Number theory - Diophantine equations

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1 General tricks

1.1 Simon's Favorite Facotring Trick

Simon's factoring trick comes from the identity

$$(x+a)(y+b) = xy + ax + by + ab.$$

When presented with an expression such as $xy + x + 3y = 15$, we can add a constant to both sides to factor the left hand side. For instance, the previous equation becomes $(x+3)(y+1) = 18$. This trick actually came up on a recent Putnam.

Example 1

Find all positive integer solutions to

$$\frac{2020}{xyz} - \frac{1}{xy} - \frac{1}{xz} - \frac{1}{yz} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

Solution. Multiplying through by xyz, we see the given equation is equivalent to

$$xyz - xy - yz - xz + x + y + z = 2020.$$

Using the factoring trick, we obtain

$$(x-1)(y-1)(z-1) = 2019.$$

WLOG, suppose $x \geq y \geq z$. Since $2019 = 3 \times 673$ is the prime factorization of 2019, we find the solutions $(2020, 2, 2)$, $(674, 4, 2)$. The rest of the solutions are obtain by permuting the entries of these solutions. □

1.2 Problems

1. Let m and n be integers such that $m^2 + 3m^2n^2 = 30n^2 + 517$. Fin $3m^2n^2$.

2. **(Putnam 2018/A1)** Find all pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

3. Let p and q be prime numbers. Find all positive integers x and y for which

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{pq}.$$

4. Find all nonnegative integer solutions to

$$x + y + z + xyz = xy + yz + xz + 2.$$

5. Let N be a positive integer. There are exactly 2005 ordered pairs (x, y) of positive integers satisfying

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}.$$

   Show that N is a perfect square.

1.3 Modular Arithmetic

If an equation can be solved in the integers, then it can be solved modulo any number. So often times, information can be gained by reducing an equation modulo some (convenient) number. For instance, reducing mod a prime dividing a coefficient can eliminate a term and give information about the remaining terms.

One of the little miracles of life is that squares behave very nicely mod 4, 8, and 16. When dealing with equations involving squares, it is sometimes beneficial to look at the equation mod one of these powers of 2.

Example 2

Show the equation $14x^2 + 15y^2 = 7^{1990}$ has no solutions in nonnegative integers x and y.

Solution. Reduce mod 8 to have

$$6x^2 + 7y^2 \equiv 1 \pmod 8.$$

Note $a^2 \equiv 0, 1, 4 \pmod 8$ - that is a square must be either 0, 1 or 4 modulo 8. Testing all 8 possible pairs for x^2 and y^2 in the above equation shows the equation cannot be solved mod 8 hence the original equation cannot be solved in the integers. □

1.4 Problems

6. Determine all nonnegative integral solutions $(n_1, n_2, \ldots, n_{14})$ if any, apart from permutations, of the Diophantine equation $n_1^4 + n_2^4 + \ldots + n_{14}^4 = 1599$.

7. Prove that the equation

$$(x+1)^2 + (x+2)^2 + (x+3)^2 + \ldots + (x+99)^2 = y^z$$

   is not solvable in integers x, y, z when $z > 1$.

8. Find all pairs of integers such that $x^3 - 4xy + y^3 = -1$.

9. **(Putnam 2001/A5)** Prove that there are unique positive integers a and n such that $a^{n+1} - (a+1)^n = 2001$.

1.5 Fermat's Infinite Descent

The method of infinite descent is an argument by contradiction. If an equation has a solution in the positive integers, then it has a solution with a "minimal" value. The infinite descent shows that if there is a solution, then there must be a "smaller" one and so there cannot be any nontrivial solution to the equation.

Example 3

Show $x^3 + 5y^3 = 25z^3$ has no nontrivial solutions in the positive integers.

Solution. Notice that if $z = 0$ then we must have $x = y = 0$. Suppose we have a solution (x, y, z) with $z \neq 0$. Suppose, moreover, that this is the solution with the smallest z value. Looking mod 5 shows x is divisible 5, hence $x = 5x_0$. The equation then becomes

$$25x_0^3 + y^3 = 5z^3.$$

This equation shows y is divisible by 5 hence $y = 5y_0$ and

$$5x_0^3 + 25y_0^3 = z^3.$$

This gives z is divisible by 5, hence $z = 5z_0$ and

$$x_0^3 + 5y_0^3 = 25z_0^3,$$

so (x_0, y_0, z_0) is a solution to the original equation with $z_0 < z$ a contradiction. □

1.6 Problems

10. Find all triples (x, y, z) of positive integers satisfying $x^3 + 3y^3 + 9z^3 - 3xyz = 0$.

11. Find all integer solutions to the equation $x^4 + y^4 + z^4 = 9u^2$.

12. Solve in the nonnegative integers the equation $2^x - 1 = xy$.

2 Linear Diophantine Equations

Theorem 1

Let a, b, c be integers. The equation

$$ax + by = c$$

has integer solutions if and only if $\gcd(a, b)$ divides.

The Euclidean algorithm gives us a way of solving equations of the form $ax + by = c$ when it is possible.

Example 4

Find a solution to the equation $4x + 18y = 32$.

Solution. Applying the Euclidean algorithm, we see $2 = \gcd(18, 4) = 18 - 4 \times 4$. Multiplying through by 16 gives $18(16) + 4(-64) = 32$. □

Theorem 2: Sylvester's theorem

If a and b are relatively prime positive integers, then the greatest c for which

$$ax + by = c$$

is not solvable in *nonnegative integers* is given by $ab - a - b$.

2.1 Problems

13. Let m, n be positive integers. Find a solution to the diophantine equation $(21n+4)x+(14n+3)y = m$.

14. Given a piece of paper, we can cut it into 8 or 12 pieces. Each resulting piece can be cut into 8 or 12 pieces and so on. Can we achieve exactly 60 pieces of paper after some finite number of steps? What about 2020?

3 Quadratic Diophantine Equations

3.1 Pythagorean Triples

A solution (x_0, y_0, z_0) to the diophantine equation

$$x^2 + y^2 = z^2$$

is called a *Pythagorean triple.* A Pythagorean triple is *primitive* if x_0, y_0, z_0 are pairwise relatively prime. Looking $\pmod 4$, we see a primitve Pythagorean triple must have exactly one of x and y be even and the other odd.

Theorem 3

If (x, y, z) is a primitive Pythagorean triple where x is even, there are relatively prime positive integers r and s for which $x = 2rs$, $y = r^2 - s^2$ and $z = r^2 + s^2$ and such that $r > s$ and $r + s$ is odd.

Example 5

Determine if there exists is a primitive Pythagorean triples with one of the legs of length 90.

Solution. We claim no such triangle exists. Indeed, if such a triangle existed, in the above theorem r and s cannot have the same parity, hence one is even. Therefore in a primitve Pythagorean triple, 4 must divide the even leg length, but $4 \nmid 90$. □

3.2 Problems

15. Prove that every Pythagorean triangle has a side whose length is divisible by 5.

16. Determine the sidelengths of a right triangle if they are integers and the product of the legs' length is three times the perimeter.

17. Prove that the following system is not solvable in positive integers

$$\begin{cases} x^2 + y^2 = u^2 \\ x^2 - y^2 = v^2 \end{cases}$$

3.3 Pell's Equation

We now turn to Pell's equation

$$x^2 - Dy^2 = 1,$$

where D positive integer which is not a square. This Diophantine equation can be solved recursively once we have found a solution. We will call a solution (x_1, y_1) to Pell's equation the *fundamental solution* if $y_1 > 0$ is minimal.

Theorem 4

Pell's equation has infinitely many nonnegative integer solutions. Moreover, if (x_1, y_1) is the fundamental solution, then the general solution is given by (x_n, y_n) where

$$x_n + \sqrt{D}y_n = (x_1 + \sqrt{D}y_1)^n.$$

Example 6

Find all all triangles whose side lengths are consecutive integers and whose area is also an integer.

Solution. Suppose the triangle has sides of lengths $n-1, n$ and $n+1$. By Heron's formula, it's area is given by

$$A = \sqrt{\left(\frac{3}{2}n\right)\left(\frac{1}{2}n\right)\left(\frac{1}{2}n+1\right)\left(\frac{1}{2}n-1\right)} = \frac{n\sqrt{3(n^2-4)}}{4}.$$

We see for the area to be an integer, n must be odd, say $n = 2m$, then $A = m\sqrt{3(m^2-1)}$, so we can write $m^2 - 1 = 3r^2$, or equivalently,

$$m^2 - 3r^2 = 1.$$

This is Pell's equation and has fundamental solution $(2, 1)$. If follows that the solutions are given by

$$m_k = \frac{1}{2}\left[(2+\sqrt{3})^n + (2-\sqrt{3})^n\right]$$
$$r_k = \frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right]$$

for $k \geq 1$. That is the required triangles have side lengths $2m_k - 1, 2m_k$ and $2m_k + 1$ for $k \geq 1$. $\square$

3.4 Problems

18. Find two nontrivial solutions to $x^2 - 8y^2 = 1$.

19. Let $t_n = 1 + 2 + 3 + \ldots + n$ denote the n^{th} triangular number. Find all triangular numbers which are perfect squares.

20. **(Putnam 2000/A2)** Prove that there exists infinitely many integers n such that $n, n+1$, and $n+2$ are each the sum of two squares of integers. [For example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

4 Additional Problems

21. **(Putnam 1992/A3)** For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy

$$(x^2 + y^2)^m = (xy)^n.$$

22. **(Putnam 2005/B2)** Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + k_2 + \ldots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \ldots + \frac{1}{k_n} = 1.$$