THE CAUCHY RADIUS.

A mathematical vignette

Ed Barbeau, University of Toronto

1. THE CAUCHY RADIUS.

In 1829, Cauchy proved that, if you have a polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

whose coefficients are complex numbers and whose constant term is nonzero, that all of its roots w must satisfy $|z| \leq r$ where r is the unique positive root of the polynomial

$$h(z) = |a_n| z^n - |a_{n-1}| z^{n-1} - \cdots - |a_1| z - |a_0|.$$

Note that the coefficients of $h(z)$ are all real, that $h(0) = -|a_0| \leq 0$ and $h(z) > 0$ when z takes a large positive real value. *Descartes' Rule of Signs* says that the number of positive roots of a polynomial with real coefficients cannot exceed the number of changes of signs in its nonzero coefficients as you read the coefficients from left to right. This result tells us that $h(z)$ has exactly one positive root. Let us call this root the *Cauchy radius*, $\rho[p]$, for p.

Thus, the Cauchy radius of $p(z)$ is an upper bound for the absolute value of the roots of $p(z)$. But for polynomials of degree greater than 1, it need not be the least upper bound. This raises the question as to whether we can multiply $p(z)$ by another polynomial $q(z)$ such that the Cauchy radius of the product $p(z)q(z)$ is strictly less than that of $p(z)$.

If we multiply $p(z)$ by another polynomial $q(z)$, then the product $p(z)q(z)$ has additional roots, so we would expect the Cauchy radius to be at least as large. So $q(z)$ would need to be carefully chosen.

2. Quadratic examples.

Let $p(z) = z^2 + 2bz + 1$. When $-1 < b < 1$, its discriminant is negative and the product of the two complex conjugate roots is 1. Thus both the roots have absolute value equal to 1.

The Cauchy radius for this polynomial is the positive root of $z^2 - |b|z - 1$, namely

$$|b| + \sqrt{b^2 + 1},$$

a number that is stricty bigger then 1 when $b \neq 0$.

A natural choice of multiplier $q(z)$ to make $\rho[pq] \leq \rho p$ is $q(z) = z^2 - bz + 1$. The roots of $q(z)$ are the negatives of the roots of $p(z)$ so that p and q have the same Caucy radius. In this situation, we find that

$$(pq)(z) = z^4 + (2 - 4b^2)z^2 + 1$$

and

$$\rho[pq](2b^2 - 1) + \sqrt{(2b^2 - 1)^2 + 1}.$$

In fact, when $b = 0, \pm(1/\sqrt{2})$, we find that $\rho[pq] = 1$. The Cauchy radius of p and pq are also easy to work out when $b = 1/2$.

Another example that is tractable to look at is

$$p(z) = z^2 - bz + (b+1); \qquad q(z) = z^2 + bz + (b+1);$$
$$(pq)(z) = z^4 - (b^2 - 2b - 2)z^2 + (b+1)^2.$$

It is easy to check that $\rho[p] = b + 1$. How does this compare with the maximum absolute value of a root of $p(z)$ and the Cauchy radius of pq?

3. A cubic example.

Let $p(z) = z^3 + z - 6$. As a possible $q(z)$, take $q(z) = z^3 - z + 6$, so that

$$p(z)q(z) = x^6 - (x-6)^2 = x^6 - x^2 + 12x - 36 = (x-2)(x^5 + 2x^4 + 4x^3 + 8x^2 + 15x + 18).$$

Since $x^3 - x - 6 = (x-2)(x^2 + 2x + 3)$, we find that $\rho[p] = \rho[q] = \rho[pq]$.

Here is another cubic example. Let $p(z) = z^3 + 2z^3 - z - 1$. Possible multipliers $q(z)$ are $z - 1$ and $z - 2$. You can approximation the Cauchy radius for $p(z)$ and $p(z)q(z)$ by finding smaller and smaller intervals $[a, b]$ with $a > 0$ for which the Cauchy polynomial is negative on a and positive on b. See the paper *Optimality of a polynomial multiplier* by Aaron Melman in the American Mathematical Monthly 125:2 (Feb., 2018), 158-168. This paper deals with a somewhat special situation, and there is likely more that can be discovered.

4. Some questions.

(1) What can be said about polynomials whose Cauchy radius is equal to the largest absolute value of a root?

(2) For which polynomials $p(z)$ does the choice of $q(z) = p(-z)$ as a multiplier lead to $\rho[pq] \leq \rho[p]$?

(3) For which polynomials will the choice of a linear multiplier lead to a lower Cauchy radius?

(4) Given a polynomial, what is the minimum degree of a multiplier that leads to a lower Cauchy radius?

(5) For a given polynomial $p(z)$ is there a value of k for which $q(z) = (z+1)^k$ will yields $\rho[pq] \leq \rho[p]$?