THE CAR, BICYCLE AND BUS

The problem. At 9 am, a car drives from A towards B. At two-thirds of the way there, it passes a bicycle going in the opposite direction. When the car arrives at B, a bus leaves towards A and arrives at A at 11 am. At three-fifths of the way from B to A, the bus passes the bicycle. All vehicles move at their own constant rates. When does the bicycle arrive at A.

Comments. This problem seems to be indeterminate, since the time at which the car arrives at and the bus leaves B is not specified. We may suspect that the arrival time of the bicycle at A is similarly variable.

The length of the road from A to B can be divided into three parts: $2/5$ from A to C, where the bus overtakes the bicycle; $4/15$ from C to D, where the car and bicycle pass; and $1/3$ from D to B.

One approach to this problem might be to look at special cases. Three in particular comes to mind.

(a) The car and the bus travel at equal speed, so that they meet at 11 am and the problem in effect becomes a two-vehicle problem with the car/bus travelling from A and B and back. Position C on the path occurs at 9:40 and position D at 10:36, at which time the bicycle has to traverse the remaining $2/5$ of the road.

(b) The car travels infinitely fast, so that the bus leaves B and the bicycle leaves position D at 9 am and the bus passes the bicycle at 11:12 am.

(c) The bus travels infinitely fast, so that the car arrives at B at 11 am and passes the bicycle at 10:20 am. At 11 am, the bicycle is at position C.

In all three cases, the arrival time of the bicycle at B turns out to be noon, so one might suspect that something is up.

The algebraic approach. Suppose that the car reaches B and the bus leaves at $15x$ minutes after 9 am. It passes the bicycle after $10x$ minutes. The bus overtakes the bicycle after

$$15x + \tfrac{3/5}{(}120 - 15x) = 72 + 6x$$

minutes. Thus the bicycle covers $4/15$ of the route from B to A in

$$(72 + 6x) - 10x = 72 - 4x$$

minutes.

(i) The bicycle covers the distance from D to A, $2/3$ of the route, in

$$\frac{2/3}{4/15}(72 - 4x) = 180 - 10x$$

minutes, so that it arrives at A

$$10x + (180 - 10x) = 180$$

minutes after 9 am.

(ii) The bicycle covers the distance from C to A, $2/5$ of the route, in

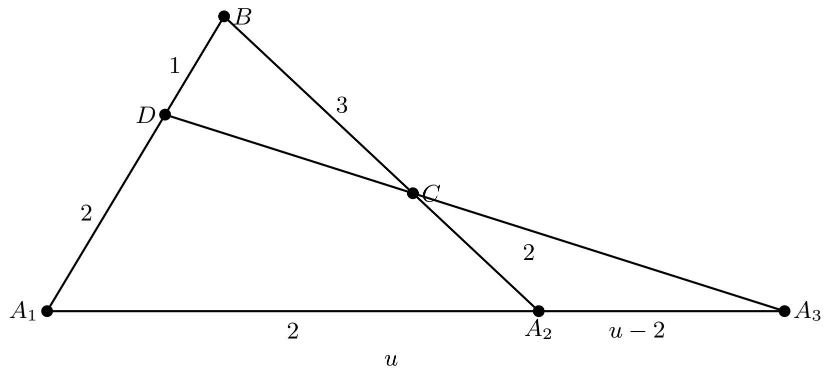
$$\tfrac{3}{2}(72 - 4x) = 108 - 6x$$

minutes, so that it arrives at A

$$(72 + 6x) + (108 - 6x) = 180$$

minutes after 9 am.

A little geometry. Is there any way we can understand why the arrival time of the bicycle is invariant, rather than it just being a happenstance from the algebra? To provide some insight, we can sketch the situation of a graph where the $x-$axis measures the time and the $y-$axis the distance from A. This is the result:



The diagram suggests that Menelaus theorem may be involved here. If we let u be the length of A_1A_3, then according to this theorem

$$\frac{2}{3} \times \frac{1}{2} \times \frac{u}{u-2} = 3,$$

which implies that $u = 3$.