Abelian Examples

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§1 4d QFTs

$4 \rightsquigarrow \#$
$3 \rightsquigarrow \mathrm{Vect}$
$2 \rightsquigarrow \mathrm{cat}$
$1 \rightsquigarrow 2\text{-cat}$

A 4d QFT T assigns

- X (alg curve / Riemann surface) $\rightsquigarrow T(X)$ a category
- $\mathring{D}_X \rightsquigarrow T(\mathring{D}_X)$ a 2-category.

The trivial 4d QFT triv assigns

- $X \rightsquigarrow T(X) := \mathrm{Vect}$
- $\mathring{D}_X \rightsquigarrow T(\mathring{D}_X) := \mathrm{DGCat}_{\mathrm{cont}}$.

Defn: A boundary condition is a map (interface) $\mathcal{B} : \mathrm{triv} \longrightarrow T$.

- $\mathcal{B}_X \in T(X)$ an object in the cat $T(X)$
- $\mathcal{B}_X \in T(\mathring{D}_X)$ an object in the 2-cat $T(\mathring{D}_X)$.

Given two boundary conditions, B_1, B_2 can form

$$\langle B_1, B_2 \rangle : \text{triv} \longrightarrow \text{triv}$$

$X \mapsto$ v.s
$\mathring{D}_X \mapsto$ cat.

which is the same data as a 3d QFT.

Basic meta-conjectures:

(1) Langlands dualities can profitably be understood as an equiv of 4-d QFTs

automorphic side $\left\{ A_G \simeq B_{G^\vee} \right\}$ spectral ("Galois") side

(2) If we compactify along $S^1 \simeq \mathring{D}$ we obtain an equiv of 3d QFTs

$$A_G[\mathring{D}] \simeq B_{G^\vee}[\mathring{D}] \quad \Big\} \text{ local (geom) Langlands}$$

The equivalence in (1) sends boundary conditions to boundary conditions and boundary conditions are preserved under compactification along $\mathring{D} \simeq S^1$.

(3) For suitable Hamiltonian G-var $M = T^*X$ $\exists$ corresponding boundary condition

$$B_M : \text{triv} \longrightarrow A_G$$

and under (1), the corresponding dual

boundary condition $\mathcal{B}^\vee_M$ is given by a dual Hamiltonian $G^\vee$-var $M^\vee$ so

$$\left(\mathcal{B}^\vee_M : \mathrm{triv} \longrightarrow \mathcal{B}_{G^\vee}\right) = \left(\mathcal{B}_{M^\vee} : \mathrm{triv} \longrightarrow \mathcal{B}_{G^\vee}\right)$$

Working w/ the compactified theory, and matching suitable boundary conditions on the A_G and $B_{G^\vee}$ sides, we arrive at

(unramified) For $(M, M^\vee)$ a pair of Hamiltonian $(G, G^\vee)$-vars.

<u>Local</u> <u>conj</u>:

$$\langle \mathcal{B}_M[\dot{D}], \mathrm{unr}_G \rangle \simeq \langle \mathcal{B}_{M^\vee}[\dot{D}], \mathrm{unr}_{G^\vee} \rangle$$

Here $\mathcal{B}_M[\dot{D}] = \mathrm{SHV}(\mathcal{L}X)$, $\mathrm{unr}_G = \mathrm{SHV}(\mathcal{L}^+G \backslash \mathcal{L}G)$

$\mathcal{B}_{M^\vee}[\dot{D}] = \mathrm{QC}(\mathrm{Maps}(\dot{D}, M^\vee/G^\vee))$, $\mathrm{unr}_{G^\vee} = \mathrm{QC}(\mathrm{Maps}(D, */G^\vee))$

i.e.

$$\mathrm{SHV}(\mathcal{L}^+G \backslash \mathcal{L}X) \simeq \mathrm{QC}(M^\vee/G^\vee)$$

missing the shearing!!

When $G = \mathbb{G}_m$, $X = \mathbb{A}^1$ w/ the usual scaling action, we can do much better.

<u>Thm</u> (Tate's thesis; deRham Feigin-Hilburn, Betti Gammage-Hilburn)

For $G = \mathbb{G}_m$, $X = \mathbb{A}^1$, $M = T^*\mathbb{A}^1$, $M^\vee = T^*(\mathbb{A}^1)^\vee$

$$\left(\mathcal{B}_M : \mathrm{triv} \longrightarrow \mathcal{A}_{\mathbb{G}_m}[\dot{D}]\right) \simeq \left(\mathcal{B}_{M^\vee} : \mathrm{triv} \longrightarrow \mathcal{B}_{\mathbb{G}_m}[B]\right)$$

i.e.

(de Rham) $\quad D^!(\mathcal{L}\mathbb{A}^1) \simeq \mathrm{Ind\,Coh}(\mathcal{L}_{dR}(\mathbb{A}^1/\mathbb{G}_m))$

(Betti) $\quad Sh^!_S(\mathcal{L}\mathbb{A}^1) \simeq \mathrm{Ind\,Coh}(\mathcal{L}_{Betti}(\mathbb{A}^1/\mathbb{G}_m))$.

compatible w/ geometric CFT.

<u>key point</u>: This is a more general version of the local conjecture in that it allows for <u>arbitrary</u> <u>ramification</u> conditions; not just $\mathrm{unr}_G \longleftrightarrow \mathrm{unr}_{G^\vee}$.

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§2 <u>Constructible</u> <u>sheaves</u> <u>on</u> <u>$\mathcal{L}\mathbb{A}^1$</u>

$$X = \mathcal{L}\mathbb{A}^1 = \Big\{ \sum_{m \gg \infty} a_m t^m : a_m \in \mathbb{C} \Big\}.$$

Then $\quad X = \mathrm{colim}\left(\cdots X_{k+1} \hookrightarrow X_k \hookrightarrow X_{k-1} \cdots \right)$

$$X_k = \Big\{ \sum_{m \geq k} a_m t^m : a_m \in \mathbb{C} \Big\}$$

and we have natural closed embeddings $X_k \hookrightarrow X_{k-1}$.

Each X_k is presented

$$X_k = \lim\left(\cdots \to X_k^2 \to X_k^1 \to X_k^0 \to 0 \right)$$

where $\quad X_k^l \simeq \Big\{ \sum_{m=k}^{k+l} a_m t^m : a_m \in \mathbb{C} \Big\}$.

Let $S :=$ stratification of X by the order of the zero / pole at the origin.

Then $\quad Sh^!_S(X) := \overset{*}{\mathrm{colim}}\ \overset{!}{\mathrm{colim}}\ Sh^!_S(X^\ell_k)$

Here $\quad Sh^!_S(X^\ell_k) =$ co-sheaves on X^ℓ_k

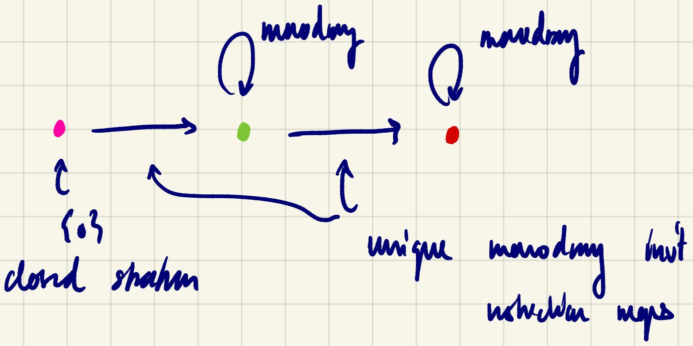
$$:= \mathrm{Fun}(\underbrace{\mathrm{Exit}(S)}_{\text{exit path category}}{}^{op}, \mathrm{Vect}).$$

<u>The key point</u>: $\quad (X^\ell_k, S) \cong (\mathbb{C}^{\ell+1}, T)$

where T is the stratification given by "successive vanishing of coordinates"

<u>Eg</u> $\quad (X^1_k, S) \cong (\mathbb{C}^2, T)$

The options for the exit paths can be summarized in the following diagram



Thus, $\quad Sh_S(X^\ell_k) \cong$

$$\begin{tikzcd} V_0 \arrow[r, "r"'] & V_1 \arrow[loop above, "m_1"] \arrow[r, "r"'] & V_2 \arrow[loop above, "m_2"] \end{tikzcd} \quad \text{s.t.}$$

r is monodromy invt, i.e.

$$mr - r = 0.$$

Closely related to nearby/vanishing cycles desc. of perverse sheaves on $\mathbb{C}^n$

see Appendix C of Achar's beautiful book.

Proposition (Prop 2.4 of G-H):

$$Sh^!_S(X^l_k) \simeq Mod_{A^l_k} \quad \text{where } A^l_k \text{ is the path alg of the quiver}$$

(1)

$$Q_{l,k} := \left(\begin{tikzcd} \underset{k}{\bullet} \arrow[loop above, "m_k^{\pm 1}"] \arrow[r, "c_{k+1}"] & \underset{k+1}{\bullet} \arrow[loop above, "m_{k+1}^{\pm}"] \arrow[r, "c_{k+2}"] & \cdots \arrow[r] & \underset{k+l-1}{\bullet} \arrow[loop above, "m^{\pm}"] \arrow[r, "c_{k+l}"] & \underset{k+l}{\bullet} \end{tikzcd} \right)$$

and relations $J^l_k := \langle c_{i+1}(1-m_i), (1-m_j)c_j \rangle$.

(2) Pull back along $X^{l+1}_k \hookrightarrow X^l_k$ is given by

$$\text{rep } V_\bullet \text{ of } A^l_k \longmapsto \begin{tikzcd} V_{\bullet-1} \arrow[r] & V_{k+l} \arrow[loop above, "id"] \arrow[r, "id"'] & V_{k+l} \end{tikzcd}$$

rep of A^{l+1}_k

(3) Push forward along $X^l_k \hookrightarrow X^{l+1}_{k-1}$ is given by

$$\text{rep } V_\bullet \text{ of } A^l_k \longmapsto \begin{tikzcd} 0 \arrow[loop above, "id"] \arrow[r, "0"'] & V_\bullet \end{tikzcd}$$

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§3 Coherent sheaves on $\mathcal{L}_{Betti}(\mathbb{A}^1/\mathbb{G}_m)$

Prop (Lemma 2.7 of G-H)

$$\mathcal{L}_B(\mathbb{A}^1/\mathbb{G}_m) \simeq \operatorname{Spec} A / \mathbb{G}_m$$

where $\quad A = \dfrac{\mathbb{C}[x, y^{\pm 1}]}{x(y-1)}, \quad |x| = 1, \ |y| = 0$

<u>Pf:</u> $\quad \mathcal{L}_B(\mathbb{A}^1/\mathbb{G}_m) \simeq \{ (x,g) \in \mathbb{A}^1 \times \mathbb{G}_m : gx = x \} / \mathbb{G}_m$

and $\quad \{ (x,g) \in \mathbb{A}^1 \times \mathbb{G}_m : gx = x \} = \mathbb{A}^1 \times \{1\} \sqcup \{0\} \times \mathbb{G}_m$

$$= \operatorname{Spec} \frac{\mathbb{C}[x, y^{\pm 1}]}{x(y-1)}$$

$\square$

Let $\mathcal{O} :=$ structure sheaf of $\mathcal{L}_B(\mathbb{A}^1/\mathbb{G}_m) = \widetilde{A}$

$\mathcal{O}_{\mathbb{C}/\mathbb{C}^\times} :=$ pushforward of struct sheaf $= \widetilde{A/y-1}$
of $\mathbb{A}^1/\mathbb{G}_m \hookrightarrow \mathcal{L}_B(\mathbb{A}^1/\mathbb{G}_m)$.

$\mathcal{O}(n) :=$ pullback of the wt n sheaf
$\mathcal{O}_{B\mathbb{G}_m}(n) \in \operatorname{Coh}(B\mathbb{G}_m)$ under

$$\mathcal{L}_B(\mathbb{A}^1/\mathbb{G}_m) = \operatorname{Spec} A/\mathbb{G}_m \longrightarrow B\mathbb{G}_m.$$

Define $\quad \mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}(n).$

To produce a functor $Sh^!(X) \to \operatorname{Ind}\operatorname{Coh}(\mathcal{L}_B(\mathbb{A}^1/\mathbb{G}_m))$,
suffices to produce compat functors

$$A^l_k\text{-mod} \simeq Sh^!(X^l_k) \longrightarrow \operatorname{Ind}\operatorname{Coh}(\mathcal{L}_B(\mathbb{A}^1/\mathbb{G}_m)).$$

i.e. map the quivers $Q_{l,k}$ into $\operatorname{Ind}\operatorname{Coh}(\mathcal{L}_B(\mathbb{A}^1/\mathbb{G}_m))$.

<u>Claim</u>: The maps $f_{k,l} : Sh^!(X^l_k) \longrightarrow \mathrm{IndCoh}(\mathcal{L}_B(\mathbb{A}^1/\mathbb{G}_m))$ given by sending a rep of A^k_l to

$$\begin{tikzcd} \mathcal{O}(k) \arrow[loop above, "y"] \arrow[r, "x"] & \mathcal{O}(k+1) \arrow[loop above, "y"] \arrow[r, "x"] & \cdots \arrow[r, "x"] & \mathcal{O}(k+l-1) \arrow[loop above, "y"] \arrow[r, "x"] & \mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}(k+l) \end{tikzcd}$$

$$\in \mathrm{IndCoh}(\mathcal{L}_B(\mathbb{A}^1/\mathbb{G}_m)).$$

does the job.

One checks that $f_{k,l}$ glue; the resulting functor is essentially surjective b.c. $\mathrm{IndCoh}(\mathcal{L}_B(\mathbb{A}^1/\mathbb{G}_m))$ is gen. by $\mathcal{O}$, $\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}$ and their twists, it is fully faithful by an easy check on projectives for each k, l and a lemma of H.R.

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§4 <u>Arithmetic</u> <u>Tate</u> <u>thesis</u>

Let F be a local field.

$S(F)$:= Schwartz-Bruhat fns on F

$S(F)'$:= tempered distributions on F

For $\omega : F^\times \longrightarrow \mathbb{C}^\times$ a quasi-character,

$$S'(\omega) := \{ \lambda \in S(F)' : r'(a)\lambda = \omega(a)\lambda \}$$

the space of ω-eigen distributions.

<u>Tate's fundamental insights:</u>

① $\dim S'(\omega) = 1$ for any ω.

② $\{S'(\omega)\}$ can be analyzed using the <u>geometry</u> of $F^\times \curvearrowright F$:

$$C_c^\infty(F^\times) \hookrightarrow S(F)$$

$$\Updownarrow$$

$$0 \longrightarrow \underbrace{S(F)'_0}_{\text{dist. supp} \subseteq 0} \longrightarrow S(F)' \longrightarrow C_c^\infty(F^\times)' \longrightarrow 0$$

$$\Downarrow \text{ apply } (-)_\omega$$

$$0 \longrightarrow S'(\omega)_0 \longrightarrow S'(\omega) \longrightarrow C_c^\infty(F^\times)'(\omega)$$

Set $\omega_s := |\cdot|^s$, $s \in \mathbb{C}$.

We can obtain natural distributions in $S'(\omega\omega_s)$ via rational

$$f \longmapsto Z(s, \omega; f) := \int_{F^\times} f(x)\, \omega\omega_s(x)\, d^\times x$$

where ω is <u>unitary</u> and the RHS cvg for $\operatorname{Re} s > 0$. In this range, we get

$$\underset{\neq 0}{Z(s, \omega)} \in S'(\omega\omega_s).$$

How do L-fns appear? As the renormalizing terms which make $Z(s, w)$ entire; i.e.

$$\frac{Z(s,w)}{L(s,w)} \text{ is entire, } \forall s.$$

How does the local fnl equation / ε-factor appear? As the constants relating $\widehat{Z(s,w)}$ (the Fourier transform) to $Z(1-s, w^{-1})$.

See Tate's thesis and Kudla's beautiful article.
