THE $4/N$ PROBLEM.

A mathematical vignette

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This purpose of this vignette is to give students experience dealing with fractions. It has been conjectured that, for any positive integer exceeding 2, the fraction $4/n$ can be written as the sum of three distinct reciprocals of positive integers, (*i.e.* three distinct fractions whose numerators are 1).[1] For example

$$\frac{4}{3} = \frac{1}{1} + \frac{1}{4} + \frac{1}{12};$$
$$\frac{4}{4} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}:$$
$$\frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}.$$

In some cases, there may be more than one way of representing the fraction.

Begin by checking the conjecture for higher values of n, looking for patterns as you go along that may help you deal with whole classes of positive integers.

Stop reading here to explore the situation, before reading on for possible approaches.

One possible approach is to write $4/n$ as an equivalent fraction $4m/nm$ in such a way that $4m$ can be written as the sum of three distinct divisors a, b, c of nm. Then write $4/n = a/nm + b/nm + c/nm$, reducing the fractions on the right side.

A second possibility is to find an integer reciprocal that is a little smaller than $4/n$, make a subtraction and then express the difference as the sum of two integer reciprocals. For small values of n, it is pretty easy with a little trial and error to hit on a representation. For example, check out $4/7$, $4/11$, $4/15$.

Note that, if we can find a representation of $4/n$ for any value of n, then we can immediately determine a representation for $4/mn$. Thus, we need only try to establish that the representation exists whenever n is an odd prime.

There is a useful rule for extending the sum of two reciprocals to a sum of three reciprocals exemplified by

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6};$$
$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12};$$
$$\frac{1}{4} = \frac{1}{5} + \frac{1}{20}.$$

Write and verify an algebraic identity generalizing this for $1/n$.

[1] This is the Erdös-Strauss Conjecture, formulated in 1948. It has been verified for donominators less that 10^{17}.

Use this identity to give a representation of $4/n$ as the sum of three distinct reciprocals of positive integers when n is even.

One approach is to start consider reciprocals $1/m$ for which $4/n > 1/m$. Then the difference has numerator $4m-n$. Search for two divisors a and b of m for which $a+b = 4m-n$. Let $u = m/a$ and $v = m/b$. Then

$$\frac{4}{n} = \frac{1}{m} + \frac{1}{nu} + \frac{1}{nv}.$$

Tables.

In this table, we present for each value of n, triples of integers whose reciprocals add to $4/n$. In some case, we give a pair of integers whose reciprocals add to $4/n$.

Value of n	Reciprocal pairs	Reciprocal triples adding to $4/n$
3		(1, 4, 12)
4	(2, 2)	(2, 3, 6)
5		(2, 5, 10)
6	(3, 3)	(2, 7, 42), (3, 4, 12)
7	(2, 14)	(2, 15, 210), (3, 6, 14)
8	(4, 4)	(3, 8, 24), (4, 5, 20)
9	(3, 9)	(3, 10, 90), (4, 9, 12)
10	(5, 5)	(5, 6, 30)
11	(3, 33)	(3, 8, 24), $(3, 34, 33 \times 34)$, (4, 11, 44)
12	(6, 6)	(6, 7, 42)
13		(4, 26, 52)
14	(7, 7)	(5, 14, 70), (7, 8, 56)
15	(5, 15)	(5, 16, 240), (6, 15, 30)
21	(7, 21)	(6, 63, 126), $(7, 22, 22 \times 23)$, (8, 21, 56)
49		(14, 105, 1470), (16, 98, 112)
73		(20, 292, 730)
97		$(28, 2 \times 97, 28 \times 97) = (28, 194, 2716)$
121		
145		
169		
193		$(50, 10 \times 193, 25 \times 193) = (50, 1930, 4825)$
217		
241		$(63, 6 \times 241, 126 \times 241), (66, 3 \times 241, , 66 \times 241)$
289		
361		
409		$(104, 16 \times 409, 208 \times 409)$

The following table will give general values of n with parameter k, and the integer triples whose reciprocals add to $4/n$. The first five cases cover all the numbers that do not leave remainder 1 when divided by 24. The last three cases cover all the numbers that do not differ from 49 or 121 by a multiple of 120.

Value of n	Reciprocal pairs	Reciprocal triples adding up to $4/n$
$2k$	(k,k)	$(k,k+1,k(k+1))$
$3k$	$(k,3k)$	$(k+1,3k,k(k+1))$, $(k,3k+1,3k(3k+1))$
$3k-1$		$(k,3k-1,k(3k-1))$
$4k-1$	$(k,k(4k-1))$	$(k+1,k(k+1),k(4k-1))$, $(k,4k^2-k+1,k(4k-1)(4k^2-k+1))$
$8k-3$		$(2k,k(8k-3),2k(8k-3))$
$25+120k$ $=5(5+24k)$		$(8+30k,(5+24k)(8+30k),(25+120k)(4+15k))$ $=(2(4+15k),2(4+15k)(5+24k),5(4+15k)(5+24k))$
$73+120k$		$(20+30k,(4+6k)(73+120k),(10+15k)(73+120k))$ $=(10(2+3k),2(2+3k)(73+120k),5(2+3k)(73+120k))$
$97+120k$		$(25+30k,(10+12k)(97+120k),(50+60k)(97+120k))$ $=(5(5+6k),2(5+6k)(97+120k),10(5+6k)(97+120k))$

If there is a counterexample to the conjecture, then the smallest counterexample for n must be an odd prime. Unfortunately, all the odd primes other than 3 and 5 are of the form $49+120k$ and $121+120k$. Note that all these primes are of the form $10k\pm3$ where $k\equiv1,2$ modulo 3 or of the form $10k\pm1$ where $k\equiv0,\pm1$ modulo 3. To see this, note that

$$(10k+3)^2-7^2=(10k-4)(10k+10)=20(5k-2)(k+1);$$
$$(10k-3)^2-7^2=(10k-10)(10k+4)=20(k-1)(5k+2);$$
$$(10k+1)^2-11^2=(10k+12)(10k-10)=20k(5k+1)-120;$$
$$(10k-1)^2-11^2=(10k-12)(10K+10)=20k(5k-1)-120.$$