Hyperspherical Varieties

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Abstract

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1 Symplectic Geometry

First we review some important constructions and examples in symplectic geometry.

1.1 The Basics

Definition 1.1. Let N be a smooth manifold. The cotangent space of $T^*_{(x,\rho)}N$ at (x,ρ) is isomorphic to $T_xN \oplus T^*_xN$ ("the N direction and the T^*_xN direction"). The canonical 1-form θ on T^*N sends (ξ,ψ) to $\rho(\xi)$.

Lemma 1.2. *Let φ be a diffeomorphism of N. It induces a diffeomorphism $\tilde{\varphi}$ of T^*N and $\tilde{\varphi}^*\theta = \theta$.*

Proof. The definition of θ only involves the smooth structure of N. □

Definition 1.3. The **standard symplectic structure** on T^*N is given by $\omega = -d\theta$. Explicitly,

$$\omega((\xi_1,\psi_1),(\xi_2,\psi_2)) = \psi_2(\xi_1) - \psi_1(\xi_2).$$

Lemma 1.4. *Let G be a Lie group acting on N by diffeomorphisms. Then the induced action on T^*N acts by symplectomorphisms with respect to the standard symplectic structure.*

Proof. $g^*\omega = g^*d\theta = dg^*\theta = d\theta = \omega$ by Lemma 1.2. □

Definition 1.5. Let (M,ω) be a symplectic variety with an action of a Lie group G. From the map $\varphi : G \to \mathrm{Diff}(M)$ we can take the induced map on Lie algebras $\xi : \mathfrak{g} \to \Theta(M)$ (we write $\Theta(M)$ for the space of vector fields on M).

Lemma 1.6. $g_*\xi_X = \xi_{Ad_g X}$ *for any $X \in \mathfrak{g}$ and $g \in G$.*

Proof. The map on Lie algebras associated to a homomorphism of Lie groups is always compatible with adjoint actions, so $\xi_{Ad_g X} = Ad_{\varphi(g)}\xi_X$. But for any $\xi \in \Theta(M)$ and $\varphi \in \mathrm{Diff}(M)$

$$\begin{aligned}\varphi(1+\varepsilon\xi)\varphi^{-1}(x) &= \varphi(\varphi^{-1}(x) + \varepsilon\xi_{\varphi^{-1}(x)}) \\ &= x + \varepsilon d\varphi(\xi_{\varphi^{-1}(x)}) \\ &= x + \varepsilon(\varphi_*\xi)_x\end{aligned}$$

so $Ad_\varphi\xi = \varphi_*\xi$. From this we see that $\xi_{Ad_g X} = g_*\xi_X$. □

Definition 1.7. A **moment map** μ for the action of G is a smooth G-equivariant map $\mu : M \to \mathfrak{g}^*$ such that $\nabla_\omega \langle \mu(m), X\rangle = \xi_X$ for all $X \in \mathfrak{g}$ (here $\mathfrak{g}^*$ is considered as a G-space via the coadjoint G-action). We will also write μ_X for the function $\langle \mu(-), X\rangle$. With this notation, the equivariance of μ is equivalent to $\mu_X(g \cdot m) = \mu_{Ad_{g^{-1}}X}(m)$ for all $X \in \mathfrak{g}$, $g \in G$ and $m \in M$.

Example 1.8. Suppose G acts on X and let $M = T^*X$ with the standard symplectic structure. Then set $\mu_X = \theta(\xi_X)$. To see that this μ is G-equivariant, note that

$$\theta_{g\cdot m}(\xi_X) = \theta_{g\cdot m}(g_* g_*^{-1}\xi_X) = (g^*\theta)_m(g_*^{-1}\xi_X) = \theta_m(\xi_{Ad_{g^{-1}}X})$$

where we've used Lemma 1.2 and Lemma 1.6 to obtain the last equality. To see that this μ is a moment map for the induced action of G on M, note that

$$\omega(\nabla_\omega \theta(\xi_X), \zeta) = \zeta \cdot \theta(\xi_X) = [di_{\xi_X}\theta](\zeta) = [\mathcal{L}_{\xi_X}\theta - i_{\xi_X}d\theta](\zeta) = \omega(\xi_X, \zeta)$$

where we've used Cartan's magic formula plus Lemma 1.2.

Example 1.9. Let (V, ω) be a symplectic vector space. By definition, $Sp(V, \omega)$ acts on V by symplectomorphisms. This action admits a moment map $\mu : V \to \mathfrak{sp}^*$ with $\mu_X(v) = \frac{1}{2}\omega(Xv, v)$. To see that μ is G-equivariant, note that

$$\frac{1}{2}\omega(X \cdot (g \cdot v), g \cdot v) = \frac{1}{2}\omega((g^{-1}Xg)v, v) = \frac{1}{2}\omega(Ad_{g^{-1}}Xv, v)$$

To check that μ is a moment map, note that

$$\begin{aligned} \omega(\nabla_\omega \mu_X(v), \zeta) &= \zeta \cdot \frac{1}{2}\omega(Xv, v) \\ &= \frac{1}{2}\omega(X\zeta, v) + \frac{1}{2}\omega(Xv, \zeta) \\ &= \omega(Xv, \zeta) \end{aligned}$$

where we've used the fact that X is a symplectomorphism to say $\omega(X\zeta, v) = \omega(Xv, \zeta)$. Since $(\xi_X)_v = Xv$ this is what we needed to show.

Definition 1.10. A symplectic variety with an action of G that admits a moment map is called a **Hamiltonian G-space**.

Example 1.11. If G acts on M by symplectomorphisms and we have $\varphi : H \to G$ then restricting along H gives an action of H on M by symplectomorphisms. Moreover, if the action of G on M admits a moment map μ then $\nu = d\varphi^* \circ \mu$ is a moment map for the action of H on M. This procedure for producing a Hamiltonian H-space from a Hamiltonian G-space is called **Hamiltonian Restriction**.

1.2 Reduction and Induction

In this section we introduce two procedures for producing to produce a new variety from a Hamiltonian G-space: Hamiltonian reduction and Hamiltonian induction. Hamiltonian reduction is a standard construction in symplectic geometry, and Hamiltonian induction is a generalization of Hamiltonian reduction that will play an essential role in the sequel. We follow [1, Sections 3.2 and 3.3].

Definition 1.12. Let M be a Hamiltonian G-space. Then the **Hamiltonian reduction** of M is written $M /\!\!/\!\!/ G$ and equals $(M \times_{\mathfrak{g}^*} 0)/G$. More generally, we can consider $M /\!\!/\!\!/_f G = (M \times_{\mathfrak{g}^*} \mathcal{O}_f)/G$ for any $f \in \mathfrak{g}^*$ (here $\mathcal{O}_f$ is the coadjoint orbit of f).

Example 1.13. Suppose G acts freely on X. The following diagram is cartesian

$$\begin{array}{ccc} T^*X \times_{\mathfrak{g}^*} 0 & \longrightarrow & T^*(X/G) \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & X/G \end{array}$$

where T^*X maps to $\mathfrak{g}^*$ by the standard moment map; that is to say, a 1-form on X is pulled back from a 1-form on X/G if and only if it vanishes on ξ_X for all $X \in \mathfrak{g}$ (recall that the standard moment map on T^*X is $\mu_X = \theta(\xi_X)$). This shows that $T^*(X/G) = (T^*X) /\!\!/\!\!/ G$.

Example 1.14. Let $\Psi \to X$ be a $\mathbf{G}_a$ bundle. Then the **twisted cotangent bundle** of X, written $T^*(X, \Psi)$ or $T^*_\Psi X$, is $T^*\Psi /\!\!/\!\!/_1 \mathbf{G}_a$. Note that by the above example $T^*\Psi /\!\!/\!\!/ \mathbf{G}_a = T^*X$. Moreover, for any M acted on by G, if we consider T^*M with the standard structure of a Hamiltonian G-space, the moment map is a homomorphism of relative groups over X. In other words, the following diagram commutes:

$$\begin{array}{ccc} T^*M \times_X T^*M & \xrightarrow{+} & T^*M \\ {\scriptstyle \mu\times\mu}\downarrow & & {\scriptstyle \mu}\downarrow \\ \mathfrak{g}^* \times \mathfrak{g}^* & \xrightarrow{+} & \mathfrak{g}^* \end{array}$$

As a result, we get a map

$$(T^*\Psi \times_{\mathfrak{g}^*} 0) \times_\Psi (T^*\Psi \times_{\mathfrak{g}^*} 1) \to (T^*\Psi \times_{\mathfrak{g}^*} 1)\,.$$

Taking the quotient of both sides by $\mathbf{G}_a$ then produces an action relative to X of the group scheme T^*X on $T^*_\Psi X$, which makes $T^*_\Psi X$ into a torsor over T^*X.

Example 1.15. Given a Hamiltonian G-space M and an element $\alpha \in \mathfrak{g}^*$ which is invariant under the coadjoint action of G we can form a new Hamiltonian G space M_α, where the G-action is the same but the moment map is $\mu_{M_\alpha}(m) = \mu_M(m) + \alpha$. Then $M /\!\!/\!\!/_\alpha G = M_\alpha /\!\!/\!\!/ G$.-

Definition 1.16. Given a map of groups $H \to G$ and a Hamiltonian H-variety S, we can perform **Hamiltonian induction** to produce a Hamiltonian G-variety $\text{h-ind}_H^G(S) = (S \times T^*G) /\!\!/\!\!/ H$.

Example 1.17. For any H we can consider the map to the trivial group $H \to 1$. Then

$$\text{h-ind}_H^1(S) = S /\!\!/\!\!/ H$$

so we see that Hamiltonian restriction is a special case of Hamiltonian induction.

Lemma 1.18. $(S \times T^*G) /\!\!/\!\!/ H = (S \times_{\mathfrak{h}^*} \mathfrak{g}^*) \times^H G$.

Lemma 1.19. *Hamiltonian induction is right-adjoint to Hamiltonian restriction in the category of Lagrangian correspondences.*

Definition 1.20. The **symplectic normal bundle** to a G-orbit O in a symplectic manifold M is a vector bundle over O whose fiber at x is $T_xO^\perp/(T_xO^\perp \cap T_xO)$.

Example 1.21. If we consider the G-orbit $H\backslash G = 0 \times^H G$ inside of $\text{h-ind}_H^G S$ then its fiber at any point is given by S.

1.3 Graded Hamiltonian G-spaces

TODO: Write something

2 Geometric Invariant Theory

In this section we work with affine varieties defined over an *algebraically closed* field *not necessarily of characteristic zero*. Although there is a running assumption in most of [1, Section 3] that they work over a field of characteristic zero, many (though not all!) of the arguments generalize to a field of arbitrary characteristic (there is a discussion rationality questions in [1, Section 3.9]). The positive characteristic setting is especially important for number-theoretic applications, so we found it reasonable to consider it here.

2.1 GIT Quotients

We follow [5, Appendix 1.C] for our material on GIT quotients.

Definition 2.1. An algebraic group G is geometrically reductive if for any finite dimensional G-representation V and any fixed vector $v \in V^G$, there is some k and some $f \in (\mathrm{Sym}^k V^*)^G$ such that $f(v) \neq 0$.

Theorem 2.2 (Haboush)**.** *Every reductive group is geometrically reductive. In fact, one can always take k to be a pth-power in the definition of geometrically reductive.*

Proof. See [5, Theorem A.1.0]. □

Lemma 2.3. *Let G be geometetrically reductive. If A is a (commutative) ring and I is a G-invariant ideal in A then for any $f \in (A/I)^G$ there is some n and some function $g \in A^G$ so that the image of g in $(A/I)^G = f^{p^n}$.*

Proof. Because f is G-invariant, it spans a 1-dimensional subrepresentation of A/I. Because I is G-invariant, we can lift this to a representation $I + k \cdot f$ inside of A. Let V be a finite-dimensional subrepresentation of $I + f \cdot f$ containing f (representations of affine algebraic groups are always locally finite so such a V must exist). Then we have a short exact sequence of G-representations

$$0 \to V \cap I \to V \to k \cdot f \to 0.$$

From this we see that in $V^\vee$, the function θ which sends f to 1 and sends $V \cap I$ to zero is G-invariant. Because G is geometrically reductive, there is some $g \in (\mathrm{Sym}^{p^n} V)^G$ such that $g(\theta) = 1$. Since the image of $\mathrm{Sym}^{p^n} V$ in A lies in $k \cdot f^{p^n} + I$, the condition $g(\theta) = 1$ forces $g = f^{p^n} \bmod I$. It is also clear that the image of g lies in A^G, so we're done. □

Definition 2.4. If $X = \mathrm{Spec}\, A$ is an affine variety over k acted on by G then the **GIT quotient** $X /\!\!/ G$ is $\mathrm{Spec}\, A^G$.

Theorem 2.5 (Nagata)**.** *If G is geometrically reductive then $X /\!\!/ G$ is an affine variety.*

Proof. The main difficulty is verifying that $X /\!\!/ G$ is finite type over k. First, note that because A is finitely generated (and the action of G on A is locally finite) we can find some finite-dimensional G-subrepresentation V containing a set of generators of A. Then $\mathrm{Sym}^* V \to A$ is a G-equivariant surjective map, so it actually suffices to prove the following: let B be a finitely-generated, non-negatively graded ring such that $B^0 = k$ and the action of G on B preserves the grading. Then

1. B^G is finitely generated.
2. If M is a Noetherian B-module with a compatible G-action, then M^G is Noetherian over B^G.

If we take these statements for granted, then we're done: A is a quotient of $B = \mathrm{Sym}^* V$, so A^G is Noetherian over B^G. Since B^G is finitely generated over k, this implies that A^G is finitely generated, too.

To prove 1, we can use Noetherian induction to reduce to the case when B^G is an integral domain and B is torsion-free as a B^G module (i.e., every $f \in B^G$ is a non-zero divisor in B). Now we procede by induction on dimension: if every homogeneous element of B^G is a unit then $B = B^0 = k$ and we're done. Otherwise let $f \in B^G$ be a homogeneous non-unit; because f is a non-zero divisor in B we know that B/fB is codimension 1 in B, so by induction $(B/fB)^G$ is finitely generated. By Lemma 2.3 we know that $[(B/fB)^G]^{p^n} \subseteq B^G/(fB)^G \subseteq (B/fB)^G$ for some n, so it follows that $B^G/(fB)^G$ is finitely generated too. By [8, Tag 07Z4], to check that B^G is finitely generated as a ring, it suffices to check that its ideal I^+ of positively-graded elements is finitely generated as an ideal. Since f is a G-invariant non-zero divisor, $(fB)^G = f(B^G)$, so I^+ is generated by f plus lifts of the generators of I^+ in $B^G/(fB)^G$.

To prove 2, we can apply a devissage argument to reduce to the the case when M is of the form $B/\mathfrak{p}$ for a G-invariant prime ideal $\mathfrak{p}$. Let K be the field of fractions of $B/\mathfrak{p}$, let K_0 be the field of fractions of $B^G/\mathfrak{p}^G$, and let L be the field of fractions of $(B/\mathfrak{p})^G$. Because B is finitely generated over k, there is a finite transcendence basis $t_1, \ldots t_n$ so that K is finite over $K_0(t_1, \ldots t_n)$. By Lemma 2.3, L is algebraic over K_0, so $L \cap K_0(t_1, \ldots, t_n) = K_0$. We then see that

$$[L : K] = [LK_0(t_1, \ldots, t_n) : K_0(t_1, \ldots, t_n)] \leq [K : K_0]$$

is a finite field extension. By 1, we know that B^G is finitely generated over a field, so the same is true of $B^G/\mathfrak{p}^G$, so its integral closure in L is finite over $B^G/\mathfrak{p}^G$. Lemma 2.3 implies that $(B/\mathfrak{p})^G$ is contained in this integral closure, so because $B^G/\mathfrak{p}^G$ is Noetherian we see that it must be finitely generated as a B^G-module.

Additional details can be found in [5, Theorem A.1.1]. □

Lemma 2.6. *Let I_α be G-stable ideals in A. If $f \in \left(\sum_\alpha I_\alpha\right)^G$ then $f^{p^n} \in \sum_\alpha I_\alpha^G$ for some n (which depends on f).*

Proof. Any given f is contained in a finite sum $I_1 + \ldots + I_k$, and by induction we can reduce to the case where $k = 2$. Suppose $f = F_1 + F_2$ where $F_1 \in I_1$ and $F_2 \in I_2$. Then mod $I_1 \cap I_2$ this decomposition is unique, so because f is G-invariant, F_1 and F_2 must both be G-invariant mod $I_1 \cap I_2$. Applying Lemma 2.3 we can find some g_1 and g_2 in A^G so that $g_i = F_i^{p^n}$ mod $I_1 \cap I_2$. Note that this automatically implies that $g_i \in I_i^G$. Now

$$f^{p^n} = (F_1 + F_2)^{p^n} = F_1^{p^n} + F_2^{p^n} = g_1 + g_2 \text{ mod } I_1 \cap I_2.$$

In other words, $f^{p^n} = g_1 + g_2 + r$, where $r \in I_1 \cap I_2$. Since f^{p^n}, g_1, and g_2 are all G-invariant, r must be too, so $f^{p^n} = (g_1 + r) + g_2$ is in $I_1^G + I_2^G$. □

Lemma 2.7. *If $X = \operatorname{Spec} A$ is an affine variety then the natural projection $\pi : X \to X /\!\!/ G$ is surjective.*

Proof. First suppose that A is an integral domain. Then for any maximal ideal $\mathfrak{m} \subseteq A^G$ if $A\mathfrak{m} = A$ then there are some $m_1, \ldots, m_k$ such that $1 \in \sum_i Am_i$. Lemma 2.6 implies $1^{p^n} = 1 \in \sum_i (Am_i)^G$, and since A is integral we know $(Am_i)^G = A^G m_i$. But this implies $1 \in \mathfrak{m}$, which is a contradiction.

Now if A is an arbitrary finitely generated k-algebra, we can form the primary decomposition of the zero ideal: $0 = \cap_i \mathfrak{q}_i$. But then $0 = \cap_i (\mathfrak{q}_i \cap A^G)$, and since $\mathfrak{m}$ is prime this implies that some $\mathfrak{q}_i \cap A^G$ is contained in $\mathfrak{m}$. If $\mathfrak{p}_i$ is the radical of $\mathfrak{q}_i$ then $\mathfrak{p}_i \cap A^G$ is the radical of $\mathfrak{q}_i \cap A^G$, so $\mathfrak{p}_i \cap A^G$ is contained in $\mathfrak{m}$. Then $A/\mathfrak{p}_i$ is an integral domain and the image of $\mathfrak{m}$ in $A^G/(\mathfrak{p}_i \cap A^G)$ is a maximal ideal, so its extension to $A/\mathfrak{p}_i$ is proper in $A/\mathfrak{p}_i$ by the above. This implies that $A\mathfrak{m}$ is proper, too. □

Lemma 2.8. *If $X = \operatorname{Spec} A$ is an affine variety then the points of $X /\!\!/ G$ are in bijection with the closed orbits of G in X.*

Proof. First note that if $f \in A^G$ and $\mathcal{O}, \mathcal{O}'$ are two G-orbits of X with $\mathcal{O} \subseteq \overline{\mathcal{O}'}$ then $f|_{\mathcal{O}} = 0$ if and only if $f|_{\mathcal{O}'} = 0$. This follows because the value of f when restricted to any G-orbit is constant, and if f is constant on a set, then it must also be constant on its closure. Let's say that a G-stable closed subscheme $Z \subseteq X$ is "replete" if for any $\mathcal{O}$ and $\mathcal{O}'$ as above, $\mathcal{O} \subseteq Z$ implies $\mathcal{O}' \subseteq Z$. Evidently, the preimage of any closed subscheme of $X /\!\!/ G$ under π must be replete.

Now note that every nonempty replete subscheme Z contains a closed orbit: since Z is G-stable and non-empty it must contain some G-orbit, and since Z is closed it must contain the closure of this orbit. But the closure of any G-orbit contains a closed orbit (any orbit with minimal dimension in the closure works). Since Lemma 2.7 implies that π is surjective we see that the preimage of any point of $X /\!\!/ G$ contains a closed orbit.

Now we claim that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are distinct closed orbits in X then their images in $X /\!\!/ G$ must be disjoint. If not, let $x \in X /\!\!/ G$ be a point in the intersection, with corresponding maximal ideal $\mathfrak{m}$. Then there is a G-invariant function f defined on $\mathcal{O}_1 \sqcup \mathcal{O}_2$ so that $f|_{\mathcal{O}_1} = 0$ and $f|_{\mathcal{O}_2} = 1$. Lemma 2.3 tells us that there is some $g \in A^G$ whose restriction to $\mathcal{O}_1 \sqcup \mathcal{O}_2$ is $f^{p^n} = f$. Because $g|\mathcal{O}_2 \neq 0$ we see that $g \notin \mathfrak{m}$, but because $g|_{\mathcal{O}_1} = 0$ we see that $\mathfrak{m} + (g) \neq A^G$. This contradicts the maximality of $\mathfrak{m}$, so we conclude that $\pi(\mathcal{O}_1) \cap \pi(\mathcal{O}_2) = \emptyset$.

Now we see that there is a map from points of $X /\!\!/ G$ to the closed orbits of X which sends each point to the unique closed G-orbit in its preimage under π. The preimages of distinct points are disjoint, so this map is injective, and if $\mathcal{O}$ is a closed orbit then the preimage of any $y \in \pi(\mathcal{O})$ contains $\mathcal{O}$ (it must contain some closed orbit $\mathcal{O}'$, and if $\mathcal{O}' \neq \mathcal{O}$ then $\pi(\mathcal{O}) \cap \pi(\mathcal{O}')$ would be nonempty). $\square$

2.2 G_m actions

Although the results of this section are originally due to Bialynicki-Birula, we adopt the perspective introduced by Drinfeld in [3].

Definition 2.9. Let Z be a k-scheme with an action of $\mathbf{G}_m$. Then the **attracting locus** of Z is the space Z^+ which represents the functor

$$\mathrm{Hom}(S, Z^+) = \mathrm{Hom}^{\mathbf{G}_m}(S \times \mathbf{A}^1, Z)$$

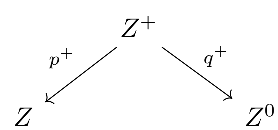
where $\mathbf{G}_m$ acts trivially on S and by the usual action on $\mathbf{A}^1$.

Definition 2.10. Let Z be a k-scheme with an action of $\mathbf{G}_m$. Then the **fixed locus** of Z is the space Z^0 which represents the functor

$$\mathrm{Hom}(S, Z^0) = \mathrm{Hom}^{\mathbf{G}_m}(S, Z)$$

where $\mathbf{A}^1$ acts trivially on S.

Theorem 2.11 (Bialynicki-Birula)**.** *Let Z be a k-scheme with an action of $\mathbf{G}_m$. Then Z^+ and Z^0 are both representable. The inclusion of Z^0 into Z is a closed embedding. In the diagram below, q^+ is affine and p^+ is a locally closed embedding on each connected component of Z^+.*



Proof. We'll prove this result under the assumption that Z is affine, since this is the case that will be relevant for us later.

Let I^- be the ideal generated by negatively graded homogeneous elements and let $A^+ = A/I^-$. Then we claim that $Z^+ = \mathrm{Spec}\, A^+$. To see this, note that if $S = \mathrm{Spec}\, B$ is affine then

$$\begin{aligned}\mathrm{Hom}^{\mathbf{G}_m}(S \times \mathbf{A}^1, Z) &= \mathrm{Hom}_{grRing}(A, B \otimes_k k[x]) \\ &= \mathrm{Hom}_{grRing^+}(A^+, B \otimes_k k[x]) \\ &= \mathrm{Hom}_{Ring}(A^+, B),\end{aligned}$$

where we've used the fact that every element of $B \otimes_k k[x]$ is non-negatively graded so every graded map from A to $B \otimes_k k[x]$ factors through A^+ and the functor $B \mapsto B \otimes_k k[x]$ is right-adjoint to the forgetful functor from non-negatively graded rings to rings.

By a similar argument, if we let I^0 be the ideal generated by the homogeneous elements with nonzero graded degree and set $A^0 = A/I^0$ then $\mathrm{Spec}\, A^0$ represents Z^0. From our constructions of Z^+ and Z^0 we see that we get closed embeddings $i^+ : Z^0 \to Z^+$ and $p^+ : Z^+ \to Z$ (so in particular p^+ is not just locally closed but a closed embedding).

To construct q^+, note that $A^+ = (A^+)^{\mathbf{G}_m} \oplus I^+$ as a k-vector space (where I^+ is the ideal generated by positively-graded elements). The natural projection from A^+ to A^0 then induces an isomorphism from $(A^+)^{\mathbf{G}_m}$ to A^0. The inverse of this map gives an inclusion $A^0 \hookrightarrow A^+$, which induces $q^+ : \mathrm{Spec}\, A^+ \to \mathrm{Spec}\, A^0$. From the construction of q^+ we see that $q^+ \circ i^+ = id_{Z_0}$. It's routine to check that the map q^+ that we've defined agrees with the map from Z^+ to Z^0 induced by the $\mathbf{G}_m$-equivariant inclusion of 0 into $\mathbf{A}^1$. $\square$

Lemma 2.12. *Let Z be a k-scheme with an action of $\mathbf{G}_m$. If x is a k-point of Z^0 then $T_x(Z^+)$ is the subspace of $T_x Z$ given by the sum of the non-negative weight spaces.*

Proof. If we let $X = \operatorname{Spec} k[\varepsilon]/\varepsilon^2$, note that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{G}_m - \operatorname{Sch}/k & \xrightarrow{-\times X} & \mathbf{G}_m - \operatorname{Sch}/k \\ \uparrow{\scriptstyle -\times \mathbf{A}^1} & & \uparrow{\scriptstyle -\times \mathbf{A}^1} \\ \operatorname{Sch}/k & \xrightarrow[-\times X]{} & \operatorname{Sch}/k \end{array}$$

Passing to right-adjoints gives $(TZ)^+ = T(Z^+)$ for all $Z \in \mathbf{G}_m - \operatorname{Sch}/k$. If x is a k-point of $\mathbf{G}_m$, then the inclusion of x into Z is $\mathbf{G}_m$-equivariant, so we get $T_x Z = x \times_Z TZ$ as a fiber product in $\mathbf{G}_m - \operatorname{Sch}/k$. Since $(-)^+$ is right-adjoint, it commutes with limits and we see that

$$(T_x Z)^+ = x \times_{Z^+} (TZ)^+ = x \times_{Z^+} T(Z^+) = T_x(Z^+).$$

Our claim then reduces to showing that for a finite-dimensional $\mathbf{G}_m$-representation V with associated $\mathbf{G}_m$-vector scheme $\underline{V}$ we have $(\underline{V})^+ = \underline{V^{\geq 0}}$, where $V^{\geq 0}$ is the subrepresentation generated by the non-negative weight spaces. But $\underline{V} = \operatorname{Spec} \operatorname{Sym} V^*$, so by our explicit construction for affine schemes in Theorem 2.11 we have $(\underline{V})^+ = \operatorname{Spec} \operatorname{Sym} V^*/I^-$. Since $V = V^{\geq 0} \oplus V^{<0}$ we see that

$$\operatorname{Sym} V^* = \operatorname{Sym}(V^{\geq 0})^* \otimes \operatorname{Sym}(V^{<0})^*$$

and $\operatorname{Sym} V^*/I^- = \operatorname{Sym}(V^{\geq 0})^*$ as desired. □

Definition 2.13. By the universal property characterizing Z^+, we know that the idenity map of Z^+ induces a $\mathbf{G}_m$-equivariant map $\alpha : \mathbf{A}^1 \times Z^+ \to Z^+$, where $\mathbf{G}_m$ only acts on $\mathbf{A}^1$ in the source. The following is a commutative diagram of $\mathbf{G}_m$-schemes:

$$\begin{array}{ccccc} 0 \times Z^+ & \longrightarrow & \mathbf{A}^1 \times Z^+ & \longleftarrow & T(\mathbf{A}^1 \times Z^+) \\ \downarrow{\scriptstyle q^+} & & \downarrow{\scriptstyle \alpha} & & \downarrow \\ Z^0 & \xrightarrow[i^+]{} & Z^+ & \longleftarrow & T(Z^+) \end{array}$$

It induces a $\mathbf{G}_m$-equivariant map on fiber products $d\alpha : T_0\mathbf{A}^1 \times TZ^+ \to T_{Z^0}Z^+$. Identifying $T_0\mathbf{A}^1$ with k, we get the tangent vector $\xi \in T_0\mathbf{A}^1$ corresponding to $1 \in k$. We can then define a map $\Lambda : Z^+ \to T_{Z^0}Z^+$ as $d\alpha \circ (\xi, s_0)$, where s_0 is the zero section $Z^+ \to TZ^+$.

Lemma 2.14. *Let $\pi : T_{Z^0}Z^+ \to Z^0$ be the projection and let Λ be as in Definition 2.13. Then $\pi \circ \Lambda = q^+$.*

Proof. If $\pi' : T_0\mathbf{A}^1 \times TZ^+ \to 0 \times Z^+$ is the projection map then by construction $\pi \circ d\alpha = q^+ \circ \pi'$. Since $\pi' \circ (\xi, s_0) = id_{Z^+}$ the result follows. □

Lemma 2.15. *The image of Λ lies inside the weight-one subbundle of $T_{Z^0}Z^+$.*

Proof. Since $d\alpha$ is linear in tangent directions and $\mathbf{G}_m$ equivariant, where $\mathbf{G}_m$ only acts on the $\mathbf{A}^1$ factor in the source,

$$\lambda \cdot \Lambda(z) = d\alpha(\lambda\xi, s_0(z)) = \lambda d\alpha(\xi, s_0(z)) = \lambda\Lambda(z).$$

□

2.2.1 A filtration on Z^+

The results of this section seem to be new, but the main ideas of the arguments are largely borrowed from [3]. Conversations with Sanath Devalapurkar were helpful for formulating Definition 2.19.

Let $\mathbf{A}^1_n$ be the n-th formal neighborhood of zero (i.e. $\operatorname{Spec} k[\varepsilon]/\varepsilon^{n+1}$). This a closed $\mathbf{G}_m$-subscheme of $\mathbf{A}^1$ with the standard $\mathbf{G}_m$ action. The following definition appears in [3, Subsection 4.2].

Definition 2.16. Write Z^+_n for the mapping space $\operatorname{Hom}^{\mathbf{G}_m}(\mathbf{A}^1_n, Z)$. In other words, $\operatorname{Hom}(S, Z^+_n) = \operatorname{Hom}^{\mathbf{G}_m}(\mathbf{A}^1_n \times S, Z)$.

Remark 2.17. I do not know whether Z_n^+ is representable by a scheme if Z is a scheme. It seems unlikely to me.

Proposition 2.18. *Let Z be a scheme smooth over k. Then the restriction map $Z^+ \to Z_n^+$ is formally smooth.*

Proof. This follows from "standard arguments", as in [3, Proposition 1.4.20]. To elaborate: we want to show that given any diagram as below with $\overline{S} \to S$ a square-zero extension, we can find a dashed arrow making the diagram commute.

$$\begin{array}{ccc} \overline{S} & \longrightarrow & Z^+ \\ \big\downarrow & \nearrow & \big\downarrow \\ S & \longrightarrow & Z_n^+ \end{array}$$

In other words, given $\mathbf{G}_m$-equivariant maps $\overline{f} : \overline{S} \times \mathbf{A}^1 \to Z$ and $g : S \times \mathbf{A}_n^1 \to Z$ we want to find a $\mathbf{G}_m$ equivariant arrow $f : S \times \mathbf{A}^1 \to Z$ as in the diagram below.

$$\begin{array}{ccccc} \overline{S} \times \mathbf{A}_n^1 & \longrightarrow & \overline{S} \times \mathbf{A}^1 & & \\ \big\downarrow & & \big\downarrow & \searrow^{\overline{f}} & \\ S \times \mathbf{A}_n^1 & \longrightarrow & S \times \mathbf{A}^1 & \overset{f}{\dashrightarrow} & Z \\ & \searrow_{g} & & & \nearrow \end{array}$$

In other words, writing $X = \overline{S} \times A^1 \sqcup_{\overline{S} \times \mathbf{A}_n^1} S \times A_n^1$, we want to show that the restriction map

$$\operatorname{Maps}^{\mathbf{G}_m}(S \times \mathbf{A}^1, Z) \to \operatorname{Maps}^{\mathbf{G}_m}(X, Z)$$

is surjective. Since the map $X \to S \times \mathbf{A}^1$ is a square-zero extension and Z is smooth, we know that

$$\operatorname{Maps}(S \times \mathbf{A}^1, Z) \to \operatorname{Maps}(X, Z)$$

is surjective, and the fiber over $\overline{g}$ in $\operatorname{Maps}(X, Z)$ is a torsor for $H^0(X; \overline{g}^* \Theta_Z \otimes \mathcal{J}_X)$, where $\mathcal{J}_X$ is the ideal defining X in $S \times \mathbf{A}^1$ (considered as a sheaf over X because the extension is square-zero). Because $\mathbf{G}_m$ is linearly reductive (true in any characteristic!) we can apply Corollary A.8 to see that

$$\operatorname{Maps}^{\mathbf{G}_m}(S \times \mathbf{A}^1, Z) \to \operatorname{Maps}^{\mathbf{G}_m}(X, Z)$$

is surjective, too. $\square$

Definition 2.19. Given a $\mathbf{G}_m$-scheme Z, define $Z^{0,\geq k}$ to be the fiber product

$$\begin{array}{ccc} Z^{0,\geq k} & \longrightarrow & Z^+ \\ \big\downarrow & \lrcorner & \big\downarrow \\ Z^0 & \longrightarrow & Z_{k-1}^+ \end{array}$$

Proposition 2.20. *Let Z be affine. Then the map $Z^{0,\geq k} \to Z^+$ is a closed embedding.*

Proof. The map $Z^0 \to Z_{k-1}^+$ is a split monomorphism, so the map $Z^{0,\geq k} \to Z^+$ must also be a monomorphism. If $S = \operatorname{Spec} B$ and $Z = \operatorname{Spec} A$, then an S-point of Z^+ is a map

$$\varphi \in \operatorname{Hom}_{grRing}(A, B \otimes k[x]) = \operatorname{Hom}_{grRing}(A^+, B \otimes k[x]).$$

Let's write $\varphi(a) = \sum_n \varphi_n(a) x^n$. The S-point lies in $Z^{0,\geq k}(S)$ when the reduction mod x^k of φ is concentrated in degree zero. In other words, $\varphi_i = 0$ for $0 < i < k$. But maps φ satisfying this condition are exactly those that factor through A^+/I_k, where I_k is the ideal generated by homogeneous elements of A^+ concentrated in degrees d with $0 < d < k$. $\square$

Remark 2.21. The assumption that Z is affine in the above proposition can probably be removed by analyzing jet schemes.

Corollary 2.22. *Let Z be a smooth affine $\mathbf{G}_m$-scheme. Then $Z^{0,\geq k}$ is smooth over Z^0.*

There is an alternative description of $Z^{0,\geq k}$ as an equivariant mapping space which will be useful.

Definition 2.23. Let C_k be the pushout $\mathbf{A}^1 \sqcup_{\mathbf{A}^1_{k-1}} *$. By Proposition B.10 we see that C_k is a $\mathbf{G}_m$-scheme. Explicitly, $C_k = \operatorname{Spec} k[t^k, t^{k+1}, t^{k+2}, \ldots]$.

Example 2.24. C_1 is $\mathbf{A}^1$. C_2 is the cuspidal cubic $\operatorname{Spec} k[t^2, t^3]$.

Proposition 2.25. $Z^{0,\geq k} = \operatorname{Maps}^{\mathbf{G}_m}(C_k, Z)$.

Proof. This follows from Corollary B.11. ☐

Proposition 2.26. *Suppose $X \to Y$ is a $\mathbf{G}_m$-equivariant closed embedding. Then*

$$X^{0,\geq k} = X \times_Y Y^{0,\geq k}.$$

Here $X^{0,\geq k}$ maps to X via the composite of the map $X^{0,\geq k} \to X^+$ and $p^+ : X^+ \to X$. Similarly,

$$X^+ = X \times_Y Y^+$$

and

$$X^0 = X \times_Y Y^0$$

Proof. Note that $X = \operatorname{Maps}^{\mathbf{G}_m}(\mathbf{G}_m, X)$ and the map $X^{0,\geq k} \to X$ is induced from the map $\mathbf{G}_m \to C_k$. This map is universally schematically dominant as a map over the base field k, so the result follows from Corollary B.6. The statements about X^+ and X^0 follow similarly by considering the universally schematically dominant maps $\mathbf{G}_m \to \mathbf{A}^1$ and $\mathbf{G}_m \to *$, respectively. ☐

2.2.2 Linear Algebraic Groups with $\mathbf{G}_m$ Actions

The results in this section are also new, but they're mostly an extension of the ideas in [2, Section 2.1]. To elaborate: there is substantial overlap in the results of our Proposition 2.32 and [2, Proposition 2.1.8] (in *loc. cit.* they assume that the $\mathbf{G}_m$ action on G is "inner", but they work over a general ring while we only work over fields). In *loc. cit.* they also prove the result "by hand", while we appeal to the general theory of Bialynicki-Birula decompositions (the conceptual contents of the proofs are basically identical). What's new is that we then go on to extend the results of Proposition 2.32 to the subgroups $G^{\geq k}$, which we define using the ideas from the previous section.

Definition 2.27. A **$\mathbf{G}_m$-algebraic group** is a group object in the category of affine schemes of finite type over k.

Proposition 2.28. *If G is a $\mathbf{G}_m$-algebraic group, then G^0, G^+, and $G^{0,\geq k}$ are all linear algebraic groups. If G is smooth then these groups are all smooth.*

Proof. The functors $Z \mapsto Z^0$ and $Z \mapsto Z^+$ are both right adjoints, so they send group objects to group objects. The functor $Z \mapsto Z^{0,\geq k}$ can be expressed as a limit of right adjoints, so it also preserves group objects. When G is smooth, the smoothness of G^0 and G^+ is shown in [3, Proposition 1.4.20]. Then the smoothness of $G^{0,\geq k}$ follows from Corollary 2.22. ☐

Definition 2.29. We say that a $\mathbf{G}_m$-scheme Z is **attractive** if $Z = Z^+$.

Note that the full subcategory of attractive $\mathbf{G}_m$-schemes is closed under the formation of products.

Lemma 2.30. *The functor $Z \mapsto Z /\!/ \mathbf{G}_m$ preserves products when restricted to attractive affine schemes.*

Proof. For any graded rings A and B, we have an inclusion $A^0 \otimes B^0 \to (A \otimes B)^0$. When A and B are both non-negatively graded, this map is an isomorphism. ☐

Definition 2.31. Let G be a $\mathbf{G}_m$-algebraic group. Define G^{++} to be the kernel of the map $q^+ : G^+ \to G^0$ (note that from the construction in Theorem 2.11 q^+ is the map from G^+ to $G^+ /\!\!/ \mathbf{G}_m$).

Proposition 2.32. *Let G be an attractive linear algebraic group. Then there is a canonical decomposition*

$$G \cong G^{++} \rtimes G^0$$

G^{++} is connected, attractive, and $(G^{++})^0 = *$. If G is smooth then G^{++} and G^0 are both smooth.

Proof. The inclusion $G^+ + \to G$ is a closed embedding by definition, so by Proposition 2.26 we see that

$$(G^{++})^+ = G^{++} \times_G G^+ = G^{++}$$

since by assumption $G = G^+$. This shows that G^{++} is attractive. The lower square in the diagram below is a pullback by definition, and the upper square is a pullback by Proposition 2.26. The composite right vertical arrow is an isomorphism, so $(G^+ +)^0 = *$.

$$\begin{array}{ccc} (G^{++})^0 & \longrightarrow & G^0 \\ \downarrow & \lrcorner & \downarrow \\ G^{++} & \longrightarrow & G \\ \downarrow & \lrcorner & \downarrow{\scriptstyle q^+} \\ * & \longrightarrow & G^0 \end{array}$$

Now we know that $G^{++} /\!\!/ \mathbf{G}_m = (G^{++})^0 = *$, so by Lemma 2.8 we see that every $\mathbf{G}_m$-orbit of G^{++} contains the identity in its closure. This shows that G^{++} is connected. The fact that $G = G^{++} \rtimes G^0$ follows from the fact that $G^{++} \times_G G^0 = *$ as in the diagram above. The smoothness of G^0 follows from [3, Proposition 1.4.20]. The smoothness of G^{++} follows from the fact that q^+ is smooth by base-change; the smoothness of q^+ is also shown in [3, Proposition 1.4.20]. □

Lemma 2.33. *Let G be an attractive $\mathbf{G}_m$-algebraic group. Then*

$$(G^{++})^{0,\geq k} = (G^{0,\geq k})^{++}$$

Proof. TODO: Write this □

Definition 2.34. In light of Lemma 2.33, define $G^{\geq k} = (G^{0,\geq k})^{++} = (G^{++})^{0,\geq k}$.

Representation Theory Now we give a Tannakian description of the group schemes $G^{\geq k}$ which will be useful both for understanding their structure theory and their actions on $\mathbf{G}_m$-schemes.

Definition 2.35. Let V be a graded vector space. Then $GL(V)$ naturally has the structure of a $\mathbf{G}_m$-algebraic group: by composing the grading map $\mathbf{G}_m \to GL(V)$ with the adjoint action of $GL(V)$ on itself, we get an action of $\mathbf{G}_m$ on $GL(V)$ by homomorphisms. Call this the **standard** $\mathbf{G}_m$ structure on $GL(V)$.

Definition 2.36. A **graded representation** of a $\mathbf{G}_m$-algebraic group G is a graded vector space V and a map $\rho : G \to GL(V)$ of $\mathbf{G}_m$-algebraic groups, where $GL(V)$ has the standard $\mathbf{G}_m$ structure.

Lemma 2.37. *A graded representation of a $\mathbf{G}_m$-algebraic group G is the same as a representation of $G \rtimes \mathbf{G}_m$.*

Proof. TODO: write this □

Definition 2.38. A graded representation of a $\mathbf{G}_m$-algebraic group G is faithful if $\ker \rho = *$.

Lemma 2.39. *Every $\mathbf{G}_m$-algebraic group has a faithful $\mathbf{G}_m$-representation.*

Proof. TODO: write this □

Definition 2.40. Let V be a graded vector space. Then let $\tau^{\geq i}V$ be direct sum of the weight subspaces of V with weight at least i.

Proposition 2.41. *Let V be a graded vector space and let $GL(V)$ have the standard $\mathbf{G}_m$ structure. Then*

- *$GL(V)^+$ is the subgroup of linear transformations that preserve every $\tau^{\geq i}V$.*
- *$GL(V)^{\geq k}$ is the subgroup of linear transformations that act trivially on each subquotient $\tau^{\geq i}V/\tau^{\geq i+k}V$.*

Proof. TODO: write this □

Theorem 2.42 (Tannakian description of $G^{\geq k}$)**.** *Let G be attractive. Then $G^{\geq k}$ is the largest subgroup of G so that for every graded representation V, the action of $G^{\geq k}$ is trivial on $\tau^{\geq i}V/\tau^{\geq i+k}V$.*

Proof. TODO: write this □

Corollary 2.43. *$G^{\geq k}$ is unipotent for every $k \geq 1$.*

Proof. Let V be a faithful graded representation of G. Then as in the proof of Theorem 2.42 we see that $G^{\geq k}$ is a closed subgroup of $GL(V)^{\geq k}$. Since $GL(V)^{\geq k}$ is unipotent this implies that $G^{\geq k}$ is unipotent, too. □

The following corollary will not be used in the remainder of the text.

Corollary 2.44. *If G is a $\mathbf{G}_m$-algebraic group which is both attractive and reductive, then $\mathbf{G}_m$ acts trivially on G.*

Proof. By Proposition 2.32 and Corollary 2.43 we see that G^{++} is a smooth connected normal unipotent subgroup of G. It must be trivial if G, so $G = G^0$. □

2.2.3 Equivariant Bialynicki-Birula Decompositions

This section is also new.

Definition 2.45. If G is a $\mathbf{G}_m$ algebraic group, then a G-equivariant $\mathbf{G}_m$-scheme Z is a $\mathbf{G}_m$ scheme with an action of G internal to the category of $\mathbf{G}_m$-schemes. In other words, we require the action map $G \times Z \to Z$ to be $\mathbf{G}_m$-equivariant.

Proposition 2.46. *Let Z be a G-equivariant $\mathbf{G}_m$-scheme. Then Z^0 is a G^0-equivariant scheme. If G is attractive then Z^+ is a G-equivariant $\mathbf{G}_m$-scheme and Z^0 is G-equivariant, where the G action factors through the quotient map to $G^0 = G/G^{++}$. Moreover, the map $q^+ : Z^+ \to Z^0$ is G-equivariant.*

Proof. TODO: write this □

Definition 2.47. Let G be attractive and let Z be a G-equivariant $\mathbf{G}_m$-scheme. Define a G-equivariant $\mathbf{G}_m$-scheme $(\mathbf{A}^1 \times Z^+)_s$ ("source") as follows: as a $\mathbf{G}_m$ scheme, the action of $\mathbf{G}_m$ is by the defining action on Z^+ and by the inverse of the standard action on $\mathbf{A}^1$. Then G acts by the given action on Z^+. The action is $\mathbf{G}_m$-equivariant because

$$\begin{aligned} g^\lambda \cdot [\lambda \cdot (\mu, z)] &= g^\lambda \cdot (\lambda^{-1}\mu, \lambda z) \\ &= (\lambda^{-1}\mu, g^\lambda \lambda z) \\ &= (\lambda^{-1}\mu, \lambda g z) \\ &= \lambda \cdot [g \cdot (\mu, z)] \end{aligned}$$

Definition 2.48. Let G be attractive and let Z be a G-equivariant $\mathbf{G}_m$-scheme. Define a G-equivariant $\mathbf{G}_m$-scheme $(\mathbf{A}^1 \times Z^+)_t$ ("target") as follows: as a $\mathbf{G}_m$ scheme, the action of $\mathbf{G}_m$ is trivial on Z^+ and by the inverse of the standard action on $\mathbf{A}^1$. Then G acts by the given action on Z^+ after first twisting by the action of $\mathbf{A}^1$ on G (which is well-defined because G is attractive). This action is $\mathbf{G}_m$ equivariant because:

$$\begin{aligned} g^\lambda \cdot [\lambda \cdot (\mu, z)] &= g^\lambda \cdot (\lambda^{-1}\mu, z) \\ &= (\lambda^{-1}\mu, g^{\lambda^{-1}\mu\lambda} z) \\ &= (\lambda^{-1}\mu, g^\mu z) \\ &= \lambda \cdot [g \cdot (\mu, z)] \end{aligned}$$

Proposition 2.49. *There is a $G \rtimes \mathbf{G}_m$-equivariant map $\alpha_{enh} : (\mathbf{A}^1 \times Z^+)_s \to (\mathbf{A}^1 \times Z^+)_t$ so that $\alpha_{enh}(\mu, z) = (\mu, \alpha(\mu, z))$, where $\alpha : \mathbf{A}^1 \times Z^+ \to Z^+$ is the action map from Definition 2.13.*

Proof. First note that α_{enh} is $\mathbf{G}_m$-equivariant:

$$\begin{aligned} \alpha_{enh}(\lambda \cdot (\mu, z)) &= \alpha_{enh}(\lambda^{-1}\mu, \lambda z) \\ &= (\lambda^{-1}\mu, \alpha(\lambda^{-1}\mu, \lambda z)) \\ &= (\lambda^{-1}\mu, \alpha(\mu, z)) \\ &= \lambda \cdot \alpha_{enh}(\mu, z) \end{aligned}$$

Now note that it's G-equivariant, too:

$$\begin{aligned} \alpha_{enh}(g \cdot (\mu, z)) &= \alpha_{enh}(\mu, gz) \\ &= (\mu, \alpha(\mu, gz)) \\ &= (\mu, g^\mu \alpha(\mu, z)) \\ &= g \cdot \alpha_{enh}(\mu, z) \end{aligned}$$

□

Lemma 2.50. *Let x be in Z^0. Then $\mathrm{Stab}(x)$ is again an attractive $\mathbf{G}_m$-algebraic group and*

$$\begin{aligned} \mathrm{Stab}(x)^0 &= G^0 \cap \mathrm{Stab}(x) \\ \mathrm{Stab}(x)^{++} &= G^{++} \end{aligned}$$

Proof. TODO: write this □

Lemma 2.51. *Let x be in Z^0. Then we get a graded representation of $\mathrm{Stab}(x)$ on the vector space $V = T_{0,x}(\mathbf{A}^1 \times Z^+)_t = T_0\mathbf{A}^1 \times T_x Z^+$. The grading is such that $T_0\mathbf{A}^1$ has weight -1 and $T_x Z^+$ has weight 0. Also, $\mathrm{Stab}(x)$ acts trivially on $T_0\mathbf{A}^1 = V/\tau^{\geq 0}V$.*

Proof. TODO: write this □

Definition 2.52. In the setting of Lemma 2.51, identifying $T_0\mathbf{A}^1$ with k, the quotient map gives a $\mathrm{Stab}(x)$-equivariant linear functional $\theta : V \to k$. We can form the standard affine space A_θ as in Definition A.4, which acquires an action of $\mathrm{Stab}(x)$ by affine-linear transformations. A_θ comes with a distinguished point a, which comes from the fact that V is split as a vector space (but note that a doesn't need to be preserved by the action of $\mathrm{Stab}(x)$!).

Proposition 2.53. *In the setting of Lemma 2.51, $G^{\geq 2}$ acts trivially on A_θ. Also, the action of $\mathrm{Stab}(x)^0$ preserves a.*

Proof. Since $V = \tau^{\geq -1}V/\tau^{\geq 1}V$, we see from Theorem 2.42 that $G^{\geq 2}$ must act trivially on V. Also, by the Tannakian description of $\mathrm{Stab}(x)^0$ (TODO: write this) we know that its action on V preserves the decomposition into weight spaces–so $V = T_0\mathbf{A}^1 \oplus T_x Z^+$ as a $\mathrm{Stab}(x)^0$ representation. □

Definition 2.54. In the spirit of Definition 2.13 and Proposition 2.49 we can define an equivariant map Λ_{enh} which maps from Z^+ to $T_{0\times Z^0}(\mathbf{A}^1 \times Z^+)$. In fact, Λ_{enh} factors through a map to $\{\xi\} \times T_{Z^0}Z^+$, so the fiber at every $x \in Z^0$ is A_θ. TODO: expand on this

2.3 Miscellaneous

In this section we collect a handful of results that are important in [1]. Our proof of Theorem 2.55 follows [6]. Theorem 2.57 and Lemma 2.58 are from [4, Section I.4]–although this paper has a running assumption that all varieties are defined over a field of characteristic zero, the results from this section do not depend on this assumption. Since we are especially interested in applying the results of [4] to the case of integral schemes, we give proofs with this additional assumption for simplicity. In this setting the arguments become much simpler, as indicated in the "Remarque" in Section I.4 of *loc. cit.*

Theorem 2.55 (Matsushima)**.** *If G is reductive then G/H is affine if and only if H is reductive.*

Proof. The reverse direction follows from Nagata's theorem since $G/H = G /\!\!/ H$. For the forward direction let $n = \dim G$ and $m = \dim H$. If $\pi : G \to G/H$ is the projection, then because G is etale-locally a product of H with G/H, we know by Artin vanishing for H that $R^q\pi_*\Lambda = 0$ for all $q > m$ (here it's important that G/H is smooth). Because G/H is assumed to be affine, we can apply Artin vanishing to conclude that $H^p_{et}(G/H; R^q\pi_*\Lambda) = 0$ whenever $p > n - m$. Now a collapse happens in the Leray spectral sequence and we get that

$$H^n_{et}(G;\Lambda) = H^{n-m}_{et}(G/H; R^m\pi_*\Lambda).$$

Since G is reductive, we can use its Chevalley model plus the Artin comparison theorem to see that

$$H^n_{et}(G;\Lambda) = H^n_{sing}(G_{\mathbf{C}};\Lambda) = \Lambda \neq 0,$$

where we've used the fact that a complex reductive group deformation retracts onto its compact real form, which is a smooth compact n-fold. This implies that $R^m\pi_*\Lambda$ must be nonzero, so some stalk of this sheaf must be nonzero. But again by the fact that G is etale-locally a product over the base we have

$$(R^m\pi_*\Lambda)_x = H^m_{et}(H;\Lambda) \neq 0.$$

If H weren't reductive, then we could quotient by its unipotent radical to realize H as an $\mathbf{A}^k$-bundle over the affine algebraic group $H/R_u(H)$. By Artin vanishing, this would contradict the fact that $H^m_{et}(H;\Lambda)$ is nonzero. □

Example 2.56. Consider $G = \mathbf{G}_m$ and $H = \mu_p$. Then the quotient is $\mathbf{G}_m$ which is affine but H may not be reductive if you assume reductive groups to be smooth.

Theorem 2.57 (G-equivariant Zariski's Main Theorem)**.** *Let X, Y be affine G-varieties and let $\varphi : X \to Y$ be a quasi-finite G-morphism. Then there is a G-variety Z so that φ factors as $\psi \circ i$, where $i : X \to Z$ is an open immersion, $\psi : Z \to Y$ is finite, and both i and ψ are G-equivariant.*

Proof (when X and Y are integral). Let Z be the normalization of Y in X (see [8, Tag 035H]). Then Z is a G-scheme since if $f \in \mathcal{O}_X$ satisfies the equation $x^k + y_{k-1}x^{k-1} + \ldots + y_0 = 0$ with $y_i \in \mathcal{O}_Y$ then $g \cdot f$ satsifies the equation $x^k + (g \cdot y_{k-1})x^{k-1} + \ldots + g \cdot y_0 = 0$. By [8, Tag 0AVK] Z is finite over Y and by [8, Tag 02LR] the map $X \to Z$ is an open embedding. □

Lemma 2.58 (Luna's Lemma)**.** *Let $\varphi : X \to Y$ be a map of affine G-varieties which is quasi-finite, sends closed orbits to closed orbits, and induces a finite map on GIT quotients. Then φ is finite.*

Proof (when X and Y are integral). If the map $\varphi /\!\!/ G : X /\!\!/ G \to Y /\!\!/ G$ is finite then every element of $\mathcal{O}_X^G$ is automatically integral over $\mathcal{O}_Y$. In the notation of the previous proof we see that $i /\!\!/ G$ is therefore an isomorphism. If the image of i in Z has a non-empty closed complement, then we can find a closed G-orbit T contained in the complement. Because $i /\!\!/ G$ is an isomorphism, there must be an orbit $\tilde{T}$ which is closed in X such that T is in the closure of $i(\tilde{T})$. Point-set topology shows that $\psi(T)$ is in the closure of $\psi \circ i(\tilde{T}) = \varphi(\tilde{T})$, but by assumption $\varphi(\tilde{T})$ is closed, so $\psi(T) = \varphi(\tilde{T})$. Since $\psi(T) = \psi \circ i(\tilde{T})$ and ψ is finite we see that T and $i(\tilde{T})$ must have the same dimension, contradicting the fact that T is in the closure of $i(\tilde{T})$. □

3 Whittaker Induction

In this section we follow [1, Sections 3.4-3.6]. For simplicity, we impose the assumption of *evenness* in our definitions and proofs in this section. Since one is mainly interested in hyperspherical varieties with a "distinguished polarization" (see Definition 4.4), which automatically implies evenness, this is not a terribly restrictive assumption. We recommend that readers interested in the odd case consult [1].

Definition 3.1. Consider a map $H \times SL_2 \to G$ (in particular, H lies in the centralizer of SL_2) and a Hamiltonian H-space S. Suppose the map $SL_2 \to G$ is even, which means that $-1 \in SL_2$ is in the kernel. Then the **Whittaker induction** of S is $\text{h-ind}_{HU}^{G} S_f$.

Definition 3.2. A graded Hamiltonian G-space X **has property h** if

- X is affine.
- X has a unique closed orbit X_0 under the action of $G \times \mathbf{G}_m$.
- The image of X_0 under μ is contained in the nilpotent cone in $\mathfrak{g}^*$
- The $\mathbf{G}_m$ action on X is "neutral". Namely, for all $x \in X_0$ it must be possible to extend $\mu(x)$ to an $\mathfrak{sl}_2$-triple so that $x \cdot \lambda = x \cdot \lambda^h$. Additionally, the cocharacter $\pi_x : \mathbf{G}_m \to G \times \mathbf{G}_m$ that sends $\lambda \mapsto (\lambda^{-h}, \lambda)$ should act by simple scaling on the fiber of the symplectic normal bundle of X_0 in X.

Lemma 3.3. *Let M be a graded Hamiltonian G-space with property h. Then the unique closed $G \times \mathbf{G}_m$ orbit M_0 is actually a single G-orbit.*

Proof. By the neutrality assumption, at every point $x \in M_0$ the $\mathbf{G}_m$ action is given by a cocharacter of G. □

Lemma 3.4. *Ignoring symplectic structures, when S is a linear graded Hamiltonian H-space* $\text{w-ind}_H^G S = V \times^H G$ *where $V = S \oplus (\mathfrak{h}^\perp \oplus \mathfrak{g}^e)$. The grading is such that $\mathbf{G}_m$ acts by the given action on S, it acts by weight $2+i$ on the part of $\mathfrak{g}^e$ with weight i with respect to h, and $\mathbf{G}_m$ acts on G by left multiplication by λ^h.*

Proof. By Lemma 1.18 we know that $\text{w-ind}_H^G = (S_f \times_{(\mathfrak{h}+\mathfrak{u})^*} \mathfrak{g}^*) \times^{HU} G$. The theory of Slodowy slices says that $(S_f \times_{(\mathfrak{h}+\mathfrak{u})^*} \mathfrak{g}^*) = (S_f \times_{\mathfrak{h}^*} \mathfrak{g}^e) \times U$. □

Lemma 3.5. *If $H \times SL_2 \to G$ is Whittaker data where H is reductive and S is a linear graded Hamiltonian H-space then* $\text{w-ind}_H^G S$ *satisfies property h.*

Proof. When H is reductive, $H \backslash G$ is affine. By Lemma 3.4, we see that $\text{w-ind}_H^G S$ is a vector bundle over $H \backslash G$, so it is affine, too. Under the $\mathbf{G}_m$ action on V, every nonzero element gets contracted to zero, so $H \backslash G = X_0$ is the unique closed $G \times \mathbf{G}_m$ orbit in $\text{w-ind}_H^G S$. The image of X_0 under the moment map is conjugate to f, and the action of $\mathbf{G}_m$ is given by scaling by λ^h. The fiber of the symplectic normal bundle to X_0 at any point is given by S, which is acted on by linear scaling under $(\lambda^{-h}, \lambda) \subseteq G \times \mathbf{G}_m$. □

Lemma 3.6. *If M_1 and M_2 are graded Hamiltonian G-spaces with property h and $\varphi : M_1 \to M_2$ is a map that is etale and an isomorphism on the closed orbits then φ is an isomorphism.*

Proof. It suffices to show that φ is finite; given this φ is finite etale, so it's a covering map, and its fibers consist of a single point because it's an isomorphism on the closed orbits. To check finiteness, we can apply Lemma 2.58: both the source and the target are affine by property h, φ is quasi-finite because it is etale, it carries closed orbits to closed orbits by assumption, and since each GIT quotient is a point the map on GIT quotients is finite. □

Lemma 3.7. *If M is a graded Hamiltonian G-space with property h and $x \in M_0$ is a point in the closed orbit, then $T_xM_0 \cap T_xM_0^{\perp} = \mathfrak{g}^f/\mathfrak{h}$, where f is the image of x under the moment map (here we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using an invariant bilinear form) and $\mathfrak{h}$ is the Lie algebra of the stabilizer of x.*

Proof. By Lemma 3.3 we know that $\mathfrak{g}$ surjects onto the tangent space of M_0 at x via the map ξ (Definition 1.5), and the kernel is exactly $\mathfrak{h}$. It remains to determine which $X \in \mathfrak{g}/\mathfrak{h}$ pair to zero with every other element under the symplectic form. But

$$\omega_x(\xi_X, \xi_Y) = \xi_Y \mu_X(x) = \mu_{[Y,X]}(x) = \langle f, [Y,X] \rangle = \langle [X,f], Y \rangle .$$

This vanishes for all $Y \in \mathfrak{g}$ if and only if $X \in \mathfrak{g}^f$. ◻

Theorem 3.8. *If M is a graded Hamiltonian G-space with property h then M is Whittaker-induced from a linear Hamiltonian H-space.*

Proof. Let M_0 be the closed orbit and let x be a point in M_0. Let H be the stabilizer of x (which is reductive by Theorem 2.55). Let f be the image of x under the moment map; by the neutrality condition there is an SL_2-triple where $\mathbf{G}_m$ acts by λ^h at x. Let S be the fiber of the symplectic normal bundle of M_0 at x. Recall that x is fixed under the $\mathbf{G}_m$ action (λ^{-h}, λ). By the Bialynicki-Birula theorem, there is a contracting scheme Z^+ for this $\mathbf{G}_m$ action, which maps to the fixed locus Z^0 by q^+. By Lemma 2.12 and the neutrality condition on the $\mathbf{G}_m$ action, we know that S automatically lies inside T_xZ^+.

Let M^+ be the fiber of q^+ at x (i.e. the points of M that contract to x under the $\mathbf{G}_m$ action). Note the slight mismatch between this notation (which matches the notation in [1]) and the notation of Section 2.2 (according to which M^+ would denote what we call Z^+ here). M^+ is stable under the HU action by Proposition 2.46 because HU is attractive for the action of λ^h.

If $\Lambda_Z : Z^+ \to T_{Z^0}Z^+$ is as in Definition 2.13, then we claim that M^+ can be expressed as the pullback in the upper square of the following diagram:

$$\begin{array}{ccc} M^+ & \longrightarrow & Z^+ \\ \Lambda\big\downarrow & \lrcorner & \big\downarrow\Lambda_Z \\ T_xZ^+ & \longrightarrow & T_{Z^0}Z^+ \\ \big\downarrow & \lrcorner & \big\downarrow\pi \\ x & \longrightarrow & Z^0 \end{array}$$

This follows because the bottom square is a pullback by definition and the outer rectangle is a pullback by Lemma 2.14. We therefore get a map $\Lambda : M^+ \to T_xZ^+$ as indicated in the diagram. It must land in the weight-one subspace of T_xZ^+ by Lemma 2.15. We claim that the weight-one subspace is exactly S. Clearly S is in the weight-one subspace by the neutrality condition on the $\mathbf{G}_m$ action, so it remains to see the other inclusion.

By the neutrality condition, $T_xM_0 = \mathfrak{g}/\mathfrak{h}$ is a $\mathbf{G}_m$-stable subspace that $\mathbf{G}_m$ acts on by λ^h. This gives a sequence of inclusions of $\mathbf{G}_m$-stable subspaces:

$$0 \subseteq V_1 \subseteq T_xM_0 \subseteq V_1^{\perp} \subseteq T_xZ^+$$

where $V_1 = T_xM_0 \cap (T_xM_0)^{\perp}$. Because $\mathbf{G}_m$ is linearly reductive we get a splitting of T_xZ^+ as a $\mathbf{G}_m$-representation into

$$T_xM_0 \oplus V_1^{\perp}/T_xM_0 \oplus T_xZ^+/V_1^{\perp}$$

Note that the weights of the $\mathbf{G}_m$ action on this subspace are non-positive, so because ω is acted on by the squared action, all of the weights of $T_xM/V_1^{\perp}$ are at least 2. By the evenness assumption, all of the weights of $\mathfrak{g}/\mathfrak{h}$ are even. So the subspace of weight one must be contained in $V_1^{\perp}/T_xM_0 = (T_xM_0 + (T_xM_0)^{\perp})/T_xM_0 = S$.

Since we assume that the SL_2-triple is even, we know that $U = U^{\geq 2}$, so by Proposition 2.53 we see that Λ is HU-equivariant, where U acts trivially on S and H acts by the usual action on T_xZ^+.

We therefore get a map $\Lambda : M_+ \to S$. We further claim that this map realizes M_+ as the fiber product

$$\begin{array}{ccc} M^+ & \xrightarrow{\Lambda} & S \\ {\scriptstyle\mu}\downarrow & & \downarrow \\ \mathfrak{g}^* & \longrightarrow & (\mathfrak{h}+\mathfrak{u})^* \end{array}$$

This amounts to saying that $d\mu$ is injective on the weight ≥ 2 subspace of $T_x M$. The kernel of $d\mu$ is given by $(T_x M_0)^{\perp}$, which is a sum of $T_x M_0 \cap (T_x M_0)^{\perp}$ and S. But the weights on both of these subspaces are < 2. Now by dimension counting we get an iso on tangent spaces. TODO: Finish this out... ☐

4 Statements of Results

In this section we state other important definitions and theorems from Sections 3.6 and 3.7 of [1].

Definition 4.1. A symplectic G-variety X is **coisotropic** if $k(X)^G$ is commutative with respect to the Poisson bracket.

Proposition 4.2. *If H is reductive then $\text{w-ind}_H^G S$ is coisotropic if and only if G/HU is spherical and S is coisotropic for the generic stabilizer of G on T^*Y.*

Definition 4.3. A graded Hamiltonian G-variety X is **hyperspherical** if

- It satsifies property h.
- It is coisotropic.
- The stabilizer of a generic point of X is connected.

Definition 4.4. A hyperspherical variety admits a **distinguished polarization** if it is even and the symplectic normal bundle to the closed orbit admits an H-stable splitting $S = S^+ \oplus S^-$.

Lemma 4.5. *If hyperspherical M admits a distinguished polarization then $M = T^*(X, \Psi)$, where $X = S^+ \times^{HU} G$ and $\Psi = S^+ \times^{HU'} G$, where $U' = \ker f$. Moreover X has to be spherical and the B-stablizer of a point in the open B orbit is connected.*

A Affine Spaces

Here we collect some basic facts about affine spaces.

Definition A.1. An **affine space** is a torsor for a vector space.

The ring of polynomial functions on any vector space V has a grading by degree; associated to this grading one can produce a filtration $F_i k[V] = \oplus_{j \leq i} k[V]_j$. If A is an affine space which is a torsor for V, we can identify A with V by choosing a point $a \in A$. This then gives an identification of the ring of functions on A and the ring of functions on V, and one can transport the grading and filtration on $k[V]$ to $k[A]$. The grading on $k[A]$ that one produces through this procedure is dependent on the choice of a, but importantly the filtration does *not* depend on the choice of a.

Definition A.2. The **filtration by degree** on $k[A]$ is the filtration defined above.

Definition A.3. The space $\text{Aff}(A, k)$ of **affine functionals** on A is $F_1 k[A]$.

One common source of affine spaces is the following construction

Definition A.4. Given a vector space V and a nonzero linear functional $\theta : V \to k$, the fiber $A_\theta = \theta^{-1}(1)$ is an affine space over the vector space $V_\theta = \ker \theta$, where the action of V_θ on A_θ is given by the addition map of V. Call such an affine space **standard**.

Lemma A.5. *Splittings of a nonzero linear functional $\theta : V \to k$ are in bijection with points of the standard affine space A_θ.*

Proof. The splitting is determined by the image of 1, which must lie in $\theta^{-1}(1) = A_\theta$. $\square$

In fact every affine space can be functorially realized as a standard affine space.

Proposition A.6. *Given an affine space A which is a torsor for a finite dimensional vector space V, there is a canonical linear embedding $\Delta : V \to \mathrm{Aff}(A,k)^\vee$ which realizes V as a hyperplane in $\mathrm{Aff}(A,k)^\vee$. There is a canonical affine embedding $\mathrm{ev} : A \to \mathrm{Aff}(A,k)^\vee$ and a canonical linear functional $\theta : \mathrm{Aff}(A,k)^\vee \to k$ so that (Δ, ev) identifies (V, A) with the standard affine space (V_θ, A_θ).*

Proof. Inside the space of affine functionals $F_1 k[A]$ there is the subspace of constant functionals $k = F_0 k[A]$. The embedding $k \to \mathrm{Aff}(A,k)$ induces a map $\mathrm{Aff}(A,k)^\vee \to k^\vee$. Identifying k with its dual gives the map $\theta : \mathrm{Aff}(a,k) \to k$. Spelling this out, for any $\lambda \in k$ and any $\xi \in \mathrm{Aff}(A,k)^\vee$, we have $\xi \cdot f_\lambda = \theta(\xi)\lambda$ (where f_λ is the constant function with value λ).

The embedding $\Delta : V \to \mathrm{Aff}(A,k)^\vee$ sends $v \in V$ to the difference operator Δ_v (so

$$\Delta_v f = f(p+v) - f(p),$$

which is independent of the choice of point $p \in A$) and the embedding $\mathrm{ev} : A \to \mathrm{Aff}(A,k)^\vee$ sends $a \in A$ to the evalutation operator ev_a (so $\mathrm{ev}_a f = f(a)$). Since the functions for which Δ_v vanishes for all v are exactly the constant functions, we see that Δ identifies V with $\ker \theta$. Clearly $\theta(\mathrm{ev}_a) = 1$, so by dimension counting we see that ev identifies A with A_θ. It remains to note that $\mathrm{ev}_a + \Delta_v = \mathrm{ev}_{a+v}$. $\square$

Since all of the above constructions are functorial, we can perform them in families. Working over BG and combining Proposition A.6 and Lemma A.5 we get the following proposition

Proposition A.7. *If A is a finite-dimensional affine space with a G action then the fixed points A^G are in bijection with the G-equivariant splittings of $\theta : \mathrm{Aff}(V,k)^\vee \to k$.*

Corollary A.8. *If G is linearly reductive then every G-affine space A has a fixed point.*

Proof. By local-finiteness considerations we can reduce to the case when A is finite-dimensional. Then by Proposition A.7 we see that the obstruction to the existence of a fixed point is given by the class of the extension

$$0 \to V \xrightarrow{\Delta} \mathrm{Aff}(V,k)^\vee \xrightarrow{\theta} k \to 0$$

in $\mathrm{Ext}^1_{\mathrm{Rep}(G)}(k, V)$. But if G is linearly reductive this Ext group vanishes. $\square$

B Mapping Spaces

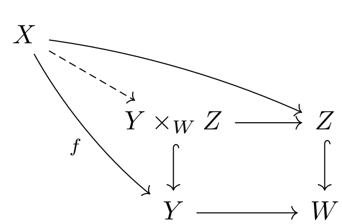
Here we collect some basic facts about mapping spaces. Let's work over a base ring k which is not necessarily a field. Although we will only be interested in the case when k is a field, we feel that it clarifies the nature of the arguments to work in this more general situation.

Definition B.1. A map of schmes $f : X \to Y$ is **schematically dominant** if the schematic image of f is equal to Y.

Proposition B.2. *Let $f : X \to Y$ be a map of schemes. The following are equivalent*

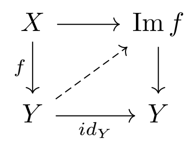
1. *f is schematically dominant.*
2. *f satisfies the left lifting property with respect to closed embeddings.*

Proof. First suppose f is schematically dominant. Then for any closed embeding $Z \to W$ and compatible maps $X \to Z, Y \to W$, we need to find a lift from Y to Z.



Since f factors through the closed embedding $Y \times_W Z \to Y$, we see that the map $Y \times_W Z \to Y$ must be an isomorphism. This gives the lift from Y to Z.

On the other hand, suppose that f satisfies the left lifting property with respect to closed embeddings. Then we can lift against the inclusion of the schematic image of f into Y as in the diagram below.

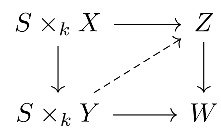


This shows that $\operatorname{Im} f = Y$. ∎

Definition B.3. A map of schemes $f : X \to Y$ is **universally schematically dominant** if it remains schematically dominant after any change of base.

Proposition B.4. *Suppose $f : X \to Y$ is universally schematically dominant and $g : Z \to W$ is closed. Then* $\operatorname{Maps}(Y, Z) = \operatorname{Maps}(Y, W) \times_{\operatorname{Maps}(X,W)} \operatorname{Maps}(X, Z)$.

Proof. We'll check equality on S-points for all S. We need to show that every diagram as below admits a lift:

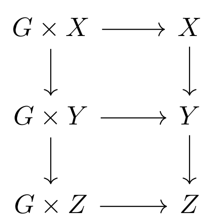


This follows since $Z \to W$ is closed and $S \times_k X \to S \times_k Y$ is schematically dominant. ∎

This proposition has a G-equivariant upgrade, based on the following

Lemma B.5. *Let X, Y, Z be G-schemes. Let $f : X \to Y$ and $g : Y \to Z$ be maps such that g is a monomorphism. Then if g is G-equivariant and $g \circ f$ is G-equivariant, so is f.*

Proof.



We want to check that the upper square in this diagram commutes. Since g is a monomorphism, we can check this after composition with g. The lower square and the outer rectangle commute because g and $g \circ f$ are G-equivariant, giving the desired result. ∎

Corollary B.6. *In the setting of Proposition B.4, if X, Y, Z, W are G-schemes and f and g are G-equivariant, then we also have* $\operatorname{Maps}^G(Y, Z) = \operatorname{Maps}^G(Y, W) \times_{\operatorname{Maps}^G(X,W)} \operatorname{Maps}^G(X, Z)$.

Proof. We need to check that if $S \times_k Z$ and $S \times_k Y$ are G-equivariant then the lift constructed in the proof of Proposition B.4 is G-equivariant. Since $Z \to W$ is a monomorphism this follows from Lemma B.5. ∎

Pushouts The inclusion of affine schemes into schemes doesn't preserve pushouts in general; however, there are special circumstances when this is true. The following is due to Schwede:

Theorem B.7. *[7, Theorem 3.4] Let X, Y, Z be affine schemes with maps $f : Y \to X$, $g : Y \to Z$. If f is a closed embedding then the pushout $X \sqcup_Y Z$ in the category of schemes exists, and is equal to the pushout in the category of affine schemes. More explicitly, if $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} B$, and $Y = \operatorname{Spec} C$, then $X \sqcup_Y Z = \operatorname{Spec} A \times_C B$.*

Lemma B.8. *In the setting of Theorem B.7, let W be another affine scheme. If either Y is flat over k or W is flat over k, then*

$$(X \sqcup_Y Z) \times W = (X \times W) \sqcup_{(Y \times W)} (Z \times W)$$

Proof. Since $A \to C$ is surjective we have the following exact sequence of k-modules

$$0 \to A \times_C B \to A \oplus B \to C \to 0$$

Writing $R = \mathcal{O}_W$, if either R or C is flat over k then this induces an exact sequence

$$0 \to (A \times_C B) \otimes R \to (A \otimes R) \oplus (B \otimes R) \to (C \otimes R) \to 0$$

which shows that

$$(A \times_C B) \otimes R = (A \otimes R) \times_{C \otimes R} (B \otimes R)$$

as desired. □

Proposition B.9. *In the setting of Theorem B.7, if Y is flat over k then for any scheme W we get $\operatorname{Maps}(X \sqcup_Y Z, W) = \operatorname{Maps}(X, W) \times_{\operatorname{Maps}(Y,W)} \operatorname{Maps}(Z, W)$.*

Proof. It suffices to check that the S-points are the same whenever $S = \operatorname{Spec} R$ is an affine scheme. But by Lemma B.8 we know that $(X \sqcup_Y Z) \times S = (X \times S) \sqcup_{Y \times S} (Z \times S)$, so this follows from the universal property of the pushout. □

This proposition also has a G-equivariant upgrade.

Proposition B.10. *Let G be a group scheme flat over k. In the setting of Theorem B.7, if X, Y, Z are G-schemes and f, g are G-equivariant, then the pushout $X \sqcup_Y Z$ exists in the category of G-schemes (and the underlying scheme is the usual pushout $X \sqcup_Y Z$).*

Proof. By Lemma B.8 we see that

$$G \times (X \sqcup_Y Z) = (G \times X) \sqcup_{G \times Y} (G \times Z)$$

so the action maps of X and Z define an action map on $X \sqcup_Y Z$. Given any G-scheme W with equivariant maps from X and Z which agree on Y, we get a map of schemes $X \sqcup_Y Z \to W$ by the universal property of the pushout. To check that the following diagram commutes

$$\begin{array}{ccc} (G \times X) \sqcup_{G \times Y} (G \times Z) & \longrightarrow & X \sqcup_Y Z \\ \downarrow & & \downarrow \\ G \times W & \longrightarrow & W \end{array}$$

it suffices to check after precomposing with the inclusions of $G \times X$ and $G \times Z$ into the pushout. But then the diagram commutes by the equivariance of the maps $X \to W$ and $Z \to W$. □

Corollary B.11. *In the setting of Proposition B.10, if Y is flat over k then for any G-scheme W we have $\operatorname{Maps}^G(X \sqcup_Y Z, W) = \operatorname{Maps}^G(X, W) \times_{\operatorname{Maps}^G(Y,W)} \operatorname{Maps}^G(Z, W)$.*

Proof. Combine proposition B.10 and Proposition B.9. □

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