INTEGER VALUES OF A RATIONAL FUNCTION

A mathematical vignette

Let

$$f(x,y,z) = \frac{x^2+y^2+z^2}{xy+yz+zx},$$

where x, y, z are integers, at least two of which are nonzero. We will investigate which integer values k are assumed by this rational function. This is equivalent to determining values of k for which the diophantine equation

$$x^2+y^2+z^2 = k(xy+yz+zx) \tag{1}$$

has a solution. Because of its homogeneity, if it does have a solution, then it must have one for which the greatest common divisor of x, y and z is 1; call such a solution *basic*.

Exercise 1. Determine solutions, if any, for $f(x,y,z) = k$ when k is equal to -2, -1, 0 and 1.

Exercise 2. Show that $f(x,y,z)$ can never be a multiple of 4.

Exercise 3. Show that $f(x,y,z)$ can never be congruent to 3 modulo 4.

Exercise 4. (a) Show that (1) is equivalent to each of

$$(x+y+z)^2 = (k+2)(xy+yz+zx) \tag{2}$$

and

$$k(x+y+z)^2 = (k+2)(x^2+y^2+z^2). \tag{3}$$

(b) Suppose that $k+2 = m^2n$ where n is divisible by no square except 1. Prove that, if $f(x,y,z) = k$, then $x^2+y^2+z^2$ is divisible by the least common multiple of k and n.

Exercise 5. Suppose that $f(x,y,z) = f(u,v,w) = k$. Show that in the ratio $(xy+yz+zx) : (uv+vw+wv)$ reduced to lowest terms, the antecedent and consequent are both squares. In particular, if there is a solution for which $x^2+y^2+z^2 = \pm k$, then, for each solution, (u,v,w), $|uv+vw+wu|$ is a square.

Exercise 6. Prove that $f(x,y,z)$ can never assume a value k that is either a positive multiple of 3 or negative integer k for which $|k| \equiv 1 \pmod 3$.

Exercise 7. Prove that $f(x,y,z)$ cannot assume a value $k = 4^r m^2 - 2$ where $r \geq 2$ and m is odd.

Exercise 8. Does the equation $f(x,y,z) = -3$ have a solution?

Exercise 9. List all the values of k for which $-10 \leq k \leq 10$ that have not been ruled out as values of $f(x,y,z)$ by the foregoing exercises.

Exercise 10. (a) Write equation (1) as a quadratic in x and determine its discriminant in terms of k, y, z. For which values of k does $f(x, y, z) = k$ have infinitely many basic solutions?

(b) If $f(x, y, z) = k$ has a solution and $k+2 = m^2 n$, where n is square free, show that -3 must be a square (mod p) for every odd prime divisor $p > 3$ of n. Use the law of quadratic reciprocity to show that $p \equiv 1 \pmod 3$.

Exercise 11. Determine infinitely basic triples (x, y, z) for which $f(x, y, z) = 2$.

Exercise 12. Determine integer values assumed by $f(x, y, z)$ when x, y, z are in arithmetic progression.

Exercise 13. One way to find integer values of $f(x, y, z)$ is to guarantee that $xy + yz + zx$ is equal to 1 or -1, or some other small value. Provide some examples.

Exercise 14. (a) Suppose that we have a solution for $f(x, y, z) = k$ for which the quadratic equation for x in terms of y, z, k has a double root. Then we would have $2x = k(y + z)$. Investigate the possibility that $x = k$.

(b) In the situation of (a), with $(x, y, z) = (k, y, z)$ and k a double root, suppose instead that $f(x, y, z) = 2k$.

Exercise 15. Investigate which values of $k < 100$ for which the only prime divisors of $k + 2$ are 2 and 3 that can be assumed by $f(x, y, z)$.

Hints and comments for the exercises.

Exercise 1. When $k = -2$, the equation is equivalent to $(x+y+z)^2 = 0$. When $k = \pm 1$, it is equivalent to

$$(x \mp y)^2 + (y \mp z)^2 + (z \mp x)^2 = 0.$$

Exercise 2. Let $f(x, y, z)$ be a multiple of 4. Since $k + 2$ is even but not a multiple of 4, $x^2 + y^2 + z^2$ must be even. But then $x + y + z$ is even as well. so that $x^2 + y^2 + z^2 \equiv 0 \pmod 4$. However, for any basic solution, at least one of x, y, z is odd and $x^2 + y^2 + z^2 \equiv 1, 2, 3 \pmod 4$.

Exercise 3. Let (x, y, z) be a basic solution of $f(x, y, z) = k$, where $k \equiv 3$ (mod 4). Then if exactly one of (x, y, z) is odd, then the left side of (1) is odd and the right is even. If exactly two of (x, y, z) are odd, then the left side is even and the right odd. Finally, if all three of (x, y, z) are odd, then the left side is congruent to 3 (mod 4) while the right side is congruent to $3 \times 3 \equiv 1 \pmod 4$.

The possibility that $k \equiv 1 \pmod 4$ remains. When x, y, z are all odd, $x^2 + y^2 + z^2 \not\equiv 7 \pmod 8$, so that, when $k \equiv 1$, then $xy + yz + zx \equiv 3$, and when $k \equiv 5$, then $xy + yz + zx \equiv 7$, modulo 8.

Exercise 6. If $k < 0$ and $|k| \equiv 1 \pmod 3$, then $k \leq -3$ and

$$|k+2| = -k-2 = |k|-2 \equiv 1-2 \equiv 2$$

modulo 3. In any case $|k|+2$ is congruent to 2 modulo 3, and therefore has a prime divisor p congruent to 2 that divides it to an odd power. Since p divides $(x+y+z)^2$, it must do so to an even power. Hence p must be a divisor of

$$4(xy+yz+zx) = 4[xy+z(y+x)] \equiv 4[xy-(y+x)^2] = (2y+x)^2 + 3x^2.$$

We may assume that x is not a multiple of 3. Then -3 must be a square, modulo p. However (using the Legendre symbol and the quadratic reciprocity law),

$$\left(\frac{-3}{p}\right) = (-1)^{(p-1)/2}(-1)^{(p-1)/2}\left(\frac{p}{3}\right) = -1,$$

from which -3 is not a square modulo p. This yields a contradiction. This was the argument in the solution of Ibrahim Aghazade, a student at ADA University in Baku, Azerbaijan. .

Exercise 7. (x,y,z) must have the form $(2^r u, 2^r v + n, 2^r w - n)$ for some odd integer n. Substituting this into the equation $(x+y+z)^2 = 4^4 m^2(xy+yz+zx)$ leads to $(u+v+w)^2 \equiv -m^2 n^2 \equiv -1 \pmod 4$, which is impossible.

Exercise 8. One can easily guess the solution $(x,y,z) = (-1,1,1)$, from which others can be found. The equation is equivalent to

$$(x^2+y^2+z^2) + 3(xy+yz+zx) = 0.$$

For a basic solution, at least one of x, y and z is odd. If any are even, then the left side is odd. Hence we may write $(x,y,z) = (2u+1, 2v+1, 2w+1)$ for some integers u, v, w. Then we obtain

$$(u^2+v^2+w^2) + 4(u+v+w) + 3(uv+vw+wu) + 3 = 0.$$

If we set $w = 0$, then we get the following quadratic in u in terms of v:

$$u^2 + (4+3v)u + (v^2+4v+3) = 0.$$

Its discriminant is $5v^2+8v+4$, which assumes a square value when $v = -1, 0, 5$. This leads to

$$f(-1,1,0) = f(-5,1,1) = f(-5,11,1) = f(-31,11,1) = -3.$$

We can fix two of the variables in equation (1) and regard it as a quadratic in the third, and in this way, generate an infinite set of solutions.

Exercise 9. $-6, -3, -2, 2, 5, 10$.

Exercise 10. The quadratic equation E_k in x

$$x^2 - k(y+z)x + (y^2 - kyz + z^2) = 0$$

has discriminant D_k equal to

$$\begin{aligned} k^2(y^2+2yz+z^2) - 4(y^2-kyz+z^2) &= (k^2-4)(y^2+z^2) + 2k(k+2)yz \\ &= (k+2)[(k-2)(y^2+z^2+yz)] + (k+2)^2 yz. \end{aligned}$$

Suppose that $p \geq 5$ is a prime divisor to an odd power of $k+2$. Since D_k must be square, it is a multiple of p^2. Since p does not divide $k-2$, p must divide y^2+z^2+yz, and therefore it divides

$$4(y^2+z^2+yz) = (2y+z)^2 + 3z^2.$$

Since we can arrange that $z \neq 0$, -3 must be a square (mod p). This can happen only if $p \equiv 1 \pmod 6$. (When $p \equiv 5 \pmod 6$, then $y^3 \equiv z^3$ if and only if $y \equiv z$.)

For example, when $k = 5$ and $(y,z) = (3,5)$, $y^2+z^2+yz = 49 = 7^2$. This puts us on track for a solution. Indeed, $D_5 = 21 \times 34 + 70 \times 15 = 1764 = 42^2$ and we find that $f(-1,3,5) = 5$.

If (1) has a solution with $k \neq 1$, then it has a solution with $y \neq z$. Since gcd $(x,y,z) = 1$, x, y, z are pairwise coprime. If $D = 0$, and $k \neq -2$, we can start with a solution and arrange y and z so that it has two distinct solutions x, and we can similar continue spawning new solutions.

Exercise 11. When $k = 2$, the discriminant D_2 is equal to $16yz$, which is square only if yz is square. Since y and z are coprime, we may take $y = u^2$ and $z = v^2$, and the equation E_2 becomes

$$0 = x^2 - 2(u^2+v^2) + (u^4 - 2u^2v^2 + v^2) = [x-(u+v)^2][x-(u-v)^2].$$

We get infinitely many solutions

$$(x,y,z) = ((u \pm v)^2, u^2, v^2),$$

where u and v are arbitrary integers.

We can form any sequence of integers satisfying the recursion $m_{n+1} = m_n + m_{n-1}$ and find that $(x,y,z) = (m_{n-1}^2, m_n^2, m_{n+1}^2)$ satisfies $f(x,y,z) = 2$. In particular $(x,y,z) = (f_{n-1}^2, f_n^2, f_{n+1}^2)$ satisfies the equation where $\{f_n\}$ is the bilateral Fibonacci sequence. In this case, determine the values of $x^2+y^2+z^2$ and $xy+yz+zx$.

Another sequence for example is

$$\{\ldots, 7^2, 5^2, 2^2, 3^2, 1^2, 4^2, 5^2, 9^2, 14^2, 23^2, \ldots.\}$$

It is interesting that the basic solutions in this case consist of squares.

Exercise 12. Suppose that $(x,y,z) = (v-u, v, v+u)$ for some integers u and v. Then

$$(k+2)u^2 = 3(k-1)v^2.$$

Since each side is divisible by 3, we assume that $k = 3l - 2$, whereupon $lu^2 = 3(l-1)v^2$. Taking $l = m^2$, we get $m^2u^2 = (3m^2-3)v^2$. Thus, $3m^2 - 3$ must be a square, $(3n)^2$ and so

$$m^2 - 3n^2 = 1.$$

This Pell's equation has infinitely many solutions (m_i, n_i) where $(m_0, n_0) = (1,0)$, $(m_1, n_1) = (2,1)$ and, for each integer i,

$$m_{i+1} = 4m_i - m_{i-1} \qquad \text{and} \qquad n_{i+1} = 4n_i - n_{i-1}.$$

(Solutions are $(1,0), (2,1), (7,4), (26,15), (97,56), \ldots.$)

When $k = 3m^2 - 2$ and $m^2 - 3n^2 = 1$, we get the solution

$$(x, y, z) = (m - 3n, m, m + 3n).$$

Thus

$$x^2 + y^2 + z^2 = 3m^2 + 18n^2 = 3(m^2 + 6n^2) = 3(m^2 + 2m^2 - 2) = 3(3m^2 - 2) = 3k,$$

and

$$xy + yz + zx = 3m^2 - 9n^2 = 3(m^2 - 3n^2) = 3.$$

When $m = 2$, $k = 10$, and we get the solutions:

$$(x, y, z) = (-1, 2, 5), (2, 5, 71), (-1, 5, 38), \ldots$$

from which we can generate an infinite supply.

When $m = 7$, $k = 145$, and we get the solution $(x, y, z) = (-5, 7, 19)$.

Exercise 13. There are several ways one can look at this. One approach is to look at situations for which $xy + yz + zx = \pm 1$. When $xy + yz + zx = 1$, then $x^2 + y^2 + z^2 = k$ and

$$x = -\left(\frac{yz - 1}{y + z}\right).$$

When $xy + yz + zx = -1$, then $x^2 + y^2 + z^2 = -k$ and

$$x = -\left(\frac{yz + 1}{y + z}\right).$$

To make x an integer in both cases, we can make $z = 1 - y$ or $z = -(y + 1)$.

In particular, we can take $(y, z) = (2r - 1, 2r + 1)$ so that $yz + 1 = 4r^2$ and $x = -2$.

Alternatively, we can note that

$$(y + x)(z + x) = x^2 \pm 1,$$

and choosing values of y and z that will make this product valid.

Interestingly, the Fibonacci numbers make another appearance. Let $\{f_n\}$ be the Fibonacci sequence with $f_0 = 0$, $f_1 = 1$. When $(x, y, z) = (-f_{n-1}, f_n, f_{n+1})$, then

$$x^2 + y^2 + z^2 = 4f_n^2 + 2(-1)^n$$

and

$$xy + yz + zx = f_n^2 - f_{n-1}f_{n+1} = (-1)^{n+1}.$$

This solution corresponds to $k = (-1)^{n+1}[4f_n^2 + 2(-1)^n]$.

Exercise 14. (a) We must have $y + z = 2$, so we may assume that $y = -(r - 1)$ and $z = r + 1$. If $(x, y, z) = (k, -(r - 1), r + 1)$ is a solution, then

$$0 = k^2 - (r^2 - 1)k + 2(r^2 + 1) = (k - (r^2 + 1))(k - 2).$$

(b) In this situation, we take $y + z = 1$, so we can assume that $y = -(r - 1)$ and $z = r$, and then solve for k.

Exercise 15. We have to investigate $k = 22, 25, 34, 46, 70$ and 94. ($k = 14, 62$ have already been dealt with.) Since 2 divides 24, 72, 96 to an odd power, then $x + y + z$ is divisible by 4 and $xy + yz + zx$ is even. Therefore, up to order, $(x, y, z) \equiv (0, 1, 3) \pmod 4$, which would make $xy + yz + zx$ odd. Hence $k = 22$, 70 and 94 are not possible.

If $k = 25$, then

$$(x + y + z)^2 = 27(xy + yz + zx).$$

Therefore $x + y + z$ is divisible by 9 and $xy + yz + zx$ is divisible by 3. Therefore, modulo 9, up to order,

$$(x, y, z) = (9u \pm 1, 9v \pm 1, 9w \pm 7), (9u \pm 1, 9v \pm 4, 9w \pm 4), (9u \pm 2, 9v \pm 2, 9w \pm 5)$$

for some integers u, v, w. Substituting any of these into the equation and dividing by 81 yields

$$(u + v + w \pm 1)^2 \equiv 2$$

(mod 3). Since this is not possible, there is no solution.

If $k = 34$, then $(x + y + z)^2 = 36(xy + yz + zx)$, so that $x + y + z = 0 \pmod 6$. Therefore $(x, y, z) \equiv (0, 1, 5), (1, 2, 3) \pmod 6$. In the first instance, if $(x, y, z) = (6u, 6v + 1, 6w - 1)$, then $(u + v + w)^2 = 36(uv + vw + wu) + 6(w - v) - 1$. But this is not possible, since -1 is not a quadratic residue, modulo 6.

If $(x, y, z) = (6u + 1, 6v + 2, 6w + 3)$, then $(u + v + w)^2 \equiv 11 \pmod 6$, which again is not possible. Therefore $k = 34$ is not a possible value.

In the case of $k = 46$,

$$(x + y + z)^2 = 16 \times 3(xy + yz + zx),$$

so that $x + y + z$ is divisible by 4. Therefore $(x, y, z) \equiv (0, 1, 3) \pmod 4$. If $(x, y, z) = (4u, 4v + 1, 4w - 1)$, then

$$(u + v + w)^2 = 3[(uv + vw + wu - v + w) - 1].$$

Since $u + v + w$ is divisible by 3, $(u, v, w) \equiv (1, 1, 1), (0, 1, -1) \pmod 3$ up to order. Therefore, in both cases,

$$uv + vw + wu - v + w - 1 \equiv -1$$

(mod 3). But then 3^2 divides the left side, but not the right side. Therefore there is no solution when $k = 46$.

Tables of possibly acceptable values

We have the following families:

k	Solution of $f(x, y, z) = k$
$r(r + 1)(r^2 + r + 4) + 2 = (r^2 + r + 2)^2 - 2$	$(-r, r + 1, r^2 + r + 1)$
$-[r^2(r + 1)^2 + 2]$	$(-r, r + 1, r^2 + r - 1)$
$-(r^2 + 2)$	$(-1, 1, r)$
$-(9r^2 + 2)$	$(-r, 2r - 1, 2r + 1)$
$r^2 + 1$	$(-(r - 1), r + 1, r^2 + 1)$
$4r(r - 1) + 2$	$(-(r - 1), r, 2r(r - 1) + 1)$

Tabulation of some numerical solutions.

Positive values of k are not possible values of $f(x,y,z)$ when k is a multiple of 3, $k \equiv 0,3 \pmod 4$, $k = 4^k m^2 - 2$, or $k+2$ is divisible to an odd power by a prime congruent to 2 (mod 3). Negative values of k are not possible when $|k|-2 \equiv 2$ (mod 3), $k \equiv 0,3 \pmod 4$ or $|k|-2$ is divisible to an odd power by a prime congruent to 2 (mod 3). The values of k that are candidates are listed below, along with solutions when available.

k	Some solution of $f(x,y,z) = k$
-102	$(-1,1,10)$
-83	$(-1,1,9),(-3,5,7),(-993,5,7)$
-66	$(-1,1,8)$
-51	$(-1,1,7)$
-38	$(-1,1,6),(-2,3,5),(-3,2,43),(-302,3,5)$
-27	$(-1,1,5)$
-18	$(-1,1,4)$
-11	$(-1,1,3),(-3,1,23),(-43,1,3)$
-6	$(-1,1,2),(-2,1,7),(-17,1,2)$
-3	$(-1,1,1),(-5,1,1),(-5,1,11),(-11,5,19),(-31,1,11)$
-2	$(-1,0,1)$
2	$(0,1,1),(1,1,4),(1,4,9),(1,9,16)$
5	$(-1,3,5),(-1,5,17),(-1,17,75),(3,5,41)$
10	$(-1,2,5),(-1,5,38),(2,5,71),(5,71,758)$
14	$(-1,2,3),(-1,2,11),(-1,3,26),(2,3,71)$
17	$(-3,5,17)$
26	$(-2,3,13)$
29	
37	$(-5,7,37)$
41	
46	
50	$(-3,4,25)$
61	
62	$(-2,3,7),(-2,3,55),(-2,7,307),(3,7,622)$
65	$(-7,9,65)$
70	
73	
74	
77	
82	$(-4,5,41)$
85	
89	
94	
98	$(-3,5,8),(-3,5,188),(5,8,1277)$
101	$(-9,11,101)$