INTERLOCKING PAIR SUMS AND PRODUCTS.

A mathematical vignette

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The pair $(2, 2)$ is interesting in that the sum and product of its two entries are equal. Are there any other pairs (a, b) of positive integers whose sum and product are equal? What is the reason for your answer? Algebraically, we are asking that $ab = a + b$.

Let us go further and ask for two pairs, (a, b) and (c, d) for which the sum of each of the pairs is equal to the product of the other: $a + b = cd$ and $ab = c + d$. If each pair has a different sum than product, then for one of the pairs, the product must be less than the sum. Why? What can we say about a pair of positive integers whose product is less than the sum? Why?

When (a, b) and (c, d) have the foregoing property, then

$$\begin{aligned} 0 &= (ab - c - d) + (cd - a - b) = (ab - a - b) + (cd - c - d) \\ &= (a - 1)(b - 1) + (c - 1)(d - 1) - 2, \end{aligned}$$

whence

$$(a - 1)(b - 1) + (c - 1)(d - 1) = 2.$$

This says that the sum of two products of nonnegtive integers is equal to 2. Thus the products are 2 and 0, or else both 1. This leads to $(a, b) = (c, d) = (2, 2)$, or else, say, $d = 1$ and $(a - 1)(b - 1) = 2$. In the latter case $(a, b) = (3, 2)$, and we are led to the pairs $(3, 2)$ and $(5, 1)$, the product of one being the sum of the other.

Another way of formulating this is to note that both the quadratics $x^2 - 5x + 6$ and $x^2 - 6x + 5$ can be factored as a product of linear polynomials with integer coefficients.

This can be generalized to obtain pairs (a, b) and (c, d) for which the product of each is k times the sum of the other, where k is a positive integer:

$$ab = k(c + d); \qquad cd = k(a + b).$$

Using the information we already have, we can note that $(a, b) = (c, d) = (2k, 2k)$ and also $(a, b) = (3k, 2k)$; $(c, d) = (5k, k)$ will work. Are there any other possibilities?

We can derive the equation

$$(a - k)(b - k) + (c - k)(d - k) = 2k^2.$$

However, this is a necessary condition, and solutions to this might not always lead to solutions of the problem.

Another way of formulating the questions is, for given integer values of k, a, b, which quadratics

$$kx^2 - abx + k^2(a + b)$$

have integer roots?