**THE UNIVERSITY OF TORONTO**

**UNDERGRADUATE MATHEMATICS COMPETITION**

*Sunday, March 16, 2003*

*Time*: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

1. Evaluate
$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2}{n^2}\right) .$$
[$\tan^{-1}$ denotes the (composition) inverse function for tan.]

2. Let $a, b, c$ be positive real numbers for which $a+b+c = abc$. Prove that
$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2} .$$

3. Solve the differential equation
$$y'' = yy' .$$

4. Show that $n$ divides the integer nearest to
$$\frac{(n+1)!}{e} .$$

5. For $x > 0$, $y > 0$, let $g(x, y)$ denote the minimum of the three quantities, $x$, $y + 1/x$ and $1/y$. Determine the maximum value of $g(x, y)$ and where this maximum is assumed.

6. A set of $n$ lightbulbs, each with an *on-off* switch, numbered $1, 2, \cdots, n$ are arranged in a line. All are initially off. Switch 1 can be operated at any time to turn its bulb on of off. Switch 2 can turn bulb 2 on or off if and only if bulb 1 is off; otherwise, it does not function. For $k \geq 3$, switch $k$ can turn bulb $k$ on or off if and only if bulb $k-1$ is off and bulbs $1, 2, \cdots, k-2$ are all on; otherwise it does not function.

   (a) Prove that there is an algorithm that will turn all of the bulbs on.

   (b) If $x_n$ is the length of the shortest algorithm that will turn on all $n$ bulbs when they are initially off, determine the largest prime divisor of $3x_n + 1$ when $n$ is odd.

7. Suppose that the polynomial $f(x)$ of degree $n \geq 1$ has all real roots and that $\lambda > 0$. Prove that the set $\{x \in \mathbf{R} : |f(x)| \leq \lambda |f'(x)|\}$ is a finite union of closed intervals whose total length is equal to $2n\lambda$.

8. Three matrices $A$, $B$ and $A + B$ have rank 1. Prove that either all the rows of $A$ and $B$ are multiples of one and the same vector, or that all of the columns of $A$ and $B$ are multiples of one and the same vector.

9. Prove that the integral
$$\int_0^{\infty} \frac{\sin^2 x}{\pi^2 - x^2} dx$$
exists and evaluate it.

10. Let $G$ be a finite group of order $n$. Show that $n$ is odd if and only if each element of $G$ is a square.

**END**

1. Evaluate

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2}{n^2}\right) .$$

[$\tan^{-1}$ denotes the (composition) inverse function for tan.]

*Solution 1.* Let $a_n = \tan^{-1} n$ for $n \geq 0$. Then

$$\tan(a_{n+1} - a_{n-1}) = \frac{(n+1)-(n-1)}{1+(n^2-1)} = \frac{2}{n^2}$$

for $n \geq 1$. Then

$$\sum_{n=1}^{m} \tan^{-1} \frac{2}{n^2} = \tan^{-1}(m+1) + \tan^{-1} m - \tan^{-1} 1 - \tan^{-1} 0 .$$

Letting $m \to \infty$ yields the answer $\pi/2 + \pi/2 - \pi/4 - 0 = 3\pi/4$.

*Solution 2.* Let $b_n = \tan^{-1}(1/n)$ for $n \geq 0$. Then

$$\tan(b_{n-1} - b_{n+1}) = \frac{2}{n^2}$$

for $n \geq 2$, whence

$$\begin{aligned}\sum_{n=1}^{m} \tan^{-1} \frac{2}{n^2} &= \tan^{-1} 2 + \sum_{n=2}^{m}(b_{n-1} - b_{n+1}) = \tan^{-1} 2 + \tan^{-1} 1 + \tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{m} - \tan^{-1} \frac{1}{m+1} \\ &= (\tan^{-1} 2 + \cot^{-1} 2) + \tan^{-1} 1 - \tan^{-1} \frac{1}{m} - \tan^{-1} \frac{1}{m+1} \\ &= \frac{\pi}{2} + \frac{\pi}{4} - \tan^{-1} \frac{1}{m} - \tan^{-1} \frac{1}{m+1}\end{aligned}$$

for $m \geq 3$, from which the result follows by letting $m$ tend to infinity.

*Solution 3.* [S. Huang] Let $s_n = \sum_{k=1}^{n} \tan^{-1}(2/k^2)$ and $t_n = \tan s_n$. Then $\{t_n\} = \{2, \infty, -9/2, -14/5, -20/9, \cdots\}$ where the numerators of the fractions are $\{-2, -5, -9, -14, -20, \cdots\}$ and the denominators are $\{-1, 0, 2, 5, 8, \cdots\}$. We conjecture that

$$t_n = \frac{-n(n+3)}{(n-2)(n+1)}$$

for $n \geq 1$. This is true for $1 \leq n \leq 5$. Suppose that it holds to $n = k-1 \geq 5$, so that $t_{k-1} = -(k-1)(k+2)/(k-3)k$. Then

$$\begin{aligned} t_k &= \frac{t_{k-1} + (2/k^2)}{1 - 2t_{k-1}k^{-2}} \\ &= \frac{-k^2(k-1)(k+2) + 2(k-3)k}{k^3(k-3) + 2(k-1)(k+2)} \\ &= \frac{-k(k+3)(k^2-2k+2)}{(k-2)(k+1)(k^2-2k+2)} = \frac{-k(k+3)}{(k-2)(k+1)} .\end{aligned}$$

The desired expression for $t_n$ holds by induction and so $\lim_{n\to\infty} t_n = -1$. For $n \geq 3$, $t_n < 0$ and $\tan^{-1}(2/n^2) < \pi/2$, so we must have $\pi/2 < s_n < \pi$ and $s_n = \pi - \tan^{-1} t_n$. Therefore

$$\lim_{n\to\infty} s_n = \tan^{-1}(\pi + \lim_{n\to\infty} t_n) = \pi - (\pi/4) = (3\pi)/4 .$$

2. Let $a, b, c$ be positive real numbers for which $a + b + c = abc$. Prove that

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2} \ .$$

*Solution 1.* Let $a = \tan\alpha$, $b = \tan\beta$, $c = \tan\beta$, where $\alpha, \beta, \gamma \in (0, \pi/2)$. Then

$$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha} = \frac{a+b+c-abc}{1-ab-bc-ca} = 0 \ ,$$

whence $\alpha + \beta + \gamma = \pi$. Then, the left side of the inequality is equal to

$$\begin{aligned}
\cos\alpha + \cos\beta + \cos\gamma &= \cos\alpha + \cos\beta - \cos(\alpha+\beta) \\
&= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - 2\cos^2\left(\frac{\alpha+\beta}{2}\right) + 1 \\
&\leq 2\cos\left(\frac{\alpha+\beta}{2}\right) - 2\cos^2\left(\frac{\alpha+\beta}{2}\right) + 1 \\
&= 2\sin\left(\frac{\gamma}{2}\right) - 2\sin^2\left(\frac{\gamma}{2}\right) + 1 \\
&= \frac{3}{2} - \frac{1}{2}(2\sin(\gamma/2) - 1)^2 \leq \frac{3}{2} \ ,
\end{aligned}$$

with equality if and only if $\alpha = \beta = \gamma = \pi/3$.

*Solution 2.* Define $\alpha$, $\beta$ and $\gamma$ and note that $\alpha + \beta + \gamma = \pi$ as in Solution 1. Since $\cos x$ is a concave function on $[0, \pi/2]$, we have that

$$\frac{\cos\alpha + \cos\beta + \cos\gamma}{3} \leq \cos\left(\frac{\alpha+\beta+\gamma}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2} \ ,$$

from which the result follows.

3. Solve the differential equation

$$y'' = yy' \ .$$

*Solution.* The equation can be rewritten

$$\frac{dy'}{dy}y' = yy'$$

from which either $y' = 0$ and $y$ is a constant, or $dy' = ydy$, whence

$$\frac{dy}{dx} = y' = \frac{1}{2}(y^2 + k)$$

for some constant $k$. (One can also get to the same place by noting that the equation can be rewritten as $2y'' = (y^2)'$ and integrating.)

In case $k = 0$, we have that $2dy/y^2 = dx$, whence $-2/y = x + c$ and $y = -2(x+c)^{-1}$ for some constant $c$.

In the case that $k = a^2 > 0$, we have that

$$\frac{2dy}{y^2 + a^2} = dx$$

whence $(2/a)\tan^{-1}(y/a) = x + c$ and

$$y = a \tan \frac{a(x+c)}{2}$$

for some constant $c$.

In the case that $k = -b^2 < 0$, we have that

$$\frac{2dy}{y^2 - b^2} = \frac{1}{b}\left[\frac{dy}{y-b} - \frac{dy}{y+b}\right] = dx$$

whence

$$\frac{1}{b} \ln \left|\frac{y-b}{y+b}\right| = x + c$$

for some constant $c$. Solving this for $y$ yields that

$$y = b\left(\frac{1 + e^{b(x+c)}}{1 - e^{b(x+c)}}\right) \qquad \text{or} \qquad y = b\left(\frac{1 - e^{b(x+c)}}{1 + e^{b(x+c)}}\right) .$$

4. Show that the positive integer $n$ divides the integer nearest to

$$\frac{(n+1)!}{e} .$$

*Solution.* By Taylor's Theorem, we have that

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n+1}\frac{1}{(n+1)!} + (-1)^{n+2}\frac{e^c}{(n+2)!}$$

where $-1 < c < 0$. Hence

$$(n+1)!e^{-1} = (n+1)! - (n+1)! + \frac{(n+1)!}{2!} - \frac{(n+1)!}{3!} + \cdots + (-1)^n[(n+1) - 1] + (-1)^{n-2}\frac{e^c}{n+2} .$$

The last term does not exceed $1/(n+2)$, which is less than $1/2$. The second last term is equal to $\pm n$, and each previous term has a factor $n$ in the numerator that is not cancelled out by the denominator. Since the sum of all but the last term is an integer divisible by $n$ and is the nearest integer to $(n+1)!/e$, the result holds.

5. For $x > 0$, $y > 0$, let $g(x,y)$ denote the minimum of the three quantities, $x$, $y + 1/x$ and $1/y$. Determine the maximum value of $g(x,y)$ and where this maximum is assumed.

*Solution 1.* Consider the attached graph showing the curves of equations $x = y + 1/x$, $x = 1/y$ and $y + 1/x = 1/y$, all three of which contain the point $P \sim (\sqrt{2}, 1/\sqrt{2})$, and the regions in which $g(x,y)$ is each one of the three given functions. When $g(x,y) = x$, the maximum value of $g(x,y)$ is equal to $\sqrt{2}$, assumed at the point $P$. When $g(x,y) = 1/y$, the maximum value of $g(x,y)$ is equal to $\sqrt{2}$, also assumed at $P$. Suppose $(x,y)$ are such that $g(x,y) = y + 1/x$. Note that the curve $y = \sqrt{2} - (1/x)$ passes through the point $P$, where it intersects each of the curves $y + (1/x) = (1/y)$ and $y + (1/x) = x$. It intersects neither of these curves at any other point, and so lies vertically above the region where $g(x,y) = y + (1/x)$. In this region, $y \le \sqrt{2} - (1/x)$ and so $g(x,y) \le \sqrt{2}$, with equality only at the point $P$. Hence the required maximum is $\sqrt{2}$, and it is assumed at $(x,y) = (\sqrt{2}, 1/\sqrt{2})$.

*Solution 2.* [R. Appel] If $x \le 1$ and $y \le 1$, then $g(x,y) \le x \le 1$. If $y \ge 1$, then $g(x,y) \le 1/y \le 1$. It remains to examine the case $x > 1$ and $y < 1$, so that $y + (1/x) < 2$. Suppose that min $(x, 1/y) = a$ and max $(x, 1/y) = b$. Then min $(1/x, y) = 1/a$ and max $(1/x, y) = 1/b$, so that

$$y + \frac{1}{x} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} .$$

Hence $g(x,y) = \min\ (a, (a+b)/(ab))$. Either $a^2 \leq 2$ or $a^2 \geq 2$. But in the latter case,

$$\frac{a+b}{ab} \leq \frac{2b}{\sqrt{2}b} = \sqrt{2}\ .$$

In either case, $g(x,y) \leq \sqrt{2}$. This maximum value is attained when $(x,y) = (\sqrt{2}, 1/\sqrt{2})$.

*Solution 3.* [D. Varodayan] By the continuity of the functions, each of the regions $\{(x,y) : 0 < x < y + (1/x), xy < 1\}$, $\{(x,y) : 0 < x, y + (1/x) < x, y + (1/x) < (1/y)\}$, and $\{(x,y) : 0 < (1/y) < x, (1/y) < y + (1/x)\}$ is an open subset of the plane; using partial derivatives, we see that none of the three functions being minimized have any critical values there. It follows that any extreme values of $g(x,y)$ must occur on one of the curves defined by the equations

$$x = y + (1/x) \tag{1}$$

$$x = 1/y \tag{2}$$

$$y + (1/x) = (1/y) \tag{3}$$

On the curve (1), $x > 1$ and

$$\begin{aligned} g(x,y) &= \min\ \left(x, \frac{x}{x^2-1}\right) \\ &= \begin{cases} x, & \text{if } x \leq \sqrt{2}; \\ \frac{x}{x^2-1}, & \text{if } x \geq \sqrt{2}. \end{cases} \end{aligned}$$

On the curve (2),

$$\begin{aligned} g(x,y) &= \min\ (x, 2/x) \\ &= \begin{cases} x, & \text{if } x \leq \sqrt{2}; \\ 2/x, & \text{if } x \geq \sqrt{2}. \end{cases} \end{aligned}$$

On the curve (3), $0 < y < 1$ and

$$\begin{aligned} g(x,y) &= \min\ \left(\frac{y}{1-y^2}, \frac{1}{y}\right) \\ &= \begin{cases} \frac{y}{1-y^2}, & \text{if } 0 < y < \frac{1}{\sqrt{2}}; \\ 1/y, & \text{if } \frac{1}{\sqrt{2}} \leq y \leq 1. \end{cases} \end{aligned}$$

On each of these curves, $g(x,y)$ reaches its maximum value of $\sqrt{2}$ when $(x,y) = (\sqrt{2}.1/\sqrt{2})$.

*Solution 4.* [J. Sparling] Let $z = 1/y$. For fixed $z$, let

$$v_z(x) = \min\ \{x, z, (1/x) + (1/z)\}$$

and

$$w(z) = \max\ \{v_z(x) : x > 0\}\ .$$

Suppose that $z \leq 1$. Then $(1/x) + (1/z) \geq z$, so $v_z(x) = \min\ \{x, z\}$ and

$$v_z(x) = \begin{cases} x, & \text{for } x \leq z; \\ z, & \text{for } x \geq z; \end{cases}$$

so that $w(z) = z$ when $z \leq 1$. Suppose that $1 < z \leq \sqrt{2}$, so that $z \leq z/(z^2-1)$. Then

$$v_z(x) = \begin{cases} x, & \text{for } x \leq z; \\ z, & \text{for } z \leq x < z/(z^2-1); \\ (1/x) + (1/z), & \text{for } z/(z^2-1) \leq x; \end{cases}$$

so that $w(z) = z$ when $1 < z \le \sqrt{2}$. Finally, suppose that $\sqrt{2} > z$. Note that $x \le (1/x) + (1/z) \Leftrightarrow zx^2 - x - z \le 0$. Then the minimum of $x$ and $(1/x)+(1/z)$ is $x$ when $zx^2 - x - z \le 0$, or $x \le (1+\sqrt{1+4z^2})/2z$. Since

$$\sqrt{2} - \left[\frac{1+\sqrt{1+4z^2}}{2z}\right] = \frac{(2\sqrt{2}z-1) - \sqrt{1+4z^2}}{2z}$$
$$= \frac{4z^2 - 4\sqrt{2}z}{2z[(2\sqrt{2}z-1)+\sqrt{1+4z^2}]}$$
$$= \frac{2(z-\sqrt{2})}{(2\sqrt{2}z-1)+\sqrt{1+4z^2}} \ge 0 \ ,$$

this minimum is always less than $z$, so that

$$v_z(x) = \begin{cases} x, & \text{for } x \le \frac{1+\sqrt{1+4z^2}}{2z} \\ \frac{1}{x} + \frac{1}{z}, & \text{for } x \ge \frac{1+\sqrt{1+4z^2}}{2z}, \end{cases}$$

so that $w(z) = (1+\sqrt{1+4z^2})/(2z) \le \sqrt{2}$ when $\sqrt{2} \le z$. Hence, the minimum value of $w(z) = \sqrt{2}$ and this is the maximum value of $g(x,y)$, assumed when $(x,y) = (\sqrt{2}, 1/\sqrt{2})$.

*Solution 5.* For $x > 0$, let

$$h_x(y) = \min\left(x, y + \frac{1}{x}, \frac{1}{y}\right) \ .$$

Suppose that $x \le \sqrt{2}$. Then $x - (1/x) \le (1/x)$ and

$$h_x(y) = \begin{cases} y + \frac{1}{x}, & \text{if } 0 < y \le x - \frac{1}{x}; \\ x, & \text{if } x - \frac{1}{x} \le y \le \frac{1}{x}; \\ \frac{1}{y}, & \text{if } \frac{1}{x} \le y; \end{cases}$$

so that the minimum value of $h_x(y)$ is $x$, and this occurs when $x - (1/x) \le y \le (1/x)$. Suppose that $x \ge \sqrt{2}$. Then $y + (1/x) \le (1/y) \Leftrightarrow xy^2 + y - x \le 0$ and

$$\sqrt{2} - \left[\frac{1+\sqrt{1+4x^2}}{2x}\right] = \frac{(\sqrt{8}x-1) - \sqrt{1+4x^2}}{2x}$$
$$= \frac{4x^2 - 4\sqrt{2}x}{2x[(\sqrt{8}x-1)+\sqrt{1+4x^2})}$$
$$= \frac{2(x-\sqrt{2})}{(\sqrt{8}x-1)+\sqrt{1+4x^2}} \ge 0 \ ,$$

so that

$$\frac{1+\sqrt{1+4x^2}}{2x} \le \sqrt{2} \le x \ .$$

$$h_x(y) = \begin{cases} y + \frac{1}{x}, & \text{when } 0 < y \le \frac{-1+\sqrt{1+4x^2}}{2x}; \\ \frac{1}{y}, & \text{when } \frac{-1+\sqrt{1+4x^2}}{2x} \le y; \end{cases}$$

so that the minimum value of $h_x(y)$ is $(1+\sqrt{1+4x^2})/(2x)$, and this occurs when $y = (-1+\sqrt{1+4x^2})/(2x)$.

Thus, we have to maximize the function $u(x)$ where

$$u(x) = \begin{cases} x, & \text{if } 0 < x \le \sqrt{2}; \\ \frac{1+\sqrt{1+4x^2}}{2x}, & \text{if } \sqrt{2} \le x. \end{cases}$$

By what we have shown, this maximum is $\sqrt{2}$ and is attained when $x = \sqrt{2}$. The result follows.

6. A set of $n$ lightbulbs, each with an *on-off* switch, numbered $1, 2, \cdots, n$ are arranged in a line. All are initially off. Switch 1 can be operated at any time to turn its bulb on of off. Switch 2 can turn bulb 2 on or off if and only if bulb 1 is off; otherwise, it does not function. For $k \geq 3$, switch $k$ can turn bulb $k$ on or off if and only if bulb $k-1$ is off and bulbs $1, 2, \cdots, k-2$ are all on; otherwise it does not function.

   (a) Prove that there is an algorithm that will turn all of the bulbs on.

   (b) If $x_n$ is the length of the shortest algorithm that will turn on all $n$ bulbs when they are initially off, determine the largest prime divisor of $3x_n + 1$ when $n$ is odd.

*Solution.* (a) Clearly $x_1 = 1$ and $x_2 = 2$. Let $n \geq 3$. The only way that bulb $n$ can be turned on is for bulb $n-1$ to be off and for bulbs $1, 2, \cdots, n-2$ to be turned on. Once bulb $n$ is turned on, then we need get bulb $n-1$ turned on. The only way to do this is to turn off bulb $n-2$; but for switch $n-2$ to work, we need to have bulb $n-3$ turned off. So before we can think about dealing with bulb $n-1$, we need to get the first $n-2$ bulbs turned off. Then we will be in the same situation as the outset with $n-1$ rather than $n$ bulbs. Thus the process has the following steps: (1) Turn on bulbs $1, \cdots, n-2$; (2) Turn on bulb $n$; (3) Turn off bulbs $n-2, \cdots, 1$; (3) Turn on bulbs $1, 2, \cdots, n-1$. So if, for each positive integer $k$, $y_k$ is the length of the shortest algorithm to turn them off after all are lit, then

$$x_n = x_{n-2} + 1 + y_{n-2} + x_{n-1} \ .$$

Since we can clearly find an algorithm for turning on or off bulb 1, an induction argument will provide an algorithm for turning any number of bulbs on or off.

We show that $x_n = y_n$ for $n = 1, 2, \cdots$. Suppose that we have an algorithm that turns all the bulbs on. We prove by induction that at each step we can legitimately reverse the whole sequence to get all the bulbs off again. Clearly, the first step is to turn either bulb 1 or bulb 2 on; since the switch is functioning, we can turn the bulb off again. Suppose that we can reverse the first $k-1$ steps and are at the $k$th step. Then the switch that operates the bulb at that step is functioning and can restore us to the situation at the end of the $(k-1)$th step. By the induction hypothesis, we can go back to having all the bulbs off. Hence, given the bulbs all on, we can reverse the steps of the algorithm to get the bulbs off again. A similar argument allows us to reverse the algorithm that turns the bulbs off. Thus, for each turning-on algorithm there is a turning-off algorithm of equal length, and vice versa. Thus $x_n = y_n$.

We have that $x_n = x_{n-1} + 2x_{n-2} + 1$ for $n \geq 3$. By, induction, we show that, for $m = 1, 2, \cdots$,

$$x_{2m} = 2x_{2m-1} \qquad \text{and} \qquad x_{2m+1} = 2x_{2m} + 1 = 4x_{2m-1} + 1 \ .$$

This is true for $m = 1$. Suppose it is true for $m \geq 1$. Then

$$\begin{aligned} x_{2(m+1)} &= x_{2m+1} + 2x_{2m} + 1 = 2(x_{2m} + 1) + 4x_{2m-1} \\ &= 2(x_{2m} + 2x_{2m-1} + 1) = 2x_{2m+1} \ , \end{aligned}$$

and

$$\begin{aligned} x_{2(m+1)+1} &= x_{2(m+1)} + 2x_{2m+1} + 1 = 2x_{2m+1} + 4x_{2m} + 3 \\ &= 2(x_{2m+1} + 2x_{2m} + 1) + 1 = 2x_{2(m+1)} + 1 \ . \end{aligned}$$

Hence, for $m \geq 1$,

$$3x_{2m+1} + 1 = 4(3x_{2m-1} + 1) = \cdots = 4^m(3x_1 + 1) = 4^{m+1} = 2^{2(m+1)} \ .$$

Thus, the largest prime divisor is 2.

7. Suppose that the polynomial $f(x)$ of degree $n \geq 1$ has all real roots and that $\lambda > 0$. Prove that the set $\{x \in \mathbf{R} : |f(x)| \leq \lambda |f'(x)|\}$ is a finite union of closed intervals whose total length is equal to $2n\lambda$.

*Solution.* Wolog, we may assume that the leading coefficient is 1. Let $f(x) = \prod_{i=1}^{k}(x - r_i)^{m_i}$, where $n = \sum_{i=1}^{k} m_i$. Then

$$\frac{f'(x)}{f(x)} = \sum_{i=1}^{k} \frac{m_i}{x - r_i} \ .$$

Note that the derivative of this function, $-\sum_{i=1}^{k} m_i(x - r_i)^{-2} < 0$, so that it decreases on each interval upon which it is defined. By considering the graph of $f'(x)/f(x)$, we see that $f'(x)/f(x) \geq 1/\lambda$ on finitely many intervals of the form $(r_i, s_i]$, where $r_i < s_i$ and the $r_i$ and $s_j$ interlace, and $f'(x)/f(x) \leq -1/\lambda$ on finitely many intervals of the form $[t_i, r_i)$, where $t_i < r_i$ and the $t_i$ and $r_j$ interlace. For each $i$, we have $t_i < r_i < s_i < t_{i+1}$.

The equation $f'(x)/f(x) = 1/\lambda$ can be rewritten as

$$\begin{aligned} 0 =& (x - r_1)(x - r_2) \cdots (x - r_k) - \lambda \sum_{i=1}^{k} m_i (x - r_1) \cdots \widehat{(x - r_i)} \cdots (x - r_k) \\ =& x^k - \left( \sum_{i=1}^{k} r_i - \lambda \sum_{i=1}^{k} m_i \right) x^{k-1} + \cdots \ . \end{aligned}$$

The sum of the roots of this polynomial is

$$s_1 + s_2 + \cdots + s_k = r_1 + \cdots + r_k - \lambda n \ ,$$

so that $\sum_{i=1}^{m}(s_i - r_i) = \lambda n$. This is the sum of the lengths of the intervals $(r_i, s_i]$ on which $f'(x)/f(x) \geq 1/\lambda$. Similarly, we can show that $f'(x)/f(x) \leq -1/\lambda$ on a finite collection of intervals of total length $\lambda n$. The set on which the inequality of the problem holds is equal to the union of all of these half-open intervals and the set $\{r_1, r_2, \cdots, r_k\}$. The result follows.

8. Three matrices $A$, $B$ and $A + B$ have rank 1. Prove that either all the rows of $A$ and $B$ are multiples of one and the same vector, or that all of the columns of $A$ and $B$ are multiples of one and the same vector.

*Solution 1.* Let $A = (a_{ij})$ and $B = (b_{ij})$. Since the image each of the matrix $A$ and the matrix $B$ as an operator on $\mathbf{C}^n$ is the span of its column vectors, there exist nonzero vectors $(a_i)$, $(b_i)$, $(u_j)$, $(v_j)$ for which

$$a_{ij} = a_i u_j \qquad \text{and} \qquad b_{ij} = b_i v_j \ .$$

Hence $a_{ij} + b_{ij} = a_i u_j + b_i v_j$. Similarly, since $A + B = (a_{ij} + b_{ij})$ has rank 1, there are vectors $(c_i)$ and $(w_j)$ for which

$$a_{ij} + b_{ij} = c_i w_j \ .$$

Thus, for all $i$ and $j$,

$$a_i u_j + b_i v_j = c_i w_j \ .$$

If the rows of $A$ and $B$ are multiples of the same row, then there exists a constant $k$ for which $v_j = k u_j$ for each $j$, and so $c_i w_j = (a_i + k b_i) u_j$, and so the $i$th row of $A + B$ is also a multiple of $(u_j)$.

Suppose if that the vectors $(u_j)$ and $(v_j)$ are linearly independent. Wolog, we may assume that $(u_1, u_2)$ and $(v_1, v_2)$ are linearly independent. Then $u_1 v_2 - u_2 v_1 \neq 0$, and for each $i$ and $j$, the system

$$u_1 a_i + v_1 b_i = w_1 c_i$$

$$u_2 a_i + v_2 b_i = w_2 c_i$$

has the solution $a_i = pc_i$, $b_i = qc_i$, where

$$p = \frac{w_1 v_2 - w_2 v_1}{u_1 v_2 - u_2 v_1}$$

and

$$q = \frac{u_1 w_2 - u_2 w_1}{u_1 v_2 - u_2 v_1}$$

are independent of $i$. The result follows.

*Solution 2.* Suppose that $A = \mathbf{u}\mathbf{a}^t$ and $B = \mathbf{v}\mathbf{b}^t$ where $\mathbf{a} = (a_j)$, $\mathbf{b} = (b_j)$, $\mathbf{u} = (u_i)$ and $\mathbf{v} = (v_i)$ are column vectors. If the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent, then the columns of $A + B$, $A$ and $B$ are multiplies of each other and the result follows. Otherwise, let $\{\mathbf{u}, \mathbf{v}\}$ be linearly independent. The $i$th columns of $A$ and $B$ are respectively $a_i\mathbf{u}$ and $b_i\mathbf{v}$. Since $A + B$ is of rank 1, there exists a column vector $\mathbf{w}$ and scalars $c_i$ for which

$$a_i\mathbf{u} + b_i\mathbf{v} = c_i\mathbf{w}$$

for each $i$. Hence, for each $i, j$,

$$\begin{aligned} c_i(a_j\mathbf{u} + b_j\mathbf{v}) - c_j(a_i\mathbf{u} + b_i\mathbf{v}) = \mathbf{0} \\ \Longrightarrow (c_i a_j - c_j a_i)\mathbf{u} + (c_i b_j - c_j b_i)\mathbf{v} = \mathbf{0} \\ \Longrightarrow c_i a_j - c_j a_i = c_i b_j - c_j b_i = 0 \\ \Longrightarrow c_i a_j = c_j a_i \quad \text{and} \quad c_i b_j = c_j b_i \\ \Longrightarrow c_i(a_j b_i - a_i b_j) = 0 \ . \end{aligned}$$

Suppose, wolog, that $c_1 \neq 0$. Then either $a_1$ or $b_1$ is nonzero; suppose the former. Then $b_i = (a_i/a_1)b_1$ and $a_i = (a_i/a_1)a_i$ for each $i$, so that the vectors $\mathbf{b}$ and $\mathbf{a}$ are parallel, and the rows of $A + B$ are sums of multiples of these parallel vectors. The result follows.

9. Prove that the integral

$$\int_0^\infty \frac{\sin^2 x}{\pi^2 - x^2} dx$$

exists and evaluate it.

*Solution.* Since $\lim_{x\to\pi} (\sin^2 x)/(\pi^2 - x^2)$ exists and equals 0 (by l'Hôpital's Rule), there is a removable singularity in the integrand at $x = \pi$. The integral on the infinite interval converges by comparison with the integral of $1/x^2$.

First, note that, for $k \geq 0$,

$$\begin{aligned} \int_{k\pi}^{(k+1)\pi} \frac{\sin^2 x}{\pi^2 - x^2} dx &= -\frac{1}{2\pi}\left[\int_{k\pi}^{(k+1)\pi} \frac{\sin^2 x}{x - \pi} dx - \int_{k\pi}^{(k+1)\pi} \frac{\sin^2 x}{x + \pi} dx\right] \\ &= -\frac{1}{2\pi}\int_{(k-1)\pi}^{k\pi} \frac{\sin^2 x}{x} dx + \frac{1}{2\pi}\int_{(k+1)\pi}^{(k+2)\pi} \frac{\sin^2 x}{x} dx \ . \end{aligned}$$

Hence

$$\int_0^{n\pi} \frac{\sin^2 x}{\pi^2 - x^2} dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin^2 x}{x} dx + \frac{1}{2\pi}\int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin^2 x}{x} dx \ .$$

The first integral on the right vanishes, because the integrand is odd, and so

$$\left|\int_0^{n\pi} \frac{\sin^2 x}{\pi^2 - x^2} dx\right| = \frac{1}{2\pi}\int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin^2 x}{x} dx \leq \frac{1}{2\pi}\int_{(n-1)\pi}^{(n+1)\pi} \frac{1}{x} dx \leq \frac{1}{n-1} \ ,$$

with the result that $\int_0^\infty \frac{\sin^2 x}{\pi^2 - x^2} dx = 0$.

10. Let $G$ be a finite group of order $n$. Show that $n$ is odd if and only if each element of $G$ is a square.

*Solution.* Suppose that $n$ is odd. Then, by Lagrange's theorem, each element of $G$ is of odd order, so that, if $a \in G$, then $a^{2k+1} = e$ (the identity) for some nonnegative integer $k$. Hence $a = (a^{k+1})^2$ is a square. On the other hand, suppose that $n$ is even. Pair off each element of $G$ with its inverse. Some elements get paired off with a distinct element, and others get paired off with themselves. Since $n$ is even and an even number of elements get paired off with a distinct element, there must be an even number of elements that get paired off with themselves. Since the identity gets paired off with itself, there must be some other element $v$ that also is its own inverse, *i.e.* $v^2 = e$. Consider the mapping $x \to x^2$ defined from $G$ to itself. Since $e$ and $v$ have the same image and since $G$ is finite, this mapping cannot be onto. Hence, there must be an element of $G$ which is not the square of another element. The result follows.