GREGARIOUS AND RECLUSIVE TRIPLES

A mathematical vignette

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PART A: A VISIT TO THE LAND OF FIBONACCI

This vignette will introduce the reader to a very prolific area of mathematical investigation that is accessible to both secondary students and their teachers. While the basic problem is quite old, there are likely more interesting discoveries to be made.

1. Triples, products and squares

For the triple of numbers $(1, 3, 8)$, the product of any pair of them is one less than a square. Similarly, the product of any two numbers in the triple $(1, 2, 5)$ is one more than a square. You may recognize the numbers in these triples as alternate terms of the Fibonacci sequence, defined by the recursion $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for each integer n. The terms with nonnegative even indices are

$$0, 1, 3, 8, 21, 55, 144, 377, 987, \ldots;$$

we find that for each three consecutive terms (x, y, z) in this sequence $xy+1$, $xz+1$ and $yz + 1$ are all squares. Likewise, for each three consecutive terms (x, y, z) in the sequence of Fibonacci numbers with positive odd indices,

$$1, 2, 5, 13, 34, 89, 233, 610, 1597, \ldots,$$

$xy - 1$, $xz - 1$ and $yz - 1$ are all squares. These are familiar Fibonacci properties.

Define a vector (x, y, z) of three integers to be a $k-$**triple** if $xy + k = c^2$, $yz + k = a^2$ and $zx + k = b^2$ for integers k, a, b, c. We have provided examples of $1-$triples and $(-1)-$triples. Both of these can be embedded in a table of sequences of $k-$triples. In this table, whose $k = 1$ row includes the foregoing Fibonacci $1-$triple:

$k \downarrow n \rightarrow$	-5	-4	-3	-2	-1	0	1	2	3	4	5
-2	54	19	9	2	3	1	6	11	33	82	219
-1	29	10	5	1	2	1	5	10	29	73	194
0	4	1	1	0	1	1	4	9	25	64	169
1	-21	-8	-3	-1	0	1	3	8	21	55	144
2	-46	-17	-7	-2	-1	1	2	7	17	46	119
3	-71	-26	-11	-3	-2	1	1	6	13	37	94
4	-96	-35	-15	-4	-3	1	0	5	9	28	69

Any three consecutive entries in the row labelled k constitute a $k-$triple. Suppose that the nth terms in this row is given by $u(k, n)$. You will observe that for these rows, any three consecutive entries constitute a $k-$ triple,

$$u(k,-2)=-k; \qquad u(k,-1)=-k+1; \qquad u(k,0)=1;$$

and

$$u(k,n+3)=2u(k,n+2)+2u(k,n+1)-u(k,n).$$

The reader is invited to conjecture a general formula for $u(k,n)$ and check out the $k-$triples. (A good place to start is with row $k=0$ and look at the value of $u(k,n)$ as n increases or decreases by 1.)

In a similar way, row $k=-1$ in the table below reproduces the $(-1)-$triples we have already seen.

$k\downarrow n\rightarrow$	-5	-4	-3	-2	-1	0	1	2	3	4	5
-4	164	61	25	8	5	1	4	5	17	40	109
-3	139	52	21	7	4	1	3	4	13	31	84
-2	114	43	17	6	3	1	2	3	9	22	59
-1	89	34	13	5	2	1	1	2	5	13	34
0	64	25	9	4	1	1	0	1	1	4	9
1	39	16	5	3	0	1	-1	0	-3	-5	-16
2	14	7	1	2	-1	1	-2	-1	-7	-14	-41
3	-11	-2	-3	1	-2	1	-3	-2	-11	-23	-66

Let $v(k,n)$ be the nth element in the kth row. In this extract, we note that the kth row consists of $k-$triples, that

$$v(k,-2)=-k+4; \qquad v(k,-1)=v(k,2)=-k+1; \qquad v(k,0)=0; \qquad v(k,1)=-k;$$

and that

$$v(k,n+3)=2v(k,n+2)+2v(k,n+1)-v(k,n).$$

Again, the reader is invited to conjecture a general formula for $v(k,n)$ and check out the occurrence of $k-$triples.

Motivated by the recursion satisfied by $u(k,n)$ and $v(k,n)$, we define the **right associate** of (x,y,z) to be the triple (y,z,w) where $w=2(y+z)-x$, the **left associate** of the triple (x,y,z) to be $(2(x+y)-z,x,y)$ and the **central associate** of (x,y,z) to be $(x,2(x+z)-y,z)$.

A $k-$triple is **gregarious** if all its associates are $k-$triples (with the same value of k). A sequence $\{u_n\}$ satisfying the **gregarious recursion** $u_{n+3}=2u_{n+2}+2u_{n+1}-u_n$ is $k-$**gregarious** if each three consecutive terms constitute a $k-$triple. Each line in the foregoing tables is a gregarious sequence.

A $k-$triple whose associates are not all $k-$triples is said to be **reclusive**. Later, we will find such triples.

Before continuing, we turn to the entries of the foregoing tables, whose entries rely on the terms of the Fibonacci sequence. Let me remind you of properties of the Finonacci sequence:

Exercise 1. Esablish the following Fibonacci identities:

$$\begin{aligned}
f_{2n+2} &= f_{2n-2} + f_{2n} + 2f_{2n-1};\\
f_{2n-2}f_{2n} + 1 &= f_{2n-1}^2;\\
f_{2n+3} &= f_{2n-1} + f_{2n+1} + 2f_{2n};\\
f_{2n-1}f_{2n+1} - 1 &= f_{2n}^2;\\
f_{n+1}f_{n-1} - f_n^2 &= (-1)^n;\\
f_{n+2}f_{n-2} - f_n^2 &= (-1)^{n-1};\\
f_{n+2}f_{n-1} - f_{n+1}f_n &= (-1)^n;\\
f_{n+1}^2 f_{n-1}^2 + f_n^4 &= 2f_{n+1}f_n^2 f_{n-1} + 1;\\
f_{n+2}^2 f_{n-2}^2 + f_n^4 &= 2f_{n+2}f_n^2 f_{n-2} + 1;\\
f_{n+2}^2 f_{n-1}^2 + f_{n+1}^2 f_n^2 &= 2f_{n+2}f_{n+1}f_n f_{n-1} + 1.
\end{aligned}$$

Exercise 2. Prove the following identities:

$$\begin{aligned}
f_{n-1}^2 - 3f_n^2 + f_{n+1}^2 &= 2(-1)^n;\\
f_{n+3}^2 &= 2(f_{n+2}^2 + f_{n+1}^2) - f_n^2;\\
f_{n+2} - 3f_n + f_{n-2} &= 0;
\end{aligned}$$

Exercise 3. Prove the following identities:

$$\begin{aligned}
(f_{n+1}^2 - kf_{n-1}^2)(f_{n+2}^2 - kf_n^2) + k &= (f_{n+2}f_{n+1} - kf_{n-1}f_n)^2;\\
(f_{n+2}^2 - kf_n^2)(f_n^2 - kf_{n-2}^2) + k &= f_n^2(f_{n+2} - f_{n-2})^2.
\end{aligned}$$

Exercise 4. Examination of the foregoing tables gives rise to the conjecture:

$$\begin{aligned}
u(k,n) &= f_{n+2}^2 - kf_n^2;\\
v(k,n) &= f_{n-1}^2 - kf_n^2.
\end{aligned}$$

Prove that $\{u(k,n)\}$ and $\{v(k,n)\}$ are gregarious $k-$sequences.

Exercise 5. Let (x, y, z) be a triple of consecutive entries in the kth row of either of the foregoing tables. What do you observe about the relationship between $xy + k$ and $z - (x + y)$?

Exercise 6. Find other $k-$triples that are not covered by the tables.

PART B: A PLETHORA OF TRIPLES AND QUADRUPLES

2. How to construct lots of $k-$triples.

Exercise 7. Suppose that x, y and c are arbitrary integers. Let $z = x + y + 2c$ and $k = c^2 - xy$. Prove that $xz + k = (x + c)^2$ and $yz + k = (y + c)^2$, so that (x, y, z) is a $k-$triple.

A $k-$triple for which z and k are related in this way is said to be *superbly gregarious* or simply *superb*.

Exercise 8. Prove that the right and left associates of a superb $k-$triple are also superb $k-$triples.

Comment. Note that for the triple $(y, z, 2(y + z) - x)$, the role of c is now played by $(y + c)$. This result allows a simple way of establish that the sequences $\{u(k, n)\}$ and $\{v(k, n)\}$ are $k-$gregarious since it necessary only to find an superbly gregarious consecutive triple in each sequence.

If we permute the terms of (x, y, z) to (x, z, y), we find that $y = x + z - 2(c + x)$ and $xz = [-(c+x)]^2$ and we can embed this triple in another sequence of $k-$triples.

We can look at this construction in three ways. Suppose we are given a triple (x, y, z) and want to know if it is a $k-$triple for some k. If $x + y + z$ is even, then $x + y$ and z have the same parity, and we can take $c = \frac{1}{2}(z - x - y)$.

Suppose we are given an integer pair (x, y) and we want to embed it into a $k-$triple (x, y, z) for some k such that $xy + k$ equal to a given square c^2. Then simply define $z = x + y + 2c$.

Finally, suppose that we are interested in $k-$triples for a specific value of k. Pick any square c^2 and chose x, y such that their product is $c^2 - k$. In this way, for example, we can find any number of $1-$triples. With $c = 5$, we find (1, 24, 35), (2, 12, 24), (3, 8, 21), (4, 6, 20). More generically, we have the infinite families $(1, c^2 - 1, (c + 1)^2 - 1)$, $(c - 1, c + 1, 4c)$, $(2, 2c(c + 1), 2(c + 1)(c + 2))$.

Thus we see that $k-$triples are prolific and many interesting infinite families of such triples can be found. For example:

k	(x, y, z)	(a, b, c)
$r^2 + s^2 + t^2 - 2(rs + st + rt)$	$(2r, 2s, 2t)$	$(-r + s + t, r - s + t, r + s - t)$
$r^2 + s^2 + t^2 - 2(rs + st + rt) - 2r$	$(2r, 2s + 1, 2t + 1)$	$(s + t - r + 1, r - s + t, r + s - t)$

3. How to construct lots of $k-$quadruples.

It is natural to ask whether, for any value of k, there are $k-$*quadruples* of numbers for which the product of any pair plus k is a square. The construction described in Section 2 makes it quite straightforward to answer this in the affirmative. If we extend the triple $(x, y, x+y+2c)$ to the left, we get the quadruple $(x+y-2c, x, y, x+y+2c)$. Since $(x+y-2c, x, y)$ and $(x, y, x+y+2c)$ are $k-$triples, it is necessary only to arrange that

$$(x+y-2c)(x+y+2c)+k = (x+y)^2 - 4c^2 + (c^2 - xy) = (x^2+xy+y^2) - 3c^2$$

is equal to d^2 for some integer d. In other words, we need to find numbers expressible in each of the forms $\phi(x,y) = x^2+xy+y^2$ and $\psi(c,d) = 3c^2+d^2$.

Exericse 9. Prove that $\phi(x,y) = \phi(x+y, -y) = \phi(-x, x+y)$.

Exercise 10. Prove that the forms $\phi(x,y) = x^2+xy+y^2$ and $\psi(c,d) = 3c^2+d^2$ take the same set of integer values, where x, y, c, d are integers. (Hint: given (c,d), let $(x,y) = (c+d, c-d)$. How can you go from (x,y) to a corresponding (c,d)?)

In order to get $k-$quadruples whose entries are distinct, we can exploit the fact the some numbers can be represented by both of the forms $\phi(x,y)$ or $\psi(c,d)$ in several ways, so that we can get numerous examples of $k-$quadruples by using each c with each of the pairs (x,y) involved.

Exercise 11. There are several ways of representing each of the numbers 49, 91 and 133 by $\phi(x,y)$ and $\psi(c,d)$. For each, use all of the possible triples (x,y,c) to construct $k-$triples.

Exercise 12. There are parametric families of $k-$quadruples. Determine $k-$quadruples when $(x,y) = (2r,s), (2r,2s), (2r,2s+1)$, where r and s are arbitrary integers.

Exercise 13. Verify that each of the following are $1-$quadruples:

$$(r-1, r+1, 4r, 4r(4r^2-1));$$
$$(1, r^2-1, r(r+2), 4r(r^3+2r^2-1));$$
$$(r, s^2-1+(r-1)(s-1)^2, s(rs+2), 4r^3s^4+8r^2(2-r)s^3+4r(r-1)(r-5)s^2+4(2r-1)(r-2)s+4(r-1));$$
$$(r, 4(r-1), r-2, 4(2r-3)(2r-1)(r-1);$$
$$(r, s, r+s+2c, 2c(r+c)(s+c)).$$

A formula given by Euler for $1-$quadruples is

$$(x,y,z,w) = (x, y, x+y+2c, 4c(x+c)(y+c)).$$

In the next exercise, we will see that it works for a rather interesting reason. When you try it for general k, there is a wrinkle.

Exercise 14. Experiment with various $1-$ and $(-1)-$ *triples* to see what happens with the triple (x,y,z,w). What happens when $(x,y,z,w) = (1, y, y+3; 8(y+1), (1, y, y+5, 24(y+2)$? Make a conjecture.

4. Reclusive $k-$triples and their families

Exercise 15. Not every $k-$triple generates a succession of $k-$triples when embedded in a sequence satisfying the congenial recurrence. For example, when

$x = y$, there are triples for which (x, x, z) is a $k-$triple, but its right associate $(x, z, x + 2z)$ is not. With $xy + k = c^2$, $zx + k = b^2$, $yz + k = a^2$, we have the examples:

k	(x, y, z)	(a, b, c)
$4r^4 + 8r^3 - 4r + 1$	$(2r + 1, 2r + 1, 2(2r + 1))$	$(2r^2 + 2r + 1, 2r^2 + 2r + 1, 2r^2 + 2r)$
$r^4 - 6r^2s^2 + s^4$	$(2rs, 2rs, 4rs)$	$(r^2 + s^2, r^2 + s^2, r^2 - s^2)$

When $z = x+y$, we can make use of Pythagorean triples to construct $k-$triples. Suppose that we have values k and c for which $xy + k = c^2$. Then we want to find a and b for which $a^2 = yz + k = x^2 + xy + k = x^2 + c^2$ and $b^2 = xz + k = y^2 + c^2$. Thus (x, c, a) and (y, c, b) are both Pythagorean triples sharing the value of a leg. (Such pairs of triples are easy to find; there are any many ways to express c^2 as a difference of squares as c^2 can be factored as a product of two integers of the same parity.) These triples allow us to isolate the values of a, b, c, x, y.

Exercise 16. The three Pythagorean triples $(5, 12, 13)$, $(9, 12, 15)$ and $(35, 12, 37)$ share the term $c = 12$. Using the three pairs of them, arrive at the reclusive $k-$triples $(5, 9, 14)$, $(5, 35, 40)$ and $(9, 35, 44)$ with values of k respectively equal to 99, -31 and -171.

The right associate of $(5, 35, 40)$ is $(35, 40, 145)$ and we note that $35 \times 145 - 31 = 71^2 + 3$, a near miss. This is not the only occurrence of this.

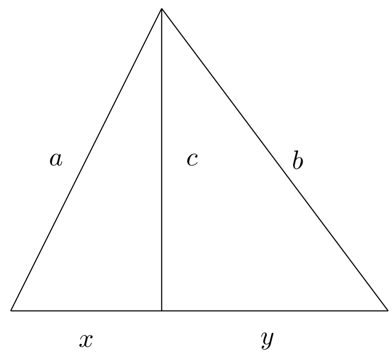
The $k-$triple $(2r+1, r^2(r+1)^2 - 1, r^2(r+1)^2 + 2r)$ with $k = -(2r^5 + r^4 - 4r^3 - 3r^2 - 2r - 1)$ has right associate

$$(r^4 + 2r^3 + r^2 - 1, r^2 + 2r^3 + r^2 + 2r, 4r^4 + 8r^3 + 4r^2 + 2r - 3).$$

We find that

$$\begin{aligned}
(r^4 + 2r^3 + r^2 - 1)(4r^4 + 8r^3 + 4r^3 + 2r - 3) - (2r^5 + r^4 - 4r^3 - 3r^2 - 2r - 1) \\
= (4r^8 + 16r^7 + 24r^6 + 18r^5 + r^4 - 12r^3 - 7r^2 - 2r + 3) \\
+ (-2r^5 - r^4 + 4r^3 + 3r^2 + 2r + 1) \\
= 4r^8 + 16r^7 + 24r^6 + 16r^5 - 8r^3 - 4r^2 + 1 + 3 \\
= (2r^4 + 4r^3 + 2r^2 - 1)^2 + 3 = [2r^2(r+1)^2 - 1]^2 + 3.
\end{aligned}$$

Finding pairs of pythagorean triples with a common leg arise in the determination of Heronian triangles, whose sides and area are all integers. Such triangle can be constructed by pasting together two right triangles that share a common leg, as shown in the diagram.



Exercise 17. (a) We can take as a particular example $x = r - 2$, $y = r + 2$, $a = 2r - 1$, $b = 2r + 1$, where r is an integer that exceeds 2. Show that, if there is a solution in integers, then $c = 3s$ where $r^2 - 3s^2 = 1$.

(b) $r^2 - 3s^2 = 1$ is an example of a Pell's equation which has infinitely many solutions. Determine the solution (r_1, s_1) with the smallest positive integers, and show that, for each n, (r_n, s_n) is a solution where

$$r_n + s_n\sqrt{3} = (r_1 + s_1\sqrt{3})^n.$$

Both the sequences $\{r_n\}$ and $\{s_n\}$ satisfy the same second order recursion; find it. Also show how each of r_{n+1} and s_{n+1} can be written as linear combinations of r_n and s_n.

PART C: FINDING TRIPLES WHEN SOME ELEMENTS ARE KNOWN

5. Further application of Pell's equation.

Once we start with a $k-$triple, (x, y, z), we can generate an infinite family of $k-$triples with the same values of x and y. We will suppose that xy is not a square and that $xy + k = c^2$. Recall that the diophantine Pell's equation $u^2 - (xy)v^2 = 1$ has infinitely many solutions (u, v) in integers.

Exercise 18. Let k, x, y, a and b be integers.

(a) Show that if there exists an integer z for which $yz + k = a^2$ and $xz + k = b^2$, then (a, b) must satisfy

$$xa^2 - yb^2 = (x - y)k.$$

(b) Suppose that (a, b) satisfies the equation in (a) and that (u, v) satisfies the diophantine equation $u^2 - (xy)v^2 = 1$. Verify that $xA^2 - yB^2 = (x - y)k$, when $(A, B) = (au + ybv, bu + xav)$.

(c) Suppose that $xy + k = c^2$ and that $xa^2 - yb^2 = (x - y)k$. Determine z so that (x, y, z) is a $k-$triple. Is this triple necessarily congenial or necessarily reclusive?

Exercise 19. Let c be an integer. The triple $(2, 4, 2c + 6)$ is a congenial $(c^2 - 8)-$triple. Use the method of Exercise 14 to construct other $k-$triples for which $(x, y) = (2, 4)$ and determine whether they are congenial or reclusive. Check for specific values of c.

Exercise 20. Determine a family of $(-1)-$triples for which $(x, y) = (1, 5)$. Look at the possible values of z and its relation to terms in the Fibonacci sequence. Make a conjecture and prove it directly. Which triples are congenial?

6. Constructing triples from the related squares

We can construct $k-$triples by starting with the squares involved. Let a, b, c be three arbitrary integers; we can factor the differences of their squares to construct a $k-$triple (x, y, z) for which $xy + k = c^2$, $xz + k = b^2$ and $yz + k = a^2$. For example, if $b^2 - c^2 = x(z - y)$, we can select different possibilities for the factors x and $z - y$.

Thus, $z - y$ will be among the divisors of $b^2 - c^2$, $y - x$ among the divisors of $a^2 - b^2$, and $z - x$ among the factors of $a^2 - c^2$. However, the choice of divisors from the three differences of squares will be constrained by the fact that

$$z - x = (z - y) + (y - x).$$

From these choices for $z - x$, $z - y$, $y - x$, we can get x, y, z from the cofactors of the square differences and check that the values are consistent with their differences.

Exercise 21. Apply this approach to $(a, b, c) = (11, 7, 3)$ to obtain $k-$triples (x, y, z) for which $xy + k = 9$, $xz + k = 49$ and $yz + k = 121$. What are the corresponding values of k?

Exercise 22. Determine $k-$triples (x, y, z) and associate squares (a, b, c) for which $x = b - c$, $y = a - c$, $z = a + b$. Are these congenial? superbly congenial?

Exercise 23. Investigate $k-$triples for which $x = b + c$, $y = a + c$, and $z = a + b$.

Exercise 24. Investigate $k-$triples for which $x = b - c$, $y = a - c$, $z = a - b$.

Exercise 25. Investigate $x = b - c$, $y = a + c$, $z = a + b$.

Exercise 26. Investigate the situation when $x = 0$ or when $y = z$.

Exercise 27. What are the possible $k-$triples when $(a, b, c) = (5, 5, 4)$? Which ones are congenial? superbly congenial? reclusive?

PART D: OTHER FAMILIES

I was impelled by a communication from Steve Hzindar to reflect further on the existence of integer triples (x, y, z) for which $xy + k$, $yz + k$ and $zx + k$ are all perfect squares for some integer value of k. Such $k-$ **triples** are **gregarious** if their left and right **associates** $(2(x+y)-z, x, y)$ and $(y, z, 2(y+z)-x)$ are also $k-$triples. They are **superbly** gregarious if, in addition, when $z-(x+y)=2c$ and $xy+k=c^2$. Otherwise, the $k-$triple is **reclusive**. Previous articles in *Crux* (50:4, 190-192; 50:5, 244-247; 50:6, 290-293) described a number of ways in which such triples can be discovered, their properties and some open questions. The purpose of this note is to carry the investigations further.

7. Triples for which $x = 1$.

Exercise 28. $(1, 3, 8)$ and $(1, 3, 120)$ are $1-$triples. Generalize these to find families of $k-$triples $(1, y, z)$. Are these triples congenial?

8. Triples in arithmetic progression.

Exercise 29. Determine $1-$triples (x, y, z), where x, y, z are in arithmetic progression.

Exercise 30. If $(x, y, z) = (v-u, v, v+u)$ is a $1-$triple in arithmetic progression and $b^2 = (v-u)(v+u)+1$, then $v^2+1 = b^2+u^2$. Investigate situations in which v^2+1 has an alternative representation as a sum of squares as to which lead to a $1-$triple. Are any of them reclusive?

Exercise 31. Find $k-$triples (x, y, z) in arithmetic progression for other values of k.

9. Triples in geometric and harmonic progression.

Exercise 32. Determine $k-$triples (x, y, z) where x, y, z are in geometric progression.

Exercise 33. (a) Determine $k-$ triples (x, y, z) where x, y, z are in harmonic progression.

(b) Verify that $(x, y, z) = (r^2-2s, r^2-s^2, r^2+rs)$ is a triple in harmonic progression. Prove that, if $yz+k=a^2$, $zx+k=b^2$ and $xy+k=c^2$ for some integers k, a, b, c, then $a^2+c^2=2b^2$ or $u^2+v^2=b^2$, where $a=u+v$, $b=u-v$. Use this to construct families of $k-$triples. Can you find any reclusive triples?

10. Triples that are pythagorean triples

Since a fundamental pythagorean triple (whose entries are coprime) must have one even and two odd entries, the sum of any pair has the same parity as the remaining one and we can use the construction of Section 2 to create a congenial $k-$triple.

Exercise 34. Check out some pythagorean triples (x, y, z) and determine k so that they are $k-$triples. Suppose that $(a^2, b^2, c^2) = (yz + k, zx + k, xy + k)$; find a relationship between (x, y, z) and (a, b, c). Can you establish a general result?

Exercise 35. In Exercise 22, triples $(x, y, z) = (b + c, c + a, a + b)$ were investigated. Determine conditions on (a, b, c) such that (x, y, z) is a pythagorean triple.

11. Additional questions, some open

Question 1. For each nonzero integer k, what is the maximum number m of entries in a set S of integers for which the values of $xy + k$ for the $\binom{m}{2}$ pairs (x, y) of distinct elements of S are all squares, with no two equal?

Question 2. Must every congenial $k-$triple be superbly congenial?

Question 3. Can a triples (x, y, z) be a congenial $k-$triple for more than one integer k.

Question 4. For each integer k we form a graph whose vertices are equivalent classes of $k-$triples. Two $k-$triples are equivalent if the terms of one are the negative of the terms of the other, the terms of one are a permutation of those of the other, or a composite of these conditions. The vertices are the equivalent classes of $k-$triples and two vertices are connected by an edge if and only if a representative triple of one is an associate of a representative triple of the other. Is the graph formed by the equivalence classes of congenial $k-$triples connected?

Question 5. Are there any $k-$triples (x, y, z) for which none of x, y, z is equal to 0 or 1 and $xyz + k$ is also a square?

Question 6. What are the possible values of the triple (k, m, d) for which there is a $k-$sequence with each term congruent to d modulo m?

For example, if m is a common divisor of r and s, then $(s^2, m, 0)$ is such a triple exemplified by the sequence

$$\ldots, r - s, 0, r + s, 4r + 8s, 9r + 21s, 25r + 55s, \ldots.$$

Are there any examples for which $d \neq 0$?

Question 7. Which $k-$triples are arithmetic progressions? geometric progressions? harmonic progressions?

Question 8. Characterize triples (x, y, z) that are not $k-$triples for any value of k?

Exercise 14. For

$$(x, y, z, w) = (x, y, x + y + 2c, 4c(x + c)(y + c)),$$

the choice of x, y, c leads to $k = c^2 - xy$ for the initial three terms. We find that

$$\begin{aligned} xy + k &= c^2; \\ yz + k &= (y + c)^2; \\ xz + k &= (x + c)^2; \\ xw + k^2 &= (c^2 + 2cx + xy)^2; \\ yw + k^2 &= (c^2 + 2cy + xy)^2; \\ zw + k^2 &= (3c^2 + 2(x + y)c + xy)^2. \end{aligned}$$

This explains why the quadruple works only when $1 = k = k^2$.

Exercise 19. In this case, we are led to the equation $a^2 - 2b^2 = -(c^2 - 8)$, where $(a, b) = (c + 4, c + 2)$ is the starting solution. The $(c^2 - 8)-$triple is $(2, 4, 12c^2 + 7c + 102)$. Here are some examples for specific values of c; they are all k triples, but some are gregarious for another value of k. The value of k for which the triple is gregarious is appended to the triple, thus $(x, y, z; k)$.

c	k	$(x, y, z; k))$
-3	1	(2, 4, 0; 1) (2, 4, 12; 1) (2, 4, 420; 42841) (2, 4, 14280; 50936761)
-2	-4	(2, 4, 2; -4) (2, 4, 10; -4) (2, 4, 290; 20156)
-1	-7	(2, 4, 4; -7) (2, 4, 8; -7) (2, 4, 44; 353) (2, 4, 184; 7913) (2, 4, 1408; 491391)
0	-8	(2, 4, 6; -8) (2, 4, 102; 2296) (2, 4, 3366; 2822392)

Exercise 20. We obtain the -1triples $(1, 5, 10)$, $(1, 5, 65)$, $(1, 5, 442)$, $(1, 5, 3026)$. These are all of the form $(1, 5, f_{2n}^2 + 1)$ with the associated squares of $(f_{2n-1} + f_{2n+1}, f_{2n}, 2)$. For the product of the second and third entries, we have

$$\begin{aligned} 5(f_{2n}^2 + 1) - 1 - (f_{2n-1} + f_{2n+1})^2 &= 5f_{2n-1}f_{2n+1} - 1 - (f_{2n-1} + f_{2n+1})^2 \\ &= f_{2n-1}(f_{2n+1} - f_{2n-1}) + (f_{2n-1} - f - 2n + 1)f_{2n+1} + f_{2n-1}f_{2n+1} - 1 \\ &= -f_{2n}(f_{2n+1} - f_{2n-1}) + f_{2n-1}f_{2n+1} - 1 = -f_{2n^2} + f_{2n-1}f_{2n+1} - 1 = 0. \end{aligned}$$

Exercise 21. We obtain the (-23)-triple $(4, 8, 18)$ and the $(-131)-$triple $(10, 14, 18)$, both congenial.

Exercise 27. $(a, b, c) = (5, 5, 4)$ gives rise to the congenial $(-65)-$sequence $\{\ldots, 61, 26, 9, 9, 10, 29, 69, \ldots\}$ and congenial $15-$sequence $\{\ldots, 61, 21, 10, 1, 1, -6, -11, -35, \ldots]$. However, the factorization $b^2 - c^2 = 3 \times 3$ yields the reclusive $7-$triple $(3, 3, 6)$. Notice that this is also a congenial $(-9)-$triple.

Exercise 28. If $(x, y, z) = (1, n^2 - k, (n + 1)^2 - k)$, then

$$yz + k = [n(n + 1) - k]^2,$$

and the triple is superbly gregarious.

We can consider $1-$triples of the form $(1,3,w^2-1)$, where $3(w^2-1)+1 = 3w^2-2$ is a square. Determining such values of w involves the solving of a Pell's equation $v^2 - 3w^2 = -2$. Some solutions are $(v,w) = (1,1),(5,3),(19,11),(71,41)$, giving rise to the $1-$triples $(1,3,0),(1,3,8),(1,3,120),(1,3,1680)$.

Related to this is Problem 10238 in the American Mathematical Monthly 99:7 (August-September, 1992), 674, which asks for a sequence of values of a_n for which $a_n + 1$ and $3a_n + 1$ are squares and $a_n a_{n+1} + 1$ is also square.

Exercise 29. This analysis is due to Steve Hzindar. Suppose that $(x,y,z) = (v-u, v, v+u)$ is a $1-$triple, with $a^2 = yz+1$, $b^2 = zx+1$ and $c^2 = xy+1$. One way to get a possible triple is to ensure that $z-(x+y) = 2u-v$ is even, and get a superb congenial triple. So we may assume that $v = 2w$. We are led to

$$a^2 = 4w^2 + 2uw + 1;$$
$$b^2 = 4w^2 - u^2 + 1;$$
$$c^2 = 4w^2 - 2uw + 1.$$

One possibility is to let $u = 2w$, which leads us to $(x,y,z) = (0,2w,4w)$ and values of w for which $8w^2+1$ is a square. Alternatively, we can create a superbly congenial triple by making $a^2 = (u-w)^2$. Since

$$a^2 = (u-w)^2 - (u^2-3w^2-1); b^2 \quad = w^2 - (u^2-3w^2-1); c^2 = (u+w)^2 - (u^2-3w^2-1),$$

we can achieve this when the Pell equation $u^2 - 3w^2 = 1$ is satisfied. The solutions are given by $(u_0,w_0)(1,0)$, $(u_1,w_1) = (2,1)$ and $(u_{n+1},w_{n+1}) = (4u_n - u_{n-1}), 4w_n - w_{n-1})$ for $n \geq 0$. An alternative recursion is

$$(u_{n+1}, w_{n+1}) = (2u_n + 3w_n, u_n + 2w_n).$$

It can be checked directly that $(x,y,z) = (w_{n-1}, 2w_n, w_{n+1})$ is a superbly congenial 1-triple.

This was the content of problem 10622, which appeared in the American Mathematical Monthly 104:9 (November, 1997), 870 and 106:9 (November, 199), 867-868.

Exercise 31. If $(x,y,z) = (2w-u, 2w, 2w+u)$, we are led to

$$a^2 = (w+u)^2 - (u^2 - 3w^2 - k);$$
$$b^2 = w^2 - (u^2 - 3w^2 - k);$$
$$c^2 = (w-u)^2 - (u^2 - 3w^2 - k).$$

Thus we can find triples whenever $u^2 - 3w^2 = k$ is solvable. For example, when $k = 13$, we are led to the solutions $(u,w) = (4,1),(5,2),(11,6),(16,9),\ldots$.

Exercise 32. Let $(x,y,z) = (s, sr, sr^2)$. For a superbly congenial $k-$triple, we need s to be even. For example, if $s = 2$, then $(x,y,z) = (2,2r,2r^2)$ is a $k-$triple with $k = (r^2+r+1)(r^2-3r+1)$ and

$$(a,b,c) = (r^2+r-1, r^2-r+1, r^2-r-1).$$

Exercise 34. We have the examples $(x,y,z) = (3,4,5)$ and $(x,y,z) = (8,15,17)$. Respectively, these are $(-11)-$triples and $(-111)-$triples with associated squares $(3,2,1)$ and $(12,5,3)$. In both cases, we find that $(x,y,z) = (b+c, c+a, a+b)$. In general, we have

(x,y,z)	k	(a,b,c)
$(2r+1, 2r^2+2r, 2r^2+2r+1)$	$-r(4r^2+5r+2)$	$(r(2r+1), r+1.r)$
$(2r, r^2-1, r^2+1)$	$-(r-1)(2r^2+r+1)$	$(r(r-1), r+1, r-1)$
$(m^2-n^2, 2mn, m^2+n^2)$	$-n(2m^3-m^2n-n^3)$	$(n(m+n), m(m-n), n(m-n))$

Exercise 35. $(b+c, c+a, a+b)$ is a congenial $k-$triple with $k = -(ab+bc+ca)$. It is a pythagorean triples if and only if $ab = (a+b+c)c$, or

$$c^2 + (a+b)c - ab = 0.$$

Since the equation $t^2 + (a+b)t + ab = 0$ has integer roots $-a$ and $-b$, this leads us to study under which there are integer coefficients p and q for which both equation $t^2+pt\pm q = 0$ have integer roots. This will happen when the discriminants $p^2-4q = r^2$ and $p^2+4q = s^3$ for some integers r and s. This leads us to the equations $r^2+s^2 = 2p^2$ and $u^2+v^2 = p^2$ where $r = u-v$ and $s = u+v$. For example

$$(u,v;r,s:p.q) = (4,3;1,7;5,6)$$

leads us to the equations $x^2+5x\pm 6 = 0$ where

$$x^2+5x+6 = (x+2)(x+3); \qquad x^2+5x-6 = (x+6)(x-1).$$

In our problem, this corresponds to $(a,b) = (2,3)$ and c equal to either 1 or -6. Both lead to pythagorean triples for (x,y,z).

More generally, we can take $(a,b) = (r+1, r(2r+1))$ and find that

$$t^2 + (2r^2+2r+1)t - r(r+1)(2r+1) = [t+(r+1)(2r+1)][t-r]$$

and arrive at the congenial triple $(2r+1, 2r^2+2r, 2r^2+2r+1)$. Alternatively, we have the pair of factorizations:

$$t^2 + (r^2+1)t + (r-1)r(r+1) = [t+(r+1)][t+r(r-1)];$$
$$t^2 + (r^2+1)t - (r-1)r(r+1) = [t-(r-1)][t+r(r+1)];$$

to arrive at the triple $(2r, r^2-1, r^2+1)$.

Another approach is to suppose that

$$t^2 + (a+b)t - ab = [t+(ab)/r][t-r].$$

Then $(ab/r) - r = a+b$, from which it follows that

$$(a-r)(b-r) = 2r^2.$$

Then, we can find a possibility for each way of factoring $2r^2$ as a product of two integers. For example, if $a-r = 2$ and $b-r = r^2$, then $(a,b) = (r+2, r(r+1))$. We find that

$$c^2 + (r^2+2r+2)c - r(r+1)(r+2) = (c+(r+1)(r+2))(c-r).$$

A final approach is to note that the discriminant of $c^2 + (a+b)c - ab$ is equal to $(a+3b)^2 - 8b^2$. Determining when this is square leads us to the Pell equation

$$\alpha^2 - 8\beta^2 = \gamma^2.$$

When $\gamma = 1$, this has solutions $(1,0),(3,1),(17,6),(99,35),\ldots$. For general γ, there are the obvious solutions

$$(\alpha,\beta) = (\gamma,0),(3\gamma,\gamma),(17\gamma,6\gamma),\ldots.$$

For many values of γ, that is all that there is. However, $\alpha^2 - 8\beta^2 = 49$ has two other fundamental solutions $(\alpha,\beta) = (9,2),(11,3)$. Since $b = \beta$ and $a + 3b = \alpha$, we can backtrack to get $(a,b) = (3,2),(2,3)$.

Likewise, since $\alpha^2 - 8\beta^2 = 17^2$ is satisfied by $(\alpha,\beta) = (19,3),(33,10)$, we are led to $(a,b) = (10,3),(3,10)$.

One approach to the problems is to note that for a $1-$triple (x,y,z), the product of $xy+1$, $yz+1$ and $zx+1$ is a square, which opens the door to an analysis using an elliptic equation. The question of when this applies to $k-$triples is settled in the short paper:

Kiran S. Kedlaya, *When is $(xy+1)(yz+1)(zx+1)$ a square?* Math. Mag. 71:1 (February, 1998), 61-63 .

The problems in this article undoubtedly have been well studied over a long period. The best historical reference I have come across is the book

Andrej Dujella, *Diophantine $m-$tuples and elliptic curves.* Springer, 2024.

This has recently been published and the author has provided a summary on the webpage https://web.math.pmf.unizg.hr/ duje/diophantine-mtuples-book.html. Here is his link to a list of open problems: https://web.math.pmf.unizg.hr/ duje/pdf/open2.pdf.

What we call a $k-$triple, he calls a $D(k)-$triple, with analogous terminology for $m-$tuples; a superb gregarious triple is, in his terms, *regular* (in my opinion, an overworked word in definitions).

Dujella dates interest in this problem to the discovery by Diophantus that $(\frac{1}{16}, \frac{33}{16}, \frac{17}{4}. \frac{105}{16})$ is a $1-$quadruple with rational entries. This is equivalent to $(1,33,68,105)$ being a $256-$quadruple. We note that the triple $(1,33,68)$ is congenial with left associate $(0,1,33)$ and right associate $(33,68,201)$, However, $(33,68,105)$ is reclusive, since neither its left or right associates, $(97,33,68)$ and $(68,105,313)$ are $256-$triples. Diophantus also discovered other examples of $k-$quadruples with $k \neq 1$.

Fermat is credited with finding the first $1-$quadruple $(1,3,8,120)$; in 1969, Baker and Davenport showed that 120 is the only value of d that makes $(1,3,8,d)$ a $1-$triple. Euler made significant progress, initiating over 200 years of intemittent and increasingly deep progress.

To construct a $k-$quadruple (x, y, z, w) by extending a $k-$triple, we have to determine w so that $v^2 = (xw + k)(yw + k)(zw + k)$, the equation of an elliptic curve in the wv-plane. Accordingly, the bulk of Dejella's book is the development of the theory of elliptic curves to support researchin this area.

For the cases $k = \pm 1$, a few results and additional references are given on pages 153-155, 157-159 of the book

Edward J. Barbeau, *Power play.* The Mathematical Association of America, 1997 ISBN 0-88385-523-2