REDUCIBLE QUADRATICS

A mathematical vignette

Ed Barbeau

The two quadratic equations $x^2 - 5x + 6 = 0$ and $x^2 - 5x - 6 = 0$ both have integer solutions. Indeed $x^2 - 5x + 6 = (x - 2)(x - 3)$ while $x^2 - 5x - 6 = (x - 6)(x + 1)$. It is not too hard by trial and error to find other examples of quadratic equations $x^2 - bx + c = 0$ and $x^2 - bx - c = 0$ with integer coefficients b and c *both* of which have integer solutions.

Students should strive to find as many examples as they can and look for possible patterns. There are different ways that students can proceed to find other examples, and teachers should be prepared to offer the minimum amount of suggestion required to avoid their getting bogged down. The mathematical analysis that follows here is to indicate to the teacher two possible directions and give some formalization of the mathematics involved and under no circumstances should be used as any kind of lesson plan for the class. Let the students create their own path.

To systematize the search, one way to approach the problem is to note that when both $x^2 - bx + c$ and $x^2 - bx - c$ have integer roots, then the discriminants of the two quadratics are squares:

$$b^2 - 4c = u^2 \qquad \text{and} \qquad b^2 + 4c = v^2$$

for some integers u and v. In other words, the squares u^2, b^2 and v^2 are in arithmetic progression with common difference $4c$. One example of three such squares is the triple: 1, 25, 49.

One way to characterize that u^2, b^2 and v^2 are in arithmetic progression is through the equation $u^2 + v^2 = 2b^2$. This can be reduced to Pythagoras' equation by making the substitution $u = x - y$ and $v = x + y$, in which case we find that

$$x^2 + y^2 = b^2.$$

This has the general solution

$$(x, y, b) = k(m^2 - n^2, 2mn, m^2 + n^2),$$

where m and n have opposite parity. This leads to $b = k(m^2 + n^2)$ and $4c = k^2(b - x + y)(b + x - y) = k^2(2n^2 + 2mn)(2m^2 - 2mn) = 4k^2mn(m^2 - n^2)$.

For example, when $(m, n, k) = (2, 1, 1)$, we find that $b = 5$ and $c = 6$. In general, we get the two quadratic polynomials:

$$x^2 - k(m^2 + n^2)x + k^2mn(m^2 - n^2) = (x - km(m - n))(x - kn(m + n))$$

and

$$x^2 - k(m^2 + n^2)x - k^2mn(m^2 - n^2) = (x + kn(m - n))(x - km(m + n)).$$

An alternative pathway into the pattern showing the reltionship between pythagorean triples and values of (b, c) is to begin with the examples: $(b, c) = (5, 6)$, $(13, 30)$, $(17, 60)$, and relate them to the pythagorean triples $(3, 4, 5), (5, 12, 13), (8, 15, 17)$ through $2 \times 6 = 3 \times 4$, $2 \times 30 = 5 \times 12$, $(2 \times 60 = 8 \times 15)$, respectively.

Another way to finding a family of pairs of quadratics is to look at the examples $x^2 - 5x \pm 6$ and $x^2 - 13x \pm 30$ and note the triangular numbers that occur in the roots of the four quadratics: $(2, 3), (-1, 6), (3, 10), (-2, 15)$. We can try to continue the pattern of root pairs to get $(4, 21), (-3, 28)$ which leads us to the quadratics $x^2 - 25x \pm 84$.

More generally, we can move to the root pairs $(t + 1, 1 + 2 + \cdots + 2t) = (t + 1, t(2t + 1))$ and $(-t, 1 + 2 + \cdots + (2t + 1)) = (-t, (t + 1)(2t + 1))$. These are the roots of the respective quadratics

$$x^2 - (2t^2 + 2t + 1)x + t(t + 1)(2t + 1) \qquad \text{and} \qquad x^2 - (2t^2 + 22t + 1) - t(t + 1)(2t + 1).$$