Equal products and sums

§1. Pairs and triples of numbers whose products equal their sums.

It is interesting that $2 + 2 = 2 \times 2$, so that the pair $(2, 2)$ has the property that its sum is equal to its product. Are there any other pairs of integers whose sum is equal to its product? Suppose we ask for pairs of rational numbers (fractions) with this proporty.

What does the graph of $xy = x + y$ look like?

We can ask the same question of triples (a, b, c) of integers: when does $a+b+c = abc$?

There are various ways we can extend the investigation:

(1) Find a set $(a_1, a_2, \cdots, a_r)$ of r integers whose sum is equal to its product? How large can the smallest integer in the set be? The second smallest?

(2) What are the sets of integers whose product is an integer multiple of the sum?

(3) For each positive integer $r \geq 2$ find two $r$tples, $(a_1, a_2, \ldots, a_r)$ and $(b_1, b_2, \ldots, b_r)$ for which the sum of each of them is equal to the product of the other.

§2. Related pairs and triples.

The investigation can be extended to couples of pairs and triples as follows:

(4) Determine couples of integer pairs $(x, y; u, v)$ for which the sum of each pair is equal to the product of the other. In other words, we want integer solutions to the system:

$$x + y = uv, \qquad xy = u + v.$$

(5) Determine couples of integer triples $(x, y, z; u, v, w)$ for which the sum of each triple is equal to the product of the other. Thus:

$$x + y + z = uvw; \qquad xyz = u + v + w.$$

§3. Discussion of the pair situation.

We are asking for solutions of the equation $x + y = xy$. Elementary students can try to guess solutions, for example, by taking various integers for x and seeing what values of y will suit. If they discover a solution for one positive integer other than 2, then they may be able to discern a pattern. Checking which solutions work is a nice exercise in fractions.

If the students know some algebra, then they can solve the equation to find that any pair $(x, x/(x - 1))$ works. In checking what other integer pairs there are, we can note that the equation $xy = x + y$ is equivalent to $(x-1)(y-1) = 1$, so it is just a matter of noting that $(x, y) = (0, 0)$ or $(x, y) = (2, 2)$. This form of the equation also helps identify the locus of the equation as a rectangular hyperbola with centre at $(1, 1)$.

§4. Discussion of the triple situation.

We may suppose that $a + b + c = abc$ where the variables are integers and $a \leq b \leq c$. Since the negative of any triple that works also works, we may suppose that $c > 0$. If one of the integers is zero, then a possible triple is $(-t, 0, t)$ for each integer t. Henceforth, let $abc \neq 0$.

From the equation, we find that

$$a = \frac{b+c}{bc-1}.$$

If $a > 0$, then $0 < bc-1 \leq b+c$, which is equivalent to $(b-1)(c-1) \leq 2$. This leads to the only possibility, $(a, b, c) = (1, 2, 3)$, of positive integers. If $a < b < 0 < c$, we get a contradiction since the fraction $c = (a+b)/(ab-1)$ has a negative numerator and a positive denominator.

We can investigate the equation $abc = k(a+b+c)$ with $c > 0$. Let $a \leq b \leq c$. Since $abc = k(a+b+c) \leq 3kc$, $ab \leq k$. Also the equation reduces to

$$(ab-k)(ac-k) = k(a^2+k),$$

so we can start with a value of a and systematically find values of b and c.

For general k, we have the triples

$$(a, b, c) = (1, 2k, 2k+1), (1, k+1, k(k+2)).$$

Here are some other triples:

$$k = 1 : (1, 2, 3)$$

$$k = 2 : (1, 3, 8), (1, 4, 5), (2, 2, 4)$$

$$k = 3 : (1, 4, 15), (1, 5, 9), (1, 6, 7), (2, 2, 12), (2, 3, 5)$$

§5. Discussion of the general situation.

For general r, we consider the equation $a_1 + \cdots + a_r = a_1 a_2 \cdots a_r$. One possible solution is given by

$$(a_1, a_2, \ldots, a_r) = (1, 1, \ldots, 1, 2, r)$$

with $r - 2$ ones.

Consider possibities where the $r$tple has a entries equal to 1 and b entries equation to 2. Then $r = a + b$ where $a + 2b = 2^b$.

$$(b, r) = (2, 2) : (2, 2)$$

$$(b, r) = (3, 5) : (1, 1, 2, 2, 2)$$

$$(b.r) = (4, 12) : (1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2) = (1_8, 2_4)$$
$$(b, r) = (5, 27) : (1_{22}, 2_5)$$
$$(b, 2^b - b) : (1_{2^b - 2b}, 2_b)$$

Other examples: (1, 1, 2, 4), (1, 1, 1, 3, 3), (1, 1, 1, 2, 5), (1, 1, 2, 2, 2), (1, 1, 1, 1, 2, 6).

When $a_1 + a_2 + \cdots + a_r = a_1 a_2 \cdots a_r$ with $a_1 \leq \cdots \leq a_r$, we have $a_1^{r-1} \leq a_1 a_2 \cdots a_{r-1} \leq r$. Thus, when $r \geq 3$, we must have $a_1 = 1$.

§6. Couples of pairs.

If we make the additional assumption that all the integers have to be positive, then the equations $a + b = uv$ and $ab = u + v$ together lead to

$$(a - 1)(b - 1) + (u - 1)(v - 1) = 2.$$

Since all the quantities in parentheses are nonnegative, we can quickly find that the only solutions are given by $(a, b; u, v)$ equal to $(2, 2; 2, 2)$, $(2, 3; 1, 5)$ and $(1, 5; 2, 3)$.

However, this equation is not so useful in tracking down solutions with non-positive entries. The key to solving the problem is to note that for one of the pairs, the absolute value of the product does not exceed the absolute value of the sum. Let us suppose that

$$|a||b| = |ab| \leq |a + b| \leq |a| + |b|.$$

If we assume that $|a| \leq |b|$, then

$$|a||b| \leq 2|b|,$$

and so $|a| \leq 2$.

If $a = 0$, then we are led to the family $(0, -t^2; t, -t)$ where t is an arbitrary integer. If $a = 1$, then we find that $(u - 1)(v - 1) = 0$, so that we essentially get the solution $(1, t; 1, t)$ for any integer t. On the other hand, if $a = -1$, we find that $(u + 1)(v + 1) = 0$, so that, say $u = -1$ and $y + v = 1$. This leads to the family $(-1, t; -1, 1 - t)$ for any integer t.

Let $|a| = 2 \leq |b|$. Then, in the foregoing chain of inequalities, we must have equality all across, so that x and y have the same sign and $|a| = |b|$. This leads to the known solution $(2, 2; 1, 5)$. When $x = y = -2$, we obtain $uv = -4$ and $u + v = 4$, which has no solution in integers.

\$ 7. Couples of triples.

Consider the pair of equations, to be solved in integers:

$$a + b + c = uvw, \qquad abc = u + v + w.$$

The absolute value of one of the products does not exceed that of the sum. Wolog, let $|abc| \leq |a| + |b| + |c|$. If we assume that $|a| \leq |b| \leq |c|$, then $|ab| \leq 3$.

Suppose, for example, that $a = 0$. Then we find that $w = -(u+v)$ and $b + c = -uv(u+v)$. This leads to the pair of triples:

$$(0, b, -uv(u+v) - b; u, v, -(u+v)),$$

where b, u, v are arbitrary integers.

If $a = b = 1$, then $uvw = u + v + w + 2$. One solution of the last equation is $u = v = -1$, and we are led to $(a, b, c; u, v, w) = (1, 1, t; -1, -1, t+2)$, for any integer t. We note in passing that if one triple of the couple is known, the other is not necessarily uniquely determined. For example,

$$(a, b, c; u, v, w) = (1, 1, 8; 1, 2, 5)$$

is another couple that works. How often does this happen?