②"'] \\ & Q(\mathcal{Z}_0) \end{tikzcd}$$

idempotent completion

$Q(\mathcal{Z}) =$ " $\widehat{\mathrm{Im}}(\mathrm{Sat}_G \to \mathrm{Perv}(LX/L^+G))$ " $\simeq \mathrm{Rep}(\check{G}_{X,GN})$

$\downarrow$ nearby cycle. $\qquad \downarrow$

$Q(\mathcal{Z}_0) =$ " $\widehat{\mathrm{Im}}(\mathrm{Sat}_G \to \mathrm{Perv}(LX_0/L^+G)^{\heartsuit})$ " $\simeq \mathrm{Rep}(\check{A}_0)$

where X_0 is horospherical

$X \rightsquigarrow X_0$ degeneration

$(\check{A}_0 = \mathrm{Im}(\check{A}_X \longrightarrow \check{A}))$

- ①, ②, ③ are $\otimes$-functors
- ① generates $Q(\mathcal{Z})$ by direct summand
- ③ $\simeq \mathrm{Rep}(\check{G}) \xrightarrow{\mathrm{res}} \mathrm{Rep}(\check{A}_0)$

§3.1. Construction of $Q(\mathcal{Z})$

$Q(\mathcal{Z})$ will be actually independent of choice of C

$C =$ smooth proj. curve$/\mathbb{C}$, $X^\circ := G/S \subset X$ open G-orbit , $K = \mathbb{C}((t))$, $\mathcal{O}_K = \mathbb{C}[[t]]$

$I =$ finite set, ${}'\mathcal{Z}_I = \{ (c_I, \mathcal{P}_G, \sigma) \mid c_I \in C^I, \mathcal{P}_G \in \mathrm{Bun}_G, \sigma : C \setminus c_I \longrightarrow \mathcal{P}_G \overset{G}{\times} X$ s.t.

- $\exists\, C^\circ \subset C$ open, $\sigma|_{C^\circ} : C^\circ \longrightarrow \mathcal{P}_G \overset{G}{\times} X^\circ \longrightarrow \mathcal{P}_G \overset{G}{\times} X$
- $\sigma|_{C^\circ} \rightsquigarrow S$-bundle on $C^\circ \rightsquigarrow \pi_0(S)$-bundle on C°

The $\underbrace{\pi_0(S)}_{= \{e\} \text{ in case } X^*(A_X)_{\mathrm{sat}} = X^*(A_X)}$-bundle is generically trivial. $\}$

$$\begin{tikzcd} & {}'\mathcal{Z}_I \arrow[dl] \\ C^I & {}'\mathcal{Z}_I^\circ \arrow[u, phantom, "\cup"] \arrow[dl] \\ C^I \setminus \Delta \arrow[u, hook] \end{tikzcd}$$

When $|I| = 1$, write ${}'\mathcal{Z}_I = {}'\mathcal{Z}$

Recall $\mathring{X}(K)/G(\mathcal{O}_K) \simeq X_*(A_X)_+$

${}'\mathcal{Z}(\mathbb{C}) \xrightarrow[\text{to } \hat{D}_c]{\text{restrict}} X_*(A_X)_+ \rightsquigarrow$ stratification ${}'\mathcal{Z}^\theta \subset {}'\mathcal{Z}$ indexed by $\theta \in X_*(A_X)_+$

$\rightsquigarrow {}'IC^\theta \in P({}'\mathcal{Z})$

Define $P('Z) \supset P_{\mathcal{H}}('Z) \supset Q(Z)$ as follows

- $P('Z)$ = cat. of perverse sheaves

$\cup$

- $P_{\mathcal{H}}('Z) = \{ \mathcal{F} \in P('Z) \mid$ that are $\underline{\text{generically Hecke equivariant}} \} \ni {}'IC^\theta$

i.e. For every

$$\begin{tikzcd} & \mathcal{H}_J \arrow[dl, "\overleftarrow{p}"'] \arrow[dr, "\overrightarrow{p}"] & \\ 'Z_J & & 'Z_J \end{tikzcd}$$

= modification @ c_J disjoint from c

one has $\overleftarrow{p}^* \mathcal{F} \xrightarrow{\sim} \overrightarrow{p}^* \mathcal{F}$ with compatibilities

$\cup$

- $Q('Z)$ = gen. by direct summand of $\operatorname{Im}(\operatorname{Sat}_C \longrightarrow P_{\mathcal{H}}('Z))$

$$S_V \longmapsto S_V * {}'IC^0$$

$\underline{\text{Fact}}$ $Q('Z)$ is semisimple with simple objects a subset of $\{'IC^\theta\}_{\theta \in (X^*(A_X)_{sat})^\vee_+} \subset X_*(A_X)_+$

$\underline{\text{Conj.}}$ Equality holds

$\underline{\text{Rmk}}$ Constructions above are still valid when $|I| > 1$

§.3.2. $\otimes$-structure on $Q(Z)$

$$Q(Z)^{\otimes I} \xrightarrow{\sim} Q(Z_I^\circ) \xrightarrow{\text{fusion}} Q(Z)$$

by composing simple objects.

induce

$$\begin{tikzcd} Z_{I^\circ} \arrow[r, hook, "j"'] \arrow[d] & Z_I \arrow[d] & Z \arrow[l, hook', "i"'] \arrow[d] \\ C^{I^\circ} \arrow[r, hook] & C^I & C \arrow[l, hook', "\Delta"'] \end{tikzcd}$$

fusion induced by $i^* \circ j_{!*}[\cdot] : P_{\mathcal{H}}(Z) \longrightarrow P_{\mathcal{H}}(Z_I^\circ)$

§3.3. Fiber functor

When X horospherical, $Q(Z) \simeq \operatorname{Rep}(\check{A}_0)$ (by explicitly computing $S_V * IC^0$)

When X not horospherical, find degeneration

horospherical

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] & X \times \mathbb{G}_m \arrow[l] \arrow[d] \\ \{0\} \arrow[r, hook] & \mathbb{A}^1 & \mathbb{G}_m \arrow[l, hook'] \end{tikzcd}$$

Relative ver. of Z $\rightsquigarrow$ $\begin{matrix} \mathfrak{Z} \\ \downarrow \\ \mathbb{A}^1 \end{matrix}$ $\overset{\text{Nearby}}{\underset{\text{cycle}}{\rightsquigarrow}}$ $\Psi : P_{\mathcal{H}}(Z) \longrightarrow P_{\mathcal{H}}(Z_0)$

technical part

$\Psi : Q(Z) \longrightarrow Q(Z_0) \simeq \operatorname{Rep}(\check{A}_0)$