PRAGUE CLOCK SEQUENCES.

A mathematical vignette

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1. The Prague Horloge. In the Old Town City Hall in Prague in the Czech Republic, there is a remarkable astronomical clock that dates back to 1410. It is the creation of clockmaker Mikulas of Kadan and mathematician Jan Sindel. A detailed description of the clock can be found on line in the article *The mathematics behind Prague's horloge* by M. Krizek, A. Selcova and L. Somer in the journal *Mathematical Culture* found online at *www.global-sci.org*. For our purposes, all we need to know is that it is a 24-hour clock that chimes the hours on the hour. The operation of the chime is governed by two related gears that determine the timing and number of chimes. The smaller gear has gaps that are separated by distances proportional to 1, 2, 3, 4, 3, 2. As this gear turns, it presents the sequence consisting of this cycle of numbers repeated over and over:

$$1, 2, 3, 4, 3, 2, 1, 2, 3, 4, 3, 2, 1, 2, 3, 4, 3, 2, \ldots.$$

The relevance of this sequence lies in the fact that we can break it into portions for which the sum of the numbers in each give us all the positive integers consecutively:

$$(1)(2)(3)(4)(3+2)(1+2+3)(4+3)(2+1+2+3)(4+3+2)\ldots.$$

This allows the gaps in the smaller gear to mesh with those in a larger gear (with gaps separated by distances 1 up to 24 consecutively) to chime the hours successively.

By continuing further along the sequence, you can satisfy yourself that you can include all the numbers in turn up to 24 by sums. How do you know that you can continue to get *every* positive integer?

The key to understanding is the observation that when you get up to $14 = 2+3+4+3+2$, you arrive at the end of one of the cycles. The number 15 will use the next complete cycle. When we come to $16 = 15 + 1$, we have to use a complete and start the next. Then $17 = 15 + 2$ takes us though a complete cycle (2, 3, 4, 5, 2, 1) plus 2. In effect, we are taken through the sequence of sums for $1, 2, 3, \ldots$ with complete cycles thrown in.

2. More general sequences. Are there other sequences with the same property as the one on the foregoing section, that is periodic with the consecutive positive integers given by sums of nonoverlapping consecutive portions. An easy example is the sequence every entry of which is 1:

$$(1)(1+1)(1+1+1)(1+1+1+1)(1+1+1+1+1)\ldots.$$

It is also easy to check that the sequence

$$1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \ldots$$

also works.

We seem to be following a pattern. However, you can see that the sequence

$$1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots$$

fails. Maybe you can perform a slight modification to find a sequence that works.

It does not seem to be an easy task to find other examples by trial and error, and we need a more systematic way of proceeding. (You might want to stop and think for a while before reading on.)

Let us go back to the sequence in section 1, whose cycle has the sum 15. We begin with a sequence of 15 dashes:

$$- - - - - - - - - - - - - - -.$$

Now we count off the positive integers by the dashes:

$$-| - -| - - - | - - - -| - - - - - |.$$

We now come to 6. So we go back to the beginning (as though the dashes were in a circle) and keep counting. We can also count 7, which takes us to a new solidus:

$$-| - -| - - - | - - - -| - - - | - -|.$$

The count of $8 = 2 + 6$ takes us "around the bend", and does not require a solidus in a new slot. Nor will we need a new solidus for any further integers; 7 is the highest number that will require a new solidus.

Now just take for our cycle the number of dashes between the solidi: 1, 2, 3, 4, 3, 2 .

This suggests that, to find other Prague clock sequences, we can follow the dash process with different numbers of dashes, corresponding to the sum of the numbers in the generating cycles. Try this out for different sums.

3. Modular arithmetic.

The dash process might seem unwieldy, and we have a mathematical tool to ease things. Return to the sequence that we started with, with sum 15. To achieve pieces of the sequences to give the first four numbers, we see that we can provisionally choose the numbers $1, 2, 3, 4, 5$ for the cycle. So far, we are ok to achieve the sum of 6:

$$(1)(2)(3)(4)(5)(1 + 2 + 3).$$

However, as things stand, we cannot get the next few numbers to add to 7, unless we break up the 5, to get the cycle $1, 2, 3, 4, 3, 2$.

To see where we break into a cycle, we have to look at the sums $1+2+3+\cdots+n$, which we have to achieve by summing the sequence from the start for each n, modulo 15. (Recall, that $a \equiv b \pmod{m}$ means that $a - b$ is divisible by m; equivalently, it means that a and b leave the same remainder when divided by m.)

We have, modulo 15, that

$$1 = 1; \quad 1+2 = 3; \quad 1+2+3+4 = 10;$$
$$1+2+3+4+5 = 15 \equiv 0;$$
$$1+2+3+4+5+6 = 21 \equiv 6;$$
$$1+2+3+4+5+6+7 = 28 \equiv 13;$$
$$1+2+3+4+5+6+7+8 = 36 \equiv 6;$$
$$1+2+3+4+5+6+7+8+9 = 45 \equiv 0.$$

The numbers $1, 3, 10, 0, 6, 13, 6$ are the sums we must achieve after running through complete cycles; in order, these are $0, 1, 3, 6, 10, 13, 15$ and this accounts for the differences $1, 2, 3, 4, 3, 2$ that define the cycle.

Suppose now that the cycle is to have sum 5. Then we can work modulo 5, and see that the sums possible counting from the beginning of the sequence must be 1, 2, $1 \equiv 1+2 = 3$ mod 5, $0 \equiv 1+2+3 = 4 \equiv 1+2+3+4+5$, and so on. This leads us to the sequence

$$1, 2, 2, 1, 2, 2, 1, 2, 2, \ldots.$$

Go through the same process for other possible cycle sums.

Here is a question for you to investigate: given cycle sum s, what is the largest number n for which $1+2+3+\cdots+n$ gives a remainder upon division by s tht forces a new splitting of a number in the provisional cycle? We saw that for $s = 15$, the answer is $n = 7$.

4. Examples. For every odd cycle sum, we can find a Prague clock sequence. When the cycle sum is even, can you show that the cycle can be decomposed into two or more smaller cycles? This table gives the cycles for small odd cycle sums.

Cycle sum	Cycle	Largest new remainder
1	1	1
3	1, 2	2
5	1, 2, 2	2
7	1, 2, 3, 1	3
9	1, 2, 3, 3	3
11	1, 2, 1, 2, 4, 1	5
13	1, 1, 1, 3, 2, 2, 3	6
15	1, 2, 3, 4, 3, 2	7
17	1, 1, 1, 1, 2, 4, 1, 4, 2	8
19	1, 1, 1, 3, 1, 2, 1, 5, 2, 2	9
21	1, 2, 3, 1, 3, 3, 2, 6	10
31	1, 2, 1, 1, 1, 4, 2, 2, 1, 1, 5, 3, 3, 1, 1, 2	