PRODUCTS THAT ARE POWERS.

A mathematical vignette

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This investigation was originally designed for school students to gain fluency with the factoring of positive integers as a product of prime powers through the challenge of optimizing sets of numbers with square products. However, like many such things, it morphed into something larger and generated some interesting more general questions.

1. Square products. A diverting pastime is to begin with a positive integer n and multiply it by any number of distinct larger integers until the product is a square. It is easy to find examples. Simply multiply n by $4n$, for instance, to get the square $(2n)^2$. So to make it a bit more challenging, ask that the largest integer that we introduce into the product is *as small as possible*.

For example, let $n = 5$. We can do better than $\{5, 20\}$, by selecting the set of integers in the product to be $\{5, 10, 18\}$ or better yet $\{5, 12, 15\}$. But as you can readily see, the set $\{5, 8, 10\}$ is one whose largest number is minimum, since we need at least two multiples of 5.

For each positive integer n, define the function $f_2(n)$ to be the smallest number k exceeding n for which there is a set of at least two distinct integers, including n, in the closed interval $[n, k]$ whose product is a square. Thus, $f_2(5) = 10$. Just to get you into proper spirit, check that $f_2(2) = 6$, $f_2(3) = 8$ and $f_2(8) = 15$. What is $f_2(12)$?

If $n = m^2$ is itself a square, then we could choose the set $\{m^2, (m + 1)^2\}$, thus showing that $f_2(m^2) \leq (m + 1)^2$. Is it possible for $f_2(m^2)$ to be strictly less than $(m + 1)^2$? Or, to ask a stronger question, are there only finitely many values of m for which $f_2(m^2) = (m + 1)^2$? Or for which $f_2(m^2) < (m + 1)^2$?

It is easy to see that $f_2(n) \leq 4n$ for each positive integer n. However, empirical investigation suggests that $f_2(n) \leq 2n$ except for $1 \leq n \leq 4$. This turns out to be so, and there are at least two approaches that you can take to establish this. First, you can show that, for sufficiently large n, the open interval $(n, 2n)$ contains a number of the form $2k^2$, and check small values of n individually. Alternatively, you can try to put inside each open interval $(n, 2n)$ a number equal to an odd power of 2 multipled by a small square. Note that, with this result in hand, that $f_2(n) = 2n$ whenever n is a prime exceeding 3.

This leads to another question. What rational values are assumed by $f_2(n)/n$ infinitely often?

Finally, is $f_2(2m) \leq 3m$ for sufficiently large values of m?

2. Higher power products. We can play the same game with higher powers, and define, for each positive integer $r \geq 2$, the function $f_r(n)$ to be the smallest value of k exceeding n for which there are at least two distinct integers, including n, in the closed interval $[n, k]$ whose product is an rth power. You may wish to verify that $f_3(6) = f_4(6) = f_5(6) = 18$ while $f_6(6) = 27$.

We can ask questions analogous to those posed for the case $n = 2$. For $n \geq 2$, show that $f_3(n) \leq 3n$. ($f_3(1) = 4$.) More generally, is it true that $f_r(n) \leq rn$ for n sufficiently large, with equality occurring when n is a prime exceeding r?

Prove that $f_r(m^r)$ is always strictly less than $(m+1)^r$ when $r \geq 3$.

What values of $f_r(n)/n$ are assumed for infinitely many values of n?

3. Comments. In finding $f_2(12)$, we must be sure that our set contains another multiple of an odd power of 3. We can try to see if we can put in either 15 or 21 (we may decide whether to include 18 depending on straightening out the power of 2). We can find the set $\{12, 14, 18, 21\}$; but $\{12, 15, 20\}$ is better. For a set with a smaller maximum, we see that we have to exclude any number divisible by a prime greater than 3, and $\{12, 18\}$ does not work. Therefore $f_2(12) = 20$.

To show that $f(n) \leq 2n$ for $n \geq 5$, we can check the cases $5 \leq n \leq 9$ by hand. Now let $x_1 = 18 = 2 \times 3^2$, $x_2 = 32 = 2^5$, $x_3 = 50 = 2 \times 5^2$ and $x_m = 4 \times x_{m-3}$ for $m \geq 4$. Each x_m is the product of an odd power of 2 and a square and $x_m < x_{m+1} < 2x_m$ for each $m \geq 1$. Then for each $n \geq 10$, we can show that for some m, $n < x_m < 2n$ for some value of $m \geq 1$.

To show that $f_3(n) \leq 3n$ for $n \geq 2$, we can check the small cases by hand and then show that for n sufficiently large, there is always a values of k for which $n < 36k^3 < 3n$ so that $n \times 2n \times 3n \times 36k^3 = (6kn)^3$ is a cube. The only fly in the ointment is that $36k^3$ might equal $2n$, so that the case $n = 18k^3$ needs special attention. However, $f_3(18) = 25 < 54$, a suitable set being $\{18, 20, 24, 25\}$, and we can derive a set that works for $18k^3$.

There is another approach to this result. It appears to be the case that, for $n \geq 9$, we can find a set of distinct integers in the closed interval $[n, 3n]$ whose cardinality is a multiple of 3, which contains n, $2n$ and $3n$, and whose elements multiply to give a cubic product. We can check this for an initial tranche of integers n and then try to prove it in general by an induction argument that involves multiplying each element in a set for n by a number u to get a set for un. The attempt to construct an argument is beset by various annoyances.

In order to show that $f_r(n) \leq rn$ for sufficiently large n, we can try to find numbers v and k such that $n < vk^r < rn$ such that $r!v$ is a rth power, vk^r is not a multiple of n, and

$$n \times 2n \times 3n \times \cdots \times rn \times (vk^r)$$

is an rth power.

The evaluation of $f_2(m^2)$ turns out to be more interesting than it first appears. Investigation of small values of m reveals that the open interval $(m^2, (m+1)^2)$ contains a set of distinct integers with a square product more often than not. It seems plausible that $f_2(m^2)$ could equal, or not equal, $(m+1)^2$ each infinitely often. Where the inequality is strict, determination of the set of integers seems to be a highly idiosyncratic process and it is hard to see how one can devise a systematic process that will cover infinitely many cases. Sometimes we find that we only need three numbers for a square product. For example, between 7^2 and 8^2, we have the triple $(50, 60, 63)$; between 12^2 and 13^2, we have the triple $(147, 150, 162)$. Does this happen infinitely often? However, it is not possible to find a pair of numbers strictly between two consecutive squares whose product is square. Can you see why?

Since $m^3 \times m^2(m+1) \times m(m+1)^2 = (m^2(m+1))^3$ is cube, $f_3(m^3) \leq m(m+1)^2$. Incidentally, note that $f_3(m^2) \leq (m+1)^2$ since $m^2 \times m(m+1) \times (m+1)^2 = (m(m+1))^3$.

More generally, since $m^r \times m^s(m+1)^{r-s} \times m^{r-s}(m+1)^s$ is an rth power for $2s \geq r$, we see that $f_r(m^r) < (m+1)^r$ for all m.

APPENDIX

This appendix gives an expanded version of the exploration along with tables of values.

Let r and n be a positive integer and let $f_r(n)$ be the minimum value of a_k over all sets $\{a_1, a_2, \ldots, a_k\}$ of distinct integers, where k is a positive integer and where
(i) $n < a_1 < a_2 < \cdots < a_k$
and
(ii) the product $na_1a_2 \ldots a_k$ is an rth power.

CONJECTURE: For each positive integer $r \geq 2$, $f_r(n) \leq rn$ for all but finitely many positive integers n.

We note that $f_r(n) = rn$ for infinitely many values of n. Let p be any prime not less than r. Then any collection of integers that contains p whose product is divisible by p^2 must contain at least r multiples of p or at least one multiple p^2. In any case, it contains a multiple of p not less than rp.

§*1. The case* $r = 1$. It is clear that $f_1(n) = n + 1$.

§*2. The case* $r = 2$.

2.1. There are two arguments to show that $f_2(n) \leq 2n$ for $n \geq 5$.

2.1.1. Define $x_k = 2k^2$ and note that $x_k < x_{k+1} < 2x_k$ for $k \geq 3$. Let $n \geq 18$, and select k so that $x_k \leq n < x_{k+1}$. Then, since $n < x_{k+1} < 2x_k \leq 2n$, it follows that the product of n, x_{k+1} and $2n$ is a square and the statement holds. If $10 \leq n \leq 17$, then the statement holds since $n \times 18 \times 2n$ is a square. If $n = 5, 6, 7$, then $n \times 8 \times 2n$ is square and the statement holds. Since $8 \times 10 \times 12 \times 15$ and 9×16 are square, the statement holds for 8 and 9. However, it can be checked that it does not hold for $n = 1, 2, 3, 4$, so 4 is the largest number for which the statement fails.

2.1.2. Define the sequence $x_1 = 18 = 2 \times 3^2$, $x_2 = 32 = 2^5$, $x_3 = 50 = 2 \times 5^2$, and $x_m = 4 \times x_{m-3}$ for $m \geq 4$. Then each x_m is the product of an odd power of 2 and a square. Furthermore, note that $x_1 < x_2 < 2x_1$, $x_2 < x_3 < 2x_2$, $x_3 < x_4 < 2x_4$, so that $x_m < x_{m+1} < 2x_m$ for each $m \geq 1$. Suppose that $10 \leq n \leq 17$. Then $n \times 18 \times 2n$ is square. Let $n \geq 18$, and suppose that m is the largest integer for which $x_m \leq n$. Then $n < x_{m+1} < 2x_m \leq 2n$, and $n \times x_{m+1} \times 2n$ is a square. Thus, the statement is true whenever $n \geq 10$. The table shows it is true for $5 \leq n \leq 9$.

2.2. It is clear that $f_2(m^2) \leq (m+1)^2$. However, we can have strict inequality. *Question: Are there infinitely many values of m for which the inequality is strict?* The issue turns on finding in the open interval $(m^2, (m+1)^2)$ a finite set of distinct integers whose product is a square. This seems to be possible for most values of m, as indicated in the table below.

Interval	Factors of square	Square root of product
$(5^2, 6^2)$	$27, 28, 30, 32, 35$	$2^4 \times 3^3 \times 5 \times 7$
$(7^2, 8^2)$	$50, 56, 63$	$2^2 \times 3 \times 5 \times 7$
$(8^2, 9^2)$	$65, 66, 70, 72, 77, 78$	$2^3 \times 3^2 \times 5 \times 7 \times 11 \times 13$
$(9^2, 10^2)$	$88, 98, 99$	$2^2 \times 3 \times 7 \times 11$
$(10^2, 11^2)$	$102, 105, 108, 119, 120$	$2^3 \times 3^3 \times 5 \times 7 \times 17$
$(11^2, 12^2)$	$125, 126, 128, 140, 143$	$2^5 \times 3 \times 5^2 \times 7$
	$128, 130, 132, 135$	$2^5 \times 3^2 \times 5 \times 11 \times 13$
$(12^2, 13^2)$	$147, 150, 162$	$2 \times 3^3 \times 5 \times 7$
$(13^2, 14^2)$	$170, 171, 176, 187, 190$	$2^3 \times 3 \times 5 \times 11 \times 17 \times 19$
$(14^2, 15^2)$	$200, 204, 208, 216, 221$	$2^6 \times 3^2 \times 5 \times 13 \times 17$
	$198, 210, 216, 220, 224$	$2^6 \times 3^3 \times 5 \times 7 \times 11$
$(15^2, 16^2)$	$228, 234, 240, 247, 250$	$2^4 \times 3^2 \times 5^2 \times 13^2 \times 19^2$
	$230, 231, 236, 240, 253$	$2^4 \times 3 \times 5 \times 7 \times 11 \times 23$
	$230, 231, 240, 242, 252, 253$	$2^4 \times 3^2 \times 5 \times 7 \times 11^2 \times 23$
	$230, 231, 240, 242, 243, 253$	$2^3 \times 3 \times 5 \times 7^2 \times 11^2 \times 23$
$(16^2, 17^2)$	$260, 264, 273, 275, 280$	$2^4 \times 3 \times 5^2 \times 7 \times 11 \times 13$
	$260, 266, 273, 285, 288$	$2^4 \times 3 \times 5 \times 13 \times 19$
$(17^2, 18^2)$	$297, 299, 308, 312, 322$	$2^3 \times 3^2 \times 7 \times 11 \times 13 \times 23$
$(18^2, 19^2)$	$330, 340, 343, 352, 357$	$2^4 \times 3 \times 7^2 \times 22 \times 17$
	$325, 336, 343, 351$	$2^4 \times 3^4 \times 5^2 \times 7^4 \times 13^2$
	$338, 343, 350$	$2^2 \times 5^2 \times 7^2 \times 13^2$
	$325, 330, 336, 340, 351, 352, 357$	$2^{12} \times 3^6 \times 5^4 \times 7^2 \times 11^2 \times 17^2$
$(19^2, 20^2)$	$363, 384, 392$	$2^{10} \times 3^2 \times 7^2 \times 11^2$
$(20^2, 21^2)$	$408, 414, 416, 418, 425, 429, 437$	$2^5 \times 3^2 \times 5 \times 11 \times 19 \times 23$
$(22^2, 23^2)$	$504, 507, 512, 525$	$2^6 \times 3^2 \times 5 \times 7 \times 13$

2.3. The product of the entries in the set $\{2n, 6k^2, 3n\}$ is a square and this shows that $f_2(2n) \leq 3n$ when k can be found with $(n/3) < k^2 < (n/2)$, or $2k^2 < n < 3k^2$. This is always possible when $n \geq 25$. For lower values of n, the inequality fails when $n = 1, 2, 3, 4, 5, 6, 8, 16$. The product of the entries in the set $\{3n, 3k^2, 4n\}$ is a square and this shows that $f_(3n) \leq 4n$ when k can be found with $n < k^2 < (4n/3)$, or $k^2 < (4n/3) < (4k^2/3)$. This always happens when $n \geq 30$.

Question: For what values of s does the equation $f(n) = sn$ have infinitely many solutions?

Question: Is $f_2(2m) \leq 3m$ for all sufficiently large m?

n	$f_2(n)$	
1	4	$\{1, 4\}$
2	6	$\{2, 3, 6\}$
3	8	$\{3, 6, 8\}$
4	9	$\{4, 9\}$
5	10	$\{5, 8, 10\}$
6	12	$\{6, 8, 12\}$
7	14	$\{7, 8, 14\}$
8	15	$\{8, 10, 12, 15\}$
9	16	$\{9, 16\}$
10	18	$\{10, 12, 15, 18\}$
11	22	$\{11, 18, 22\}$
12	20	$\{12, 15, 20\}$
13	26	$\{13, 18, 26\}$
14	21	$\{14, 15, 18, 20, 21\}$
15	24	$\{15, 18, 20, 24\}$
16	25	$\{16, 25\}$
17	34	$\{17, 18, 34\}; \{17, 32, 34\}$
18	27	$\{18, 24, 27\}$
19	38	$\{19, 32, 38\}$
20	30	$\{20, 24, 30\}$
21	28	$\{21, 27, 28\}$
22	33	$\{22, 24, 33\}$
23	46	$\{23, 32, 46\}$
24	32	$\{24, 27, 32\}$
25	35	$\{25, 27, 28, 30, 32, 35\}$
26	39	$\{26, 27, 32, 39\}$
27	35	$\{27, 28, 30, 32, 35\}$
28	40	$\{28, 32, 25, 40\}$
29	58	$\{29, 32, 58\}; \{29, 50, 58\}$
30	42	$\{30, 35, 42\}$
31	62	$\{31, 32, 62\}; \{31, 50, 62\}$
32	50	$\{32, 50\}$
33	48	$\{33, 44, 48\}$
34	51	$\{34, 35, 40, 42, 51\}$
35	48	$\{35, 40, 42, 48\}$
36	49	$\{36, 49\}$
37	74	$\{37, 50, 74\}; \{37, 72, 74\}$
38	57	$\{38, 48, 50, 57\}$
39	52	$\{39, 48, 52\}$
40	50	$\{40, 45, 50\}$
41	82	$\{41, 50, 82\}; \{41, 72, 82\}$
42	56	$\{42, 48, 56\}$
43	86	$\{43, 50, 85\}; \{43, 72, 86\}$
44	55	$\{44, 45, 55\}$
45	60	$\{45, 50, 54, 60\}$
46	69	$\{46, 54, 69\}$
47	94	$\{47, 50, 94\}; \{47, 72, 94\}$
48	54	$\{48, 50, 54\}$
49	63	$\{49, 50, 56, 63\}$
50	63	$\{50, 56, 63\}$

n	$f_2(n)$	
51	68	$\{51, 54, 56, 63, 68\}$
52	65	$\{52, 54, 56, 60, 63, 65\}$
53	106	$\{53, 72, 106\}$; $\{53, 98, 106$
54	75	$\{54, 55, 60, 66, 75\}$
55	72	$\{55, 60, 66, 72\}$
56	77	$\{56, 60, 63, 66, 70, 72, 77\}$
57	76	$\{57, 75, 76\}$
58	87	$\{58, 66, 75, 77, 84, 87\}$
59	118	$\{59, 72, 118\}$; $\{59, 98, 118\}$
60	88	$\{60, 66, 88\}$
61	122	$\{61, 72, 122\}$; $\{61, 98, 122\}$
62	93	$\{62, 65, 78, 80, 93\}$
63	85	$\{63, 68, 70, 72, 85\}$
64	80	$\{64, 66, 70, 75, 77, 80\}$
65	84	$\{65, 70, 78, 84\}$
80	98	$\{80, 90, 98\}$
81	99	$\{81, 88, 98, 99\}$
100	121	$\{100, 121\}$
121	143	$\{121, 125, 126, 130, 132, 135, 140, 143\}$
144	162	$\{144, 147, 150, 162\}$
169	190	$\{169, 170, 171, 176, 187, 190\}$
196	221	$\{196, 200, 204, 208, 216, 221\}$
225	250	$\{225, 228, 234, 240, 247, 250\}$
256	280	$\{256, 260, 264, 273, 275, 280\}$
289	322	$\{289, 297, 299, 308, 312, 322\}$
324	350	$\{324, 338, 343, 350\}$
361	392	$\{361, 363, 384, 392\}$
400	437	$\{400, 408, 414, 416, 418, 425, 429, 437\}$
484	523	$\{484, 504, 507, 512, 525\}$

§3. *The case $r = 3$.*

n	$f_3(n)$	Optimal sets	Other efficient sets
1	4	$\{1, 2, 4\}$	
2	4	$\{2, 4\}$	
3	9	$\{3, 9\}$	
4	9	$\{4, 6, 9\}$	
5	15	$\{5, 6, 10, 12, 15\}$	
6	18	$\{6, 16, 18\}$	
7	21	$\{7, 9, 10, 14, 15, 16, 18, 20, 21\}$	
8	16	$\{8, 9, 12, 16\}$	
9	16	$\{9, 12, 16\}$	
10	28	$\{10, 14, 15, 20, 21, 24, 28\}$	
11	33	$\{11, 12, 22, 24, 33\}$	
12	18	$\{12, 18\}$	
13	39	$\{13, 16, 36, 39\}$	
14	28	$\{14, 18, 20, 21, 25, 28\}$	
15	32	$\{15, 18, 25, 32\}$	
16	25	$\{16, 20, 25\}$	
17	51	$\{17, 34, 36, 51\}$	
18	25	$\{18, 20, 24, 25\}$	
19	57	$\{19, 36, 38, 57\}$	
20	42	$\{20, 21, 28, 30, 40, 42\}$	$\{20, 30, 45\}; \{20, 50\}\}$
22	44	$\{22, 24, 25, 28, 33, 35, 42, 44\}$	
23	69	$\{23, 36, 46, 69\}$	
24	45	$\{24, 30, 36, 40, 45\}$	
25	36	$\{25, 30, 36\}$	
26	52	$\{26, 32, 39, 45, 50, 52\}$	
27	48	$\{27, 36, 48\}$	
28	45	$\{28, 35, 40, 42, 45\}$	
29	87	$\{29, 36, 58, 87\}$	
30	45	$\{30, 32, 40, 45\}$	
31	93	$\{31, 36, 62, 93\}$	
32	54	$\{32, 54\}$	
33	55	$\{33, 40, 44, 45, 54, 55\}$	
34	68	$\{34, 49, 51, 63, 68\}$	
35	70	$\{35, 60, 63, 70\}$	
36	48	$\{36, 48\}$	$\{36, 42, 49\}$
37	111	$\{37, 45, 74, 100, 111\}$	
38	76	$\{38, 49, 57, 63, 76\}$	
41	123	$\{41, 45, 82, 100, 123\}$	
43	129	$\{43, 45, 86, 100, 129\}$	
47	141	$\{47, 63, 94, 98, 128, 141\}$	
53	159	$\{53, 63, 98, 106, 128, 159\}$	
59	177	$\{59, 63, 98, 118, 128, 177\}$	
61	183	$\{61, 63, 98, 122, 128, 183\}$	

In the list so far, we see that, for $n \geq 2$, $f_3(n) \leq 3n$. We will establish this. When n is prime, there must be in fact equality.

3.1. We use induction to obtain a stronger result for $n \geq 10$, to wit: *There exists a set $\{n, 2n, 3n; b_1, b_2, \ldots, b_{3r}\}$ of distinct positive integers whose product is a cube and for which $n < b_i < 3n$ for each i.* (Note that the number of entries is a multiple of 3.)

For $10 \leq n \leq 15$, then set $\{n, 2n, 3n; 16, 18, 27\}$ is suitable, the product of its entries being $2^6 \times 3^6 \times n^3$. For $16 \leq n \leq 20$, the set

$$\{n, 2n, 3n; 21, 24, 25, 27, 28, 35\}$$

is suitable, the product of its entries being $2^6 \times 3^6 \times 5^3 \times 7^3 \times n^3$. Thus the result holds for $10 \leq n \leq 20$.

Now suppose that the result holds for $10 \leq n \leq 2m - 1$, where $m \geq 10$, and let

$$\{m, 2m, 3m; b_1, b_2, \ldots, b_{3r}\}$$

be a suitable set for $n = m$ with product $6b_1b_2 \ldots b_{3r}m^3 = u^3m^3$. Then

$$\{2m, 4m, 6m; 2b_1, 2b_2, \ldots, 2b_{3r}\}$$

is a suitable set for $n = 2m$ with product $(2^{r+1})^3u^3m^3$. For $n = 2m + 1$, consider the set

$$\{2m + 1, 2(2m + 1), 3(2m + 1), 2b_1, 2b_2, \ldots, 2b_{3r}\}$$

We have that $2m < 2b_i < 6m < 3(2m + 1)$, so that $2m + 1 < 2b_i$ for each i. Also the product of the numbers in the set is $(2^r)^3u^3(2m + 1)^3$, so that this set is suitable for $n = 2m + 1$. **We need to deal with the possibility that $2b_i = 2(2m + 1), 3(2m + 1)$.**

Here is a list of sets whose cardinalities are a multiple of 3, whose maximum element is not greater than three times the minimum element, and whose elements multiply to a cube.

n	The set
6	$\{6, 16, 18\}$
8	$\{8, 12, 18\}$
9	$\{9, 12, 16\}$
10	$\{10, 12, 14, 15, 18, 20, 21, 24, 28\}$
11	$\{11, 12, 22, 24, 27, 33\}$
12	$\{12, 18, 27\}$
13	$\{13, 16, 18, 26, 27, 39\}$
14	$\{14, 18, 20, 21, 25, 28\}$
15	$\{15, 18, 20, 27, 30, 36\}$
16	$\{16, 20, 25\}$
17	$\{17, 25, 34, 36, 40, 57\}$
18	$\{18, 24, 32\}; \{18, 30, 50\}$
19	$\{19, 25, 36, 38, 40, 57\}$
20	$\{20, 27, 50\}; \{20, 21, 28, 30, 40, 42\}$
22	$\{22, 24, 25, 27, 28, 33, 35, 42, 44\}$
23	$\{23, 25, 36, 40, 46, 69\}$
24	$\{24, 27, 30, 36, 40, 45\}$
25	$\{25, 30, 36\}$
26	$\{26, 32, 39, 45, 50, 52\}$
27	$\{27, 36, 48\}$
29	$\{29, 36, 49, 56, 58, 87\}$
36	$\{36, 42, 49\}$
37	$\{37, 45, 64, 74, 100, 111\}$
38	$\{38, 49, 57, 63, 64, 76\}$
41	$\{41, 45, 64, 82, 100, 123\}$
43	$\{43, 45, 64, 86, 100, 129\}$
47	$\{47, 63, 94, 98, 128, 141\}$
53	$\{53, 63, 98, 106, 128, 159\}$
59	$\{59, 63, 98, 118, 128, 177\}$
61	$\{61, 63, 98, 122, 128, 183\}$

3.2. A second approach is to augment the set $\{n, 2n, 3n\}$ by additional numbers between n and $3n$. The following cases cover $n \geq 13$:

$\{n, 2n, 3n; 36\}$	$13 \leq n \leq 35; n \neq 18$
$\{n, 2n, 3n; 45, 100\}$	$34 \leq n \leq 44$
$\{n, 2n, 3n; 60, 75\}$	$31 \leq n \leq 59$
$\{m, 2n, 3n; 63, 98, 128\}$	$43 \leq n \leq 62$
$\{n, 2n, 3n; 72, 108\}$	$37 \leq n \leq 71; n \neq 54$
$\{n, 2n, 3n; 48, 126\}$	$43 \leq n \leq 97; n \neq 63$
$\{n, 2n, 3n; 128, 144\}$	$55 \leq n \leq 127; n \neq 64, 72$
$\{n, 2n, 3n; 110, 225\}$	$76 \leq n \leq 159$
$\{n, 2n, 3n; 243, 256\}$	$86 \leq n \leq 242; n \neq 128$
$\{n, 2n, 3n; 352, 363, 375\}$	$126 \leq n \leq 351; n \neq 176$
$\{n, 2n, 3n; 972\}$	$325 \leq n \leq 971; n \neq 486$

The last entry is a special case of $\{n, 2n, 3n; 36k^3\}$ which works for $12k^3 + 1 \leq n \leq 36k^3 - 1$, $n \neq 18k^3$. The sets of validity for the sets overlap when $k \geq 3$ since

$12(k+1)^3+1 \leq 36k^3-1$. The missing values of n, namely $18k^3$ and $176 = 2^3 \times 22$ can be settled from the cases $n = 18, 22$ respectively.

An interesting examples of a set is

$$\{n, 2n, 3n; 2^{3k+1}, 3^2 \cdot 2^{3k-2}\}$$

which works for $3 \cdot 2^{3k-2}+1 \leq n \leq 2^{3k+1}-1$, $n \neq 2^{3k}, 3^2 \cdot 2^{3k-3}, 3 \cdot 2^{3k-2}$.

3.3. $f_3(m^2) \leq (m+1)^2$, since the product of the numbers $\{m^2, m(m+1), (m+1)^3\}$ is a cube. Equality does not hold when $m+1$ is a cube. In that case $f_3(m^2) = m(m+1)$; an example is that $f(49) = 56$.

3.4. $f_3(m^3) \leq m(m+1)^2$, since the product of the numbers is the set $\{m^3, m^2(m+1), m(m+1)^2\}$ is a cube. Are there infinitely many values of m for which the inequality is strict? Are there infinitely many values of m for which there is equality?

Question. Is $f(n) \leq 2n$ whenever n is even? When n is twice a prime, we must have $f(n) \geq 2n$.

§4. *The case $r = 4$.*

n	$f_4(n)$	
1	8	$\{1, 2, 8\}$
2	8	$\{2, 8\}$
3	9	$\{3, 6, 8, 9\}$
4	12	$\{4, 6, 9, 12\}$
5	20	$\{5, 6, 8, 10, 15, 18, 20\}$
6	18	$\{6, 12, 18\}$
7	28	$\{7, 9, 10, 14, 15, 21, 25, 28\}$
8	18	$\{8, 9, 18\}$
9	25	$\{9, 10, 18, 21, 25\}$
10	32	$\{10, 20, 25, 32\}$
11	44	$\{11, 22, 27, 32, 33, 44\}$
12	48	$\{12, 36, 48\}$
13	52	$\{13, 26, 27, 32, 39, 52\}$
16	32	$\{16, 20, 25, 27, 30, 32\}$

For the case $n = 16$, we have the following sets whose maximum is less than 64:

$$\{16, 24, 25, 27, 50\}, \quad \{16, 25, 27, 30, 40\}, \quad \{16, 20, 25, 27, 30, 32\}$$

For larger numbers:

$\{n, 2n, 3n, 4n; 27, 32\}$	$9 \leq n \leq 26; n \neq 16$
$\{n, 2n, 3n, 4n; 54\}$	$14 \leq n53; n \neq 18, 27$
$\{n, 2n, 3n, 4n; 96, 144\}$	$37 \leq n \leq 95; n \neq 48, 73$
$\{n, 2n, 3n, 4n; 162, 192\}$	$49 \leq n \leq 161; n \neq 54, 64, 81, 96$
$\{n, 2n, 3n, 4n; 125, 270\}$	$69 \leq n \leq 124$
$\{n, 2n, 3n, 4n; 256, 360, 375\}$	$94 \leq n \leq 255; n \neq 120, 125, 128, 180$
$\{n, 2n, 3n, 4n; 864\}$	$217 \leq n \leq 863; n \neq 288, 432$
$\{n, 2n, 3n, 4n; 1944, 2304, 2500\}$	$626 \leq n \leq 1943; n \neq 768, 972, 1152, 1250$
$\{n, 2n, 3n, 4n; 4374\}$	$1094 \leq n \leq 4374; n \neq 1458, 2187$
$\{n, 2n, 3n, 4n; 13824\}$	$3457 \leq n \leq 13823$

For $k \geq 3$, the set $\{n, 2n, 3n, 4n; 54k^4\}$ covers all $n \geq 1094$ with the exception of those that duplicate $54k^3$, and these can b derived in other ways.

§5. *The case $r = 5$.*

n	$f_5(n)$	
1	8	$\{1, 4, 8\}$
2	16	$\{2, 16\}$
3	18	$\{3, 9, 16, 18\}$
4	8	$\{4, 8\}$
5	25	$\{5, 6, 10, 12, 15, 18, 25\}$
6	18	$\{6, 8, 9, 18\}$
7	35	$\{7, 14, 16, 20, 21, 25, 27, 28, 30, 35\}$
8	36	$\{8, 10, 15, 16, 18, 24, 25, 27, 30, 36\}$
9	27	$\{9, 27\}$
10	27	$\{10, 12, 15, 20, 25, 27\}$
11	55	$\{11, 15, 16, 18, 22, 27, 30, 33, 44, 45, 50, 55\}$
12	36	$\{12, 18, 36\}$
13	65	$\{13, 15, 16, 18, 26, 27, 30, 39, 45, 50, 52, 65\}$
14	45	$\{14, 21, 25, 28, 30, 35, 42, 45\}$
15	40	$\{15, 16, 18, 20, 25, 27, 40\}$
16	27	$\{16, 18, 27\}$
18	45	$\{18, 25, 30, 40, 45\}$
20	45	$\{20, 25, 27, 40, 45\}$
27	48	$\{27, 30, 48\}$
32	81	$\{32, 40, 50, 64, 75, 81\}$
81	96	$\{81, 96\}$

$\{n, 2n, 3n, 4n, 5n; 50, 54, 75\}$	$16 \leq n \leq 49; n \neq 18, 25, 27$
$\{n, 2n, 3n, 4n, 5n; 150, 200, 208\}$	$42 \leq n \leq 149; n \neq 50, 52, 75, 100, 104$
$\{n, 2n, 3n, 4n, 5n; 512, 625, 648\}$	$130 \leq n \leq 511; n \neq 162, 216, 256, 324$

§6. The case $r = 6$.

n	$f_6(n)$	
1	8	$\{1, 2, 4, 8\}$
2	8	$\{2, 4, 8\}$
3	18	$\{3, 8, 9, 12, 18\}$
4	16	$\{4, 16\}$
5	27	$\{5, 8, 9, 10, 15, 20, 25, 27\}$
6	24	$\{6, 8, 9, 16, 18, 24\}$
6	27	$\{6, 16, 18, 27\}; \{6, 12, 24, 27\}$
7	42	$\{7, 10, 14, 16, 20, 21, 25, 27, 28, 30, 35, 42\}$
8	27	$\{8, 9, 24, 27\}; \{8, 12, 18, 27\}$
9	24	$\{9, 12, 18, 24\}$
10	32	$\{10, 15, 20, 24, 25, 27, 32\}$
11	66	$\{11, 15, 16, 18, 20, 22, 25, 30, 33, 44, 55, 66\}$
12	40	$\{12, 15, 18, 20, 24, 25, 30, 40\}$
13	78	$\{13, 15, 16, 18, 20, 25, 26, 30, 39, 52, 65, 78\}$
14	56	$\{14, 15, 21, 25, 27, 28, 32, 35, 42, 50, 56\}$
15	40	$\{15, 18, 20, 25, 27, 30, 32, 40\}$
16	32	$\{16, 18, 24, 32\}$
18	54	$\{18, 48, 54\}$
20	45	$\{20, 25, 27, 30, 40, 45\}$

General r.

Proposition. *For all but finitely many positive integers n, $f_r(n) \leq rn$ and equality occurs infinitely often.*

Proof. We note that the product of the numbers in the set

$$\{n, 2n, 3n, \ldots, rn; (r!)^{r-1}k^r\}$$

is an rth power for each positive integer k. Let $s = (r!)^{r-1}$. The smallest number in the set is n and the largest rn if and only if $n \leq pk^r \leq rn$. We have to show that when k is sufficiently large, then the sets work for all n. **However, we have to deal with the possibility that pk^r is equal to one of the in.**

Note that, since $\{m^2, m^3(m+1)^{r-s}, m^{r-s}(m+1)^s\}$ has a product which is an rth power, $f_r(m^r) < (m+1)^2$.