RUNNERS ON A TRACK

A mathematical vignette

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1. There is a well-known mathematical fact that, given any two people who live sufficiently long, the age last birthday (number of complete years lived) of the elder is twice the age last birthday of the younger for periods of time totalling one year. Another way of forumlating this is to imagine two runners on a circular track. One begins to run around the track and the second begins to run, starting at any point on the track, at a later time. Both run at the same constant speed. Then for a distances totally the length of one lap, the earlier runner has completed twice as many laps as the later one.

2. Now imagine that the two runners, A and B, start at the same point on the track. The second runner, B, begins later than the first A, but starts when the first one crosses the starting point, so that the two runners now are running side by side. Now let n be any positive integer. How many laps must the A cover before he is joined by the B so that, for each integer k with $2 \leq k \leq n$, the A has completed exactly k times as many laps as the second one.

If $n = 2$, this can always be achieved. If A runs r laps before being joined by B, then A has always run r more laps than B. So when B has run r laps, A has run $2r$ laps.

Let let $n = 3$, so that we want to have a situation where A has completed twice as many laps as B and another situation when B has completed three times as many laps as A. If B starts after A has completed d laps, then there are always d completed laps between them. If after t laps after B begins, A has accumulated three times as many laps, then $d + t = 3t$, so that $d = 2t$ must be even.

More generally, if we want a situation in which A has completely r times as many laps as B, then we are led to $d + t = rt$, from which d must be a multiple of $r - 1$. On the other hand, suppose that $d = t(r - 1)$ for some integer t. Then after $d/(r - 1) = t$ further laps, B has completed t laps and A has $d + t = rt$ laps.

We can conclude that, A has completely exactly d laps when B starts out, and if at some future time A has completed exactly r times as many laps as A for $1 \leq r \leq n$, then it is necessary and sufficient that d is a multiple of $k = r - 1$ for $1 \leq k \leq n - 1$.

3. Now suppose that we have the two runners as before, but this time B starts out when A has completed exactly $d + \frac{1}{2}$ laps. In this case, for half of a lap of each curcuit, A has completed d more laps than A, while for the remaining half-lap, A has completed $d + 1$ more laps.

Then, given any nonnegative integer n, it is the case that A will have completed r times as many laps as B for $1 \leq r \leq n + 1$ if and only if either d or $d + 1$ is divisible by $k = r - 1$ for $1 \leq k \leq n$. Then we can ask about the values of d that satisfy this. In particular, $d(d+1)$ is a multiple of the least common multiple $f(n)$ of the first n positive integers.

In the table below, we list values of n; $f(n)$, the least common multiple of the first n natural numbers; pairs $(d, d+1)$ for which one of the numbers is divisible by k for each k satisfying $1 \leq k \leq n$; $g(n)$, the product $d(d+1)$; and $g(n)/f(n)$.

n	$f(n)$	$(d, d+1)$	$g(n)$	$g(n)/f(n)$
3	6	(2, 3)	6	1
4	12	(3, 4)	12	1
	12	(8, 9)	72	6
5	60	(15, 16)	240	4
	60	(20, 21)	420	7
	60	(24, 25)	600	10
	60	(35, 36)	1260	21
	60	(39, 40)	1560	230
	60	(44, 45)	1980	330
6	60	(24, 25)	600	10
	60	(35, 36)	1260	21
7	420	(35, 36)	1260	3
	420	(84, 85)	7140	17
	420	(119, 120)	14280	34
	420	(300, 301)	90300	215
	420	(335, 336)	112560	268
	420	(384, 385)	147840	352
8	840	(119, 120)	14280	17
	840	(335, 336)	112560	134
	840	(384, 385)	147840	176
	840	(455, 456)	207480	247
	840	(504, 505)	254520	303
	840	(720, 721)	519120	618
9	2520	(279, 280)	78120	31
10	2520	(279, 280)	78120	31
11	27720			

As an example, consider the row corresponding to $n = 7$ and $(d, d+1) = (35, 36)$ and its relevance to the situation where A has completed k times as many laps as B for $1 \leq r \leq 8$ over a space of a half-lap.

In this table, we list the divisor k of either d or $d + 1$, the particular number in the pair it divides, the number of laps completed by A and B when A has completed $k + 1$ as many laps as B. We have also included multiples greater than 7.

k	Multiple of k	Completed laps
2	36	(54, 18)
3	36	(48, 12)
4	36	(45, 9)
5	35	(42, 7)
6	36	(42, 6)
7	35	(40, 5)
9	36	(40, 4)
12	36	(39, 3)
18	36	(38, 2)

Since $d(d+1)$ has to be a multiple of $f(n)$, we can check those multiples that are products of consecutive integers. For example,

$$13 \times 840 = 10920 = 104 \times 105;$$

$$17 \times 840 = 14280 = 119 \times 120;$$

$$31 \times 2520 = 78120 = 279 \times 280;$$

$$7 \times 27720 = 194040 = 440 \times 441.$$

These often yield cases for much smaller values of n.

4. Given any positive integer n, it is always possible to find a pair of consecutive integers such that, for each k with $1 \leq k \leq n$, k will divide one of them. To see this, consider the least common multiple $f(n)$ and write it as the product ab where a and b are coprime positive integers. Then there are positive integers x and y for which $ax - by = 1$ and positive integers u and v for which $au - bv = -1$, Then $(d, d+1) = (by, ax), (au, bv)$ will do the job.

Suppose that $(d', d'+1)$ is a second pair of integers; then $d' - d$ must be a multiple of $f(n)$. Hence, for each factorization ab of $f(n)$, there is a unique pair with $1 \leq d < f(n)$ for which a divides d and b divides $d+1$.

An ancillary question is the representation of $f(n)$ as a product of pairwise coprime factors (exceeding 1). We have:

$$\begin{aligned}
f(3) &= 2 \times 3; \\
f(4) &= 3 \times 4; \\
f(5) &= 3 \times 4 \times 5; \\
f(6) &= 5 \times 12; \\
f(7) &= 5 \times 7 \times 12; \\
f(8) &= 5 \times 7 \times 24; \\
f(9) &= 5 \times 7 \times 72; \\
f(10) &= 7 \times 720; \\
f(11) &= 7 \times 11 \times 720.
\end{aligned}$$

5. Sometimes multiples of $f(n)$ can be written as products of integers that differ by 2 or 3. Here are some examples.

$$\begin{gathered}840 = 28 \times 30; 2 \times 840 = 1680 = 40 \times 42;\\ 2 \times 2520 = 5040 = 70 \times 72; 13 \times 2520 = 32760 = 180 \times 182;\\ 25 \times 2520 = 63000 = 250 \times 252;\\ 10 \times 27720 = 277200 = 525 \times 528.\end{gathered}$$

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This suggests investigation of pairs $(d, d+2)$ for which every odd integer between 1 and n, inclusive, divides one or other of d and $d+2$.