# THE UNIVERSITY OF TORONTO
# UNDERGRADUATE MATHEMATICS COMPETITION

*March 12, 2006*

*Time*: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

It is not necessary to do all the problems. Complete solutions to fewer problems are preferred to partial solutions to many.

1. (a) Suppose that a $6 \times 6$ square grid of unit squares (chessboard) is tiled by $1 \times 2$ rectangles (dominoes). Prove that it can be decomposed into two rectangles, tiled by disjoint subsets of the dominoes.

   (b) Is the same thing true for an $8 \times 8$ array?

2. Let $\mathbf{u}$ be a unit vector in $\mathbf{R}^3$ and define the operator $P$ by $P(\mathbf{x}) = \mathbf{u} \times \mathbf{x}$ for $\mathbf{x} \in \mathbf{R}^3$ (where $\times$ denotes the cross product).

   (a) Describe the operator $I + P^2$.

   (b) Describe the action of the operator $I + (\sin\theta)P + (1 - \cos\theta)P^2$.

3. Let $p(x)$ be a polynomials of positive degree $n$ with $n$ distinct real roots $a_1 < a_2 < \cdots < a_n$. Let $b$ be a real number for which $2b < a_1 + a_2$. Prove that

$$2^{n-1}|p(b)| \geq |p'(a_1)(b - a_1)| \ .$$

4. Two parabolas have parallel axes and intersect in two points. Prove that their common chord bisects the segments whose endpoints are the points of contact of their common tangent.

5. Suppose that you have a $3 \times 3$ grid of squares. A *line* is a set of three squares in the same row, the same column or the same diagonal; thus, there are eight lines.

   Two players $A$ and $B$ play a game. They take alternate turns, $A$ putting a 0 in any unoccupied square of the grid and $B$ putting a 1. The first player is $A$, and the game cannot go on for more than nine moves. (The play is similar to noughts-and-crosses, or tictactoe.) A move is *legitimate* if it does not result in two lines of squares being filled in with different sums. The winner is the last player to make a legitimate move.

   (For example, if there are three 0s down the diagonal, then $B$ can place a 1 in any vacant square provided it completes no other line, for then the sum would differ from the diagonal sum. If there are two zeros at the top of the main diagonal and two ones at the left of the bottom line, then the lower right square cannot be filled by either player, as it would result in two lines with different sums.)

   (a) What is the maximum number of legitimate moves possible in a game?

   (b) What is the minimum number of legitimate moves possible in a game that would not leave a legitimate move available for the next player?

   (c) Which player has a winning strategy? Explain.

6. Suppose that $k$ is a positive integer and that

$$f(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \cdots + a_k e^{\lambda_k t}$$

   where $a_1, \cdots, a_k, \lambda_1, \cdots, \lambda_k$ are real numbers with $\lambda_1 < \lambda_2 < \cdots < \lambda_k$. Prove that $f(t) = 0$ has finitely many real solutions. What is the maximum number of solutions possible, as a function of $k$?

7. Let $A$ be a real $3 \times 3$ invertible matrix for which the sums of the rows, columns and two diagonals are all equal. Prove that the rows, columns and diagonal sums of $A^{-1}$ are all equal.

8. Let $f(x)$ be a real function defined and twice differentiable on an open interval containing $[-1, 1]$. Suppose that $0 < \alpha \leq \gamma$ and that $|f(x)| \leq \alpha$ and $|f''(x)| \leq \gamma$ for $-1 \leq x \leq 1$. Prove that

$$|f'(x)| \leq 2\sqrt{\alpha\gamma}$$

for $-1 \leq x \leq 1$. (Part marks are possible for the weaker inequality $|f'(x)| \leq \alpha + \gamma$.)

9. A high school student asked to solve the surd equation

$$\sqrt{3x-2} - \sqrt{2x-3} = 1$$

gave the following answer: *Squaring both sides leads to*

$$3x - 2 - 2x - 3 = 1$$

*so* $x = 6$. The answer is, in fact, correct.

Show that there are infinitely many real quadruples $(a, b, c, d)$ for which this method leads to a correct solution of the surd equation

$$\sqrt{ax-b} - \sqrt{cx-d} = 1 \ .$$

10. Let $P$ be a planar polygon that is not convex. The vertices can be classified as either *convex* or *concave* according as to whether the angle at the vertex is less than or greater than $180°$ respectively. There must be at least two convex vertices. Select two consecutive convex vertices (*i.e.*, two interior angles less than $180°$ for which all interior angles in between exceed $180°$) and join them by a segment. Reflect the edges between these two convex angles in the segment to form along with the other edges of $P$ a polygon $P_1$. If $P_1$ is not convex, repeat the process, reflecting some of the edges of $P_1$ in a segment joining two consecutive convex vertices, to form a polygon $P_2$. Repeat the process. Prove that, after a finite number of steps, we arrive at a polygon $P_n$ that is convex.

**Solutions.**

1. (a) Suppose that a $6 \times 6$ square grid of unit squares (chessboard) is tiled by $1 \times 2$ rectangles (dominoes). Prove that it can be decomposed into two rectangles, tiled by disjoint subsets of the dominoes.

   (b) Is the same thing true for an $8 \times 8$ array?

*Solution.* (a) There are 18 dominoes and 10 interior lines in the grid. For the decomposition not to occur, each of the lines must be straddled by at least one domino. We argue that, in fact, at least two dominos must straddle each line. Since no domino can straddle more than one line, this would require 20 dominos and so yield a contradiction.

Each interior line has six segments. For a line next to the side of the square grid, an adjacent domino between it and the side must either cross one segment or be adjacent to two segments. Since the number of segments is even, evenly many dominos must cross a segment. For the next line in, an adjacent domino must be adjacent to two segments, be adjacent to one segment and cross the previous line, or cross one segment. Since the number of dominoes straddling the previous line is even, there must be evenly many that cross the segment. In this way, we can work our way from one line to the next.

(b) Number the squares in the grid by pairs $ij$ of digits where the square is in the $i$th row and $j$th column. Here is a tiling with dominos in which each interior line is straddled and no decomposition into subrectangles is possible:

$$(11-12),(13-14),(15-16),(17-27),(18-28),(21-31),(22-32),(23-33)$$

$$(24-34),(25-26),(35-45),(36-46),(37-38),(41-42),(43-44),(47-57)$$

$$(48-58),(51-61),(52-53),(54-55),(56-66),(62-72),(63-73),(64-74)$$

$$(65-75),(67-68),(71-81),(76-77),(78-88),(82-83),(84-85),(86-87)\ .$$

2. Let $\mathbf{u}$ be a unit vector in $\mathbf{R}^3$ and define the operator $P$ by $P(\mathbf{x}) = \mathbf{u} \times \mathbf{x}$ for $\mathbf{x} \in \mathbf{R}^3$ (where $\times$ denotes the cross product).

   (a) Describe the operator $I + P^2$.

   (b) Describe the action of the operator $I + (\sin\theta)P + (1 - \cos\theta)P^2$.

*Solution 1.* Recall that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ for any 3-vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

(a) We have that

$$P^2(\mathbf{x}) = \mathbf{u} \times (\mathbf{u} \times \mathbf{x}) = (\mathbf{u} \cdot \mathbf{x})\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{x} = (\mathbf{u} \cdot \mathbf{x})\mathbf{u} - \mathbf{x}$$

from which we see that $(I + P^2)(\mathbf{x}) = (\mathbf{u} \cdot \mathbf{x})\mathbf{u}$. Thus, $I + P^2$ is an orthogonal projection onto the one-dimensional spaces spanned by $\{\mathbf{u}\}$.

(b) Let $R = I + (\sin\theta)P + (1 - \cos\theta)P^2$. Observe that $P(\mathbf{u}) = \mathbf{o}$. Then $R(\mathbf{u}) = \mathbf{u}$, so that the one-dimensional space spanned by $\{\mathbf{u}\}$ is invariant under $R$. Let $\mathbf{v}$ be a unit vector orthogonal to $\mathbf{u}$ and $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. Then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is an orthonormal basis for $\mathbf{R}^3$. We have that

$$R(\mathbf{v}) = \mathbf{v} + (\sin\theta)\mathbf{w} + (1 - \cos\theta)(-\mathbf{v}) = (\cos\theta)\mathbf{v} + (\sin\theta)\mathbf{w}$$

and

$$R(\mathbf{w}) = \mathbf{w} + (\sin\theta)(\mathbf{u} \times \mathbf{w}) + (1 - \cos\theta)(-\mathbf{w}) = (-\sin\theta)\mathbf{v} + (\cos\theta)\mathbf{w}\ .$$

It follows that the plane through $\mathbf{o}$ orthogonal to $\mathbf{u}$ is rotated about the axis $\mathbf{u}$ through an angle $\theta$; by linearity, the same holds for all of $\mathbf{R}^3$.

*Solution 2.* Wolog, we may assume that $\mathbf{u} = (1,0,0)$. Let $\mathbf{x} = (x,y,z)$. The $\mathbf{u} \times \mathbf{x} = (0,-z,y)$ and $\mathbf{u} \times (\mathbf{u} \times \mathbf{x}) = (0,-y,-z)$. Hence $(I + P^2)\mathbf{x} = (x,0,0)$ and

$$(I + (\sin\theta)P + (1-\cos\theta)P^2)\mathbf{x} = (x,0,0) + (0,(\cos\theta)y - (\sin\theta)z, (\sin\theta)y + (\cos\theta)z) \ .$$

Thus, $I + P^2$ is an orthogonal projection onto $\mathbf{u}$ and $I + (\sin\theta)P + (1-\cos\theta)P^2$ is a rotation about the axis $\mathbf{u}$ through the angle $\theta$.

*Solution 3.* Let $\mathbf{v}$ be the image of $\mathbf{u}$ after a $90°$ rotation and let $\mathbf{w} = \mathbf{u}\times\mathbf{v}$. Then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is an orthonormal basis of $\mathbf{R}^3$ and $\mathbf{u} \times \mathbf{w} = -\mathbf{v}$. Hence, if $\mathbf{x} = x\mathbf{u} + y\mathbf{v} + x\mathbf{w}$, then we find that $(I + P^2)\mathbf{x} = x\mathbf{u} = (\mathbf{x}\cdot\mathbf{u})\mathbf{u}$ and $(I + (\sin\theta)P + (1-\cos\theta)P^2)(\mathbf{x}) = x\mathbf{u} + [(\cos\theta)y - (\sin\theta)z]\mathbf{v} + [(\cos\theta)z + (\sin\theta)y]\mathbf{w}$, and we reach the same conclusion as before.

3. Let $p(x)$ be a polynomials of positive degree $n$ with $n$ distinct real roots $a_1 < a_2 < \cdots < a_n$. Let $b$ be a real number for which $2b < a_1 + a_2$. Prove that

$$2^{n-1}|p(b)| \geq |p'(a_1)(b - a_1)| \ .$$

*Solution.* Wolog, let $p(x)$ have leading coefficient 1. Observe that, for $i > 1$,

$$a_i - a_1 = a_i - b + b - a_1 < a_i - b + a_2 - b < 2(a_i - b)$$

and that $p(x) = \prod(x - a_i)$, from which $p'(a_1) = \prod_{i\geq 2}(a_1 - a_i)$. Then

$$\begin{aligned}|p(b)| &= |b - a_1| \prod_{i\geq 2} |b - a_i| \\ &\geq |b - a_1| \prod_{i\geq 2} \frac{1}{2}(a_i - a_1) = |b - a_1||p'(a_i)|2^{-(n-1)} \ .\end{aligned}$$

4. Two parabolas have parallel axes and intersect in two points. Prove that their common chord bisects the segments whose endpoints are the points of contact of their common tangent.

*Solution 1.* Wolog, we may assume that the parabolas have the equations $y = ax^2$ and $y = b(x-1)^2 + c$. The common chord has equation

$$0 = a[b(x-1)^2 + c - y] - b[ax^2 - y] = 0 \ ,$$

or

$$(a - b)y + 2abx - a(b + c) = 0 \ . \tag{1}$$

Consider a point $(u, au^2)$ on the first parabola. The tangent at this point has equation $y = 2aux - au^2$. The abscissa of the intersection point of this tangent with the parabola of equation $y = b(x-1)^2 + c$ is given by the question

$$bx^2 - 2(b + au)x + (au^2 + b + c) = 0 \ .$$

This has coincident roots if and only if

$$(b + au)^2 = b(au^2 + b + c) \Longleftrightarrow a(a - b)u^2 + 2abu - bc = 0 \ . \tag{2}$$

In this situation, the coincident roots are $x = 1 + (au)/b$ and the point of contact of the common tangent with the second parabola is

$$\left(1 + \frac{au}{b}, \frac{a^2u^2}{b} + c\right) \ .$$

The midpoint of the segment joining the two contact points is

$$\left(\frac{b+au+bu}{2b}, \frac{abu^2+a^2u^2+bc}{2b}\right) .$$

Plugging this into the left side of (1) and using (2) yields that

$$\begin{aligned}[1/(2b)][(a-b)a(a+b)u^2 + (a-b)bc + 2ab^2 + 2ab(a+b)u - 2ab(b+c)] \\ = [(a+b)/(2b)][a(a-b)u^2 + 2abu - bc] = 0 .\end{aligned}$$

Thus, the coordinates of the midpoint of the segment satisfy (1) and the result follows.

*Solution 2.* [A. Feizmohammadi] Let the two parabolas have equations $y = ax(x-u)$ and $y = bx(x-v)$. Since the two parabolas must open the same way for the situation to occur, wolog, we may suppose that $a, b > 0$. The parabolas intersect at the points $(0,0)$ and $((au-bv)/(a-b), (ab(au-bv)(u-v)/(a-b)^2)$, and the common chord has equation $(a-b)y - ab(u-v)x = 0$.

Let $y = mx+k$ be the equation of the common tangent. Then both of the equations $ax^2-(au+m)x-k = 0$ and $bx^2 - (bv+m)x - k = 0$ have double roots. Therefore $(au+m)^2 + 4ak = (bv+m)^2 + 4bk = 0$, from which (by eliminating $k$),

$$ab(au^2 - bv^2) + 2ab(u-v)m + (b-a)m^2 = 0 .$$

The common tangent of equation $y = mx + k$ touches the first parabola at

$$\left(\frac{au+m}{2a}, \frac{m^2-a^2u^2}{4a}\right)$$

and the second parabola at

$$\left(\frac{bv+m}{2b}, \frac{m^2-b^2v^2}{4b}\right) .$$

The midpoint of the segment joining these two points is

$$\left(\frac{ab(u+v)+(a+b)m}{4ab}, \frac{(a+b)m^2-ab(au^2+bv^2)}{8ab}\right) .$$

Using these coordinates as the values of $x$ and $y$, we find that

$$\begin{aligned}8ab[(a-b)y - ab(u-v)x] &= (a-b)[(a+b)m^2 - ab(au^2+bv^2)] - 2a^2b^2(u-v)(u+v) \\ &\quad + 2ab(a+b)(u-v)m \\ &= (a^2-b^2)m^2 - 2ab(a+b)(u-v)m \\ &\quad - [(a-b)a^2bu^2 + (a-b)ab^2v^2 + 2a^2b^2u^2 - 2a^2b^2v^2] \\ &= (a^2-b^2)m^2 - 2ab(a+b)(u-v)m - [a^3bu^2 + a^2b^2u^2 - a^2b^2v^2 - ab^3v^2] \\ &= (a+b)[(a-b)m^2 - 2ab(u-v)m] - [ab(a(a+b)u^2 - ab(b(a+b)v^2] \\ &= (a+b)[(a-b)m^2 - 2ab(u-v)m - ab(au^2-bv^2)] = 0 .\end{aligned}$$

5. Suppose that you have a $3 \times 3$ grid of squares. A *line* is a set of three squares in the same row, the same column or the same diagonal; thus, there are eight lines.

   Two players $A$ and $B$ play a game. They take alternate turns, $A$ putting a 0 in any unoccupied square of the grid and $B$ putting a 1. The first player is $A$, and the game cannot go on for more than nine moves. (The play is similar to noughts-and-crosses, or tictactoe.) A move is *legitimate* if it does not result in two lines of squares being filled in with different sums. The winner is the last player to make a legitimate move.

(For example, if there are three 0s down the diagonal, then $B$ can place a 1 in any square provided it completes no other line, for then the sum would differ from the diagonal sum. If there are two zeros at the top of the main diagonal and two ones at the left of the bottom line, then the lower right square cannot be filled by either player, as it would result in two lines with different sums.)

(a) What is the maximum number of legitimate moves possible in a game?

(b) What is the minimum number of legitimate moves possible in a game that would not leave a legitimate move available for the next player?

(c) Which player has a winning strategy? Explain.

*Solution.* (a) A game cannot continue to nine moves. Otherwise, the line sum must be three times the value on the centre square of the grid (why?) and so must be 0 or 3. But some line must contain both zeros and ones, yielding a contradiction. [An alternative argument is that, if the array is filled, not all the rows can have the same numbers of 0s and 1s, and therefore cannot have the same sums.] However, an 8-move grid is possible, in which one player selects the corner squares and the other the squares in the middle of the edges.

(b) Consider any game after four moves have occurred and it is A's turn to play a zero. Suppose, first of all, that no lines have been filled with numbers. The only way an inaccessible square can occur is if it is the intersection of two lines each having the other two squares filled in. This can happen in at most one way. So $A$ would have at least four possible squares to fill in. On the other hand, if three of the first four moves complete a line, the fourth number can bar at most three squares for A in the three lines determined by the fourth number and one of the other three. Thus, $A$ would have at least two possible positions to fill. Thus, a game must go to at least five moves.

A five-move game can be obtained when A has placed three 0's down the left column and B has 1 in the centre square and another square of the middle column. Each remaining position is closed to B as it would complete a line whose sum is not 0.

(c) A has a winning strategy. Let A begin by putting 0 in the centre square. After three moves, A can guarantee either a row, column or diagonal with two 0's and one 1. Suppose that we have 0, 0, 1 down a diagonal. Wherever B puts his 1, A can complete the line through the centre containing it. For each of B's responses, A can make a move that stymies B. Suppose that we have 1, 0, 0 down a column through the centre parallel to a side of the grid. If B puts 1 in a row with a 0, then A can complete that row. Whatever B does next, A has a move that will stymie him. If B puts 1 in a row with another 1, A puts a 0 at the other end of the diagonal. Whatever B does next, A has a move that will stymie him.

6. Suppose that $k$ is a positive integer and that

$$f(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \cdots + a_k e^{\lambda_k t}$$

where $a_1, \cdots, a_k, \lambda_1, \cdots, \lambda_k$ are real numbers with $\lambda_1 < \lambda_2 < \cdots < \lambda_k$. Prove that $f(t) = 0$ has finitely many real solutions. What is the maximum number of solutions possible as a function of $k$?

*Solution.* We prove by induction that the equation can have at most $k-1$ solutions. It holds for $k = 1$. Suppose that it holds up to $k$, and that $a_1 e^{\lambda_1 t} + \cdots + a_{k+1} e^{\lambda_{k+1} t}$ vanishes for $m$ distinct values of $t$. Then

$$g(t) \equiv a_1 e^{(\lambda_1 - \lambda_{k+1})t} + a_2 e^{(\lambda_2 - \lambda_{k+1})t} + \cdots + a_k e^{(\lambda_k - \lambda_{k+1})t} + a_{k+1}$$

has the same roots, so that, by Rolle's theorem,

$$g'(t) \equiv a_1(\lambda_1 - \lambda_{k+1}) e^{(\lambda_1 - \lambda_{k+1})} t + a_2(\lambda_2 - \lambda_{k+1}) e^{(\lambda_2 - \lambda_{k+1})t} + \cdots + a_k(\lambda_k - \lambda_{k+1}) e^{(\lambda_k - \lambda_{k+1})t}$$

has $m-1$ distinct roots. Hence $m-1 \leq k-1$ and $m \leq k$.

For $k \geq 2$, let $p_k(x) = (x-1)(x-2)(x-3)\cdots(x-\overline{k-1}) \equiv a_1 + a_2x + \cdots + a_kx^{k-1}$. Then $f_k(t) = e^t p_k(e^t) = a_1e^t + \cdots + a_ke^{kt}$ has $k-1$ roots, namely $t = \log i$ $(1 \leq i \leq k-1)$. Hence $k-1$ roots is possible.

7. Let $A$ be a real $3 \times 3$ invertible matrix for which the sums of the rows, columns and two diagonals are all equal. Prove that the rows, columns and diagonal sums of $A^{-1}$ are all equal.

*Solution 1.* Let $r$ be the common sum and let $\mathbf{e} = (1,1,1)^{|bft}$. Then $A\mathbf{e} = r\mathbf{e}$. Since $A$ is invertible, $r \neq 0$ and $A^{-1}\mathbf{e} = r^{-1}\mathbf{e}$. The other eigenvalues (counting repetitions) of $A$ are $u$ and $-u$, since the trace of $A$ is equal to $r$. Since $A$ is invertible, $u \neq 0$.

The eigenvalues of $A^{-1}$ are $r^{-1}$, $u^{-1}$ and $-u^{-1}$, and the trace of $A^{-1}$ is equal to $r^{-1}+u^{-1}-u^{-1} = r^{-1}$. Since also $A^{-1}\mathbf{e} = r^{-1}\mathbf{e}$, the row and main diagonal sums are $r^{-1}$.

Since $(A^{-1})^{\mathbf{t}} = (A^{\mathbf{t}})^{-1}$, we can apply the same reasoning to $A^{\mathbf{t}}$ to find that the row sums of $A^{\mathbf{t}}$ and, hence, the column sums of $A^{-1}$ are equal to $r^{-1}$.

Finally, let $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then $B^{-1} = A^{-1}P$ and we can apply the result to $B$, which has the same row, column and diagonal sums as $A$. The main diagonals of $B$ and $B^{-1}$ are respectively equal to the cross diagonals of $A$ and $A^{-1}$, so we find that the cross diagonal of $A^{-1}$ is equal to $r^{-1}$. The desired result follows.

*Solution 2.* Since the common sum of the rows, columns and diagonals of a magic square is three times the central entry, the general form of the matrix A is

$$\begin{pmatrix} a & 3c-a-b & b \\ c+b-a & c & c+a-b \\ 2c-b & a+b-c & 2c-a \end{pmatrix} .$$

The adjugate, adj $A$ of this matrix is

$$\begin{pmatrix} 3c^2-a^2+b^2-ac-2bc & b^2-a^2-6c^2+5ac+bc & 3c^2+b^2-a^2+2ac-5bc \\ 5ac-5bc+b^2-a^2 & b^2-a^2+2ac-2bc & b^2-a^2-ac+bc \\ b^2-a^2-3c^2+2ac+bc & 6c^2+b^2-a^2-ac-5bc & b^2-a^2-3c^2+5ac-2bc \end{pmatrix} .$$

The sum of each row, column and diagonal of adj $A$ is $3(b-a)(b+a-2c)$. Since $(\det A)A^{-1} = \text{adj } A$, the desired result follows.

*Comment.* A direct solution for the rows and columns can be obtained from the relationships between the elements of $A$ and $A^{-1}$. Let $A = (a_{ij})$ and $A^{-1} = (b_{ij})$. Then, for $1 \leq i, k \leq 3$,

$$\sum_{i=1}^{3} a_{ij}b_{jk} = \delta_{ik} ,$$

where $\delta_{ij}$ is the Kronecker delta (1 if its subscripts agree, 0 otherwise). Let the common sums for $A$ be $s$. Then

$$s\sum_{j=1}^{3} b_{jk} = \sum_{j=1}^{3}(\sum_{i=1}^{3} a_{ij})b_{jk} = \sum_{i=1}^{3}(\sum_{j=1}^{3} a_{ij}b_{jk}) = \sum_{i=1}^{3} \delta_{ik} = 1 ,$$

whence $s \neq 0$ and $\sum_{j=1}^{3} b_{jk} = 1/s$. Hence, each column sum of $A^{-1}$ is $1/s$. A similar argument holds for the row sums.

8. Let $f(x)$ be a real function defined and twice differentiable on an open interval containing $[-1,1]$. Suppose that $0 < \alpha \leq \gamma$ and that $|f(x)| \leq \alpha$ and $|f''(x)| \leq \gamma$ for $-1 \leq x \leq 1$. Prove that

$$|f'(x)| \leq 2\sqrt{\alpha\gamma}$$

for $-1 \le x \le 1$. (Part marks are possible for the weaker inequality $|f'(x)| \le \alpha + \gamma$.)

*Solution.* Let $|f'(x)|$ attain its maximum value on $[-1,1]$ at $x = u$. Then, for $-1 \le a \le u \le b \le 1$, we have, by Taylor's theorem, for some $v, w$ in $[a,b]$,

$$f(a) - f(u) = (a-u)f'(u) + \frac{1}{2}(a-u)^2 f''(v) ,$$

$$f(b) - f(u) = (b-u)f'(u) + \frac{1}{2}(b-u)^2 f''(w) .$$

Therefore

$$(b-a)f'(u) = f(b) - f(a) + \frac{1}{2}[(a-u)^2 f''(v) - (b-u)^2 f''(w)]$$

so that

$$\begin{aligned}(b-a)|f'(u)| &\le |f(a)| + |f(b)| + \frac{1}{2}[(b-u)^2|f''(w)| + (a-u)^2|f''(v)|] \\ &\le 2\alpha + \frac{\gamma}{2}[(b-u)^2 + (a-u)^2] \le 2\alpha + \frac{\gamma}{2}(b-a)^2 ,\end{aligned}$$

since

$$(b-a)^2 - (b-u)^2 - (a-u)^2 = 2(b-u)(u-a) \ge 0 .$$

Select $a$ and $b$ so that $b - a = 2\sqrt{\alpha/\gamma}$. Then

$$|f'(u)| \le \left(\sqrt{\frac{\gamma}{\alpha}}\right)(\alpha) + \gamma\sqrt{\frac{\alpha}{\gamma}} = 2\sqrt{\alpha\gamma} .$$

*Comment.* If $a = -1$ and $b = 1$, then we get in the above that $2|f'(u)| \le 2\alpha + 2\gamma$.

9. A high school student asked to solve the surd equation

$$\sqrt{3x-2} - \sqrt{2x-3} = 1$$

gave the following answer: *Squaring both sides leads to*

$$3x - 2 - 2x - 3 = 1$$

*so* $x = 6$. The answer is, in fact, correct.

Show that there are infinitely many real quadruples $(a,b,c,d)$ for which this method leads to a correct solution of the surd equation

$$\sqrt{ax-b} - \sqrt{cx-d} = 1 .$$

*Solution 1.* Solving the general equation properly leads to

$$\sqrt{ax-b} - \sqrt{cx-d} = 1 \Longrightarrow ax - b = 1 + cx - d - 2\sqrt{cx-d}$$

$$\Longrightarrow (a-c)x = (b+1-d) - 2\sqrt{cx-d} .$$

To make the manipulation simpler, specialize to $a = c+1$ and $d = b+1$. Then the equation becomes

$$x^2 = 4(cx-d) \Longrightarrow 0 = x^2 - 4cx + 4d .$$

Using the student's "method" to solve the same equation gives $ax - b - cx - d = 1$ which yields $x = (1+b+d)/(a-c) = 2d$. So, for the "method" to work, we need

$$0 = 4d^2 - 8cd + 4d = 4d(d - 2c + 1)$$

which can be achieved by making $2c = d + 1$. So we can take

$$(a, b, c, d) = (t + 1, 2(t - 1), t, 2t - 1)$$

for some real $t$. The original problem corresponds to $t = 2$.

The equation

$$\sqrt{(t+1)x - 2(t-1)} - \sqrt{tx - (2t-1)} = 1$$

is satisfied by $x = 2$ and $x = 4t - 2$. The first solution works for all values of $t$, while the second is valid if and only if $t \geq \frac{1}{2}$. The equation $(t + 1)x - 2(t - 1) - tx - (2t - 1) = 1$ is equivalent to $x = 4t - 2$.

*Solution 2.* [G. Goldstein] *Analysis.* We want to solve simultaneously the equations

$$\sqrt{ax - b} - \sqrt{cx - d} = 1 \tag{1}$$

and

$$ax - b - cx - d = 1 \ . \tag{2}$$

From (1), we find that

$$ax - b = 1 + (cx - d) + 2\sqrt{cx - d} \ . \tag{3}$$

From (2) and (3), we obtain that $d = \sqrt{cx - d}$, so that $x = (d^2 + d)/c$. From (2), we have that $x = (1 + b + d)/(a - c)$.

Select $a, c, d$ so that $d > 0$ and $ac(a - c) \neq 0$, and choose $b$ to satisfy

$$\frac{d^2 + d}{c} = \frac{1 + b + d}{a - c} \ .$$

Let

$$x = \frac{d^2 + d}{c} = \frac{1 + b + d}{a - c} = \frac{(d^2 + d) + (1 + b + d)}{c + (a - c)} = \frac{(d + 1)^2 + b}{a} \ .$$

Then

$$\sqrt{ax - b} - \sqrt{cx - d} = \sqrt{(d + 1)^2} - \sqrt{d^2} = (d + 1) - d = 1$$

and

$$ax - b - cx - d = (d + 1)^2 + b - b - d^2 - d - d = 1 \ .$$

*Comments.* In Solution 2, if we take $c = d = 1$, we get the family of parameters $(a, b, c, d) = (a, 2a - 4, 1, 1)$. R. Barrington Leigh found the set of parameters given in Solution 1. A. Feizmohammadi provided the parameters $(a, b, c, d) = (2c, 0, c, 1)$.

10. Let $P$ be a planar polygon that is not convex. The vertices can be classified as either *convex* or *concave* according as to whether the angle at the vertex is less than or greater than $180°$ respectively. There must be at least two convex vertices. Select two consecutive convex vertices (*i.e.*, two interior angles less than $180°$ for which all interior angles in between exceed $180°$) and join them by a segment. Reflect the edges between these two convex angles in the segment to form along with the other edges of $P$ a polygon $P_1$. If $P_1$ is not convex, repeat the process, reflecting some of the edges of $P_1$ in a segment joining two consecutive convex vertices, to form a polygon $P_2$. Repeat the process. Prove that, after a finite number of steps, we arrive at a polygon $P_n$ that is convex.

*Solution.* Note that for the problem to work, we need to assume that, at each stage, we create a polygon that does not cross itself. Suppose that $P$ is a $m-$gon. Then each $P_i$ has at most $m$ sides, as some angles of $180°$ may be created at the vertices. Also, the perimeter of $P_i$ is the same as that of $P$ for each index $i$. Also $P \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_i \subseteq \cdots$. Therefore there is a disc that contains all of the polygons $P_i$ (the perimeter is at least as long as any line segment contained within the polygon).

Let $a$ be a vertex of $P$, and let $a_i$ be the corresponding vertex obtained from $a$ in $P_i$. each $a_i$ being equal to its predecessor or obtained from it by a reflection that takes it outside of $P_{i-1}$. In the latter case, $a_i$ must lie outside of $P_j$ when $j < i$. Because the sequence $\{a_i\}$ is bounded, it must have an accumulation point $a'$.

Let $\epsilon > 0$ be small enough that an $\epsilon-$neighbourhood about $a$ excludes all other vertices of $P$. Consider an $\epsilon-$neighbourhood of $a'$. It contains a point $a_l$. If $a_{l+1} \notin U$, then $a_{l+1}$ is the reflection of $a_l$ in a line joining two other vertices of $P_l$ and lying on the opposite side of the centre of $U$ to $a_l$, and so $U \subseteq P_{l+1}$. This entails that $a_k \notin U$ for $k \geq l+1$, contradicting that $a'$ is an accumulation point. Hence $a_k \in U$ for $k \geq l$. It follows that $\lim_{i\to\infty} a_i = a'$. We show, in fact, that $a_i = a'$ for sufficiently large $i$.

Let $P'$ be the limiting polygon with vertices $a'$. Suppose, if possible that $a'$ is a concave vertex. We can find a neighbourhood $V_{x'}$ around each vertex $x'$ of $P'$ such that, selecting any point from each of these neighbourhoods $V_{x'}$ will create a vertex with a convex vertex in $V_{a'}$. Thus, there is an index $r$ such that the polygon $P_r$ has a vertex inside each of these neighbourhoods. If the convex vertex $a_r$ is reflected about a line joining two other vertices in $P_r$ to $a_{r+1}$, then $a_r$ will be an interior point of $P_{r+1}$ and each subsequent $a_i$ will lie outdie of the interior of $P_{r+1}$, and so outside of $U_{a'}$, yielding a contradiction.

Hence $P'$ has acute angles at all of its vertices. But then for sufficiently large $n$, so will $P_n$ and so $a_i = a_n$ for $i \geq n$. The result follows.