THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh

March 8, 2015

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

The grading is designed to encourage only the stronger students to attempt more than five problems. Each solution is graded out of 10. If the sum of the scores for the solutions to the five best problems does not exceed 30, this sum will be the final grade. If the sum of these scores does exceed 30, then all solutions will be graded for credit.

1. Suppose that u and v are two real-valued functions defined on the set of reals. Let $f(x) = u(v(x))$ and $g(x) = u(-v(x))$ for each real x. If $f(x)$ is continuous, must $g(x)$ also be continuous?

2. Given $2n$ distinct points in space, the sum S of the lengths of all the segments joining pairs of them is calculated. Then n of the points are removed along with all the segments having at least one endpoint from among them. Prove that the sum of the lengths of all the remaining segments is less that $\frac{1}{2}S$.

3. Let $f : [0,1] \longrightarrow \mathbf{R}$ be continuously differentiable. Prove that
$$\left| \frac{f(0)+f(1)}{2} - \int_0^1 f(x)dx \right| \leq \frac{1}{4} \sup\{|f'(x)| : 0 \leq x \leq 1\}.$$

4. Determine all the values of the positive integer $n \geq 2$ for which the following statement is true, and for each, indicate when equality holds.

   For any nonnegative real numbers $x_1, x_2, \cdots, x_n$,
$$(x_1 + x_2 + \cdots + x_n)^2 \geq n(x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1),$$
   where the right side has n *summands.*

5. Let $f(x)$ be a real polynomial of degree 4 whose graph has two real inflection points. There are three regions bounded by the graph and the line passing through these inflection points. Prove that two of these regions have equal area and that the area of the third region is equal to the sum of the other two areas.

6. Using the digits 1, 2, 3, 4, 5, 6, 7, 8, each exactly once, create two numbers and form their product. For example, $472 \times 83156 = 39249632$. What are the smallest and the largest values such a product can have?

7. Determine
$$\int_0^2 \frac{e^x dx}{e^{1-x} + e^{x-1}}.$$

8. Let $\{a_n\}$ and $\{b_n\}$ be two *decreasing* positive real sequences for which
$$\sum_{n=1}^{\infty} a_n = \infty$$

and

$$\sum_{n=1}^{\infty} b_n = \infty.$$

Let I be a subset of the natural numbers, and define the sequence $\{c_n\}$ by

$$c_n = \begin{cases} a_n, & \text{if } n \in I \\ b_n, & \text{if } n \notin I. \end{cases}$$

Is it possible for $\sum_{n=1}^{\infty} c_n$ to converge?

9. What is the dimension of the vector subspace of $\mathbf{R}^n$ generated by the set of vectors

$$(\sigma(1), \sigma(2), \sigma(3), \cdots, \sigma(n))$$

where σ runs through all $n!$ of the permutations of the first n natural numbers.

10. (a) Let

$$g(x,y) = x^2y + xy^2 + xy + x + y + 1.$$

We form a sequence $\{x_0\}$ as follows: $x_0 = 0$. The next term x_1 is the unique root -1 of the linear equation $g(t,0) = 0$. For each $n \geq 2$, x_n is the root other than x_{n-2} of the equation $g(t, x_{n-1}) = 0$.

Let $\{f_n\}$ be the Fibonacci sequence determined by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Prove that, for any nonnegative integer k,

$$x_{2k} = \frac{f_k}{f_{k+1}} \qquad \text{and} \qquad x_{2k+1} = -\frac{f_{k+2}}{f_{k+1}}.$$

(b) Let

$$h(x,y) = x^2y + xy^2 + \beta xy + \gamma(x+y) + \delta$$

be a polynomial with real coefficients β, γ, δ. We form a bilateral sequence $\{x_n : n \in \mathbf{Z}\}$ as follows. Let $x_0 \neq 0$ be given arbitrarily. We select x_{-1} and x_1 to be the two roots of the quadratic equation $h(t, x_0) = 0$ in either order. From here, we can define inductively the terms of the sequence for positive and negative values of the index so that x_{n-1} and x_{n+1} are the two roots of the equation $h(t, x_n) = 0$. We suppose that at each stage, neither of these roots os zero.

Prove that the sequence $\{x_n\}$ has period 5 (*i.e.* $x_{n+5} = x_n$ for each index n) if and only if $\gamma^3 + \delta^2 - \beta\gamma\delta = 0$.

Solutions

1. Suppose that u and v are two real-valued functions defined on the set of reals. Let $f(x) = u(v(x))$ and $g(x) = u(-v(x))$ for each real x. If $f(x)$ is continuous, must $g(x)$ also be continuous?

Solution 1. The answer is *no.* Let

$$v(x) = \begin{cases} 0, & \text{when } x \leq 0 \\ 1, & \text{when } x > 0. \end{cases}$$

Suppose that u is any function for which $u(-1) = 1$ and $u(0) = u(1) = 0$. Then $f(x) = 0$ for all real x while

$$g(x) = \begin{cases} 0, & \text{when } x \leq 0 \\ 1, & \text{when } x > 0. \end{cases}$$

Solution 2. (Xuan Tu) An example for which $g(x)$ fails to be continuous at -1 is $(u(x), v(x)) = ((1+x)^{-1}, x^2)$.

2. Given $2n$ distinct points in space, the sum S of the lengths of all the segments joining pairs of them is calculated. Then n of the points are removed along with all the segments having at least one endpoint from among them. Prove that the sum of the lengths of all the remaining segments is less than $\frac{1}{2}S$.

Solution 1 (by B. Galvao-Sousa.) Let $P_1, P_2, \cdots, P_n$ be the points that remain after the points $Q_1, Q_2, \cdots, Q_n$ are removed. By repeated application of the triangle inequality, for each of the $\binom{n}{2}$ pairs (i,j), we have that

$$|P_iP_j| \leq |P_iQ_j| + |Q_jQ_i| + |Q_iP_j|,$$

so that

$$2|P_iP_j| \leq |P_iP_j| + |P_iQ_j| + |Q_jQ_i| + |Q_iP_j|.$$

Summing over all pairs (i,j) and adding the n lengths $|P_iQ_i|$ to the right side leads to

$$2\sum |P_iP_j| \leq \sum |P_iP_j| + \sum(|P_iQ_j| + |P_jQ_i|) + \sum |Q_iQ_j| < S$$

as required.

Solution 2. Define P_i and Q_i as before, For each triple i, j, k with $i \neq j$ and $1 \leq i, j, k \leq n$, write the triangle inequality

$$|P_iP_j| \leq |P_iQ_k| + |P_jQ_k|.$$

There are $n\binom{n}{2}$ inequalities in all; for each of the $\binom{n}{2}$ choices of P_iP_j we have an inequality for each of the n choices of Q_k. Each $|P_iP_j|$ appears in n of the inequalities. There are $2n\binom{n}{2} = n^2(n-1)$ terms on the right side of the equalities and each of the n^2 terms of the form $|P_iQ_k|$ appears $n-1$ times. Let T be the sum of all the lengths $|P_iP_j|$. Then the sum of the lengths of all the intervals involving at least one Q_k is $S - T$, and this includes all the intervals of the form P_iQ_k. Adding all the inequalities yields that $nT \leq (n-1)(S-T)$, from which we find that

$$(2n-1)T \leq (n-1)S.$$

The desired result follows.

3. Let $f : [0,1] \longrightarrow \mathbf{R}$ be continuously differentiable. Prove that

$$\left| \frac{f(0)+f(1)}{2} - \int_0^1 f(x)dx \right| \leq \frac{1}{4} \sup\{|f'(x)| : 0 \leq x \leq 1\}.$$

Solution 1. Integrating by parts, we find that

$$\int_0^1 \left(x - \frac{1}{2}\right) f'(x)dx = \left[\left(x - \frac{1}{2}\right) f(x)\right]_0^1 - \int_0^1 f(x)dx = \frac{f(1)+f(0)}{2} - \int_0^1 f(x)dx.$$

Since

$$\left| \int_0^1 \left(x - \frac{1}{2}\right) f'(x)dx \right| \leq \sup|f'(x)| \int_0^1 \left| x - \frac{1}{2} \right| dx = \frac{1}{4} \sup|f'(x)|,$$

the desired result follows.

Solution 2. Since

$$\frac{f(1)-f(0)}{2} - \int_0^1 (f(x) - f(0))dx = \frac{f(0)+f(1)}{2} - \int_0^1 f(x)dx,$$

we have that

$$\frac{f(1)}{2} - \int_0^1 f(x)dx = \frac{1}{2}\int_0^1 f'(x)dx - \int_0^1 \int_0^x f'(y)dydx = \frac{1}{2}\int_0^1 f'(y)dy - \int_0^1 f'(y)\int_y^1 dxdy$$
$$= \frac{1}{2}\int_0^1 f'(y)dy - \int_0^1 f'(y)(1-y)dy \int_0^1 \left(y - \frac{1}{2}\right) f'(y)dy$$

and we cab conclude as in Solution 1.

Comment. A weaker result can be obtained by noting that the integral mean value theorem provides a value $c \in [0,1]$ for which $f(c) = \int_0^1 f(x)dx$. The left side is thus equal to

$$\frac{1}{2}|(f(1)-f(c)) + (f(c)-f(0))| \leq \frac{1}{2}(|f(1)-f(c)| + |f(c)-f(0)|) = \frac{1}{2}(\left|\int_c^1 f'(x)dx\right| + |\int_0^c f'(x)dx|$$
$$\leq \int_0^1 |f'(x)|dx \leq \frac{1}{2}\sup|f'(x)|.$$

[This problem was contributed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.]

4. Determine all the values of the positive integer $n \geq 2$ for which the following statement is true, and for each, indicate when equality holds.

   For any nonnegtive real numbers $x_1, x_2, \cdots, x_n$,

$$(x_1 + x_2 + \cdots + x_n)^2 \geq n(x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1),$$

   where the right side has n *summands.*

Solution. Let $x_1 = x_2 = 1$ and $x_3 = x_4 = \cdots = x_n = 0$. Then the left side of the inequality is equal to 4 and the right side to n. Therefore a necessary condition for the inequality to hold for all sets of x_i is $n \leq 4$.

For $n = 2$, we find that

$$(x_1 + x_2)^2 - 2(x_1x_2 + x_2x_1) = (x_1 - x_2)^2 \geq 0,$$

so the inequality holds with equality if and only if $x_1 = x_2$.

For $n = 3$, we find that

$$2[(x_1 + x_2 + x_3)^2 - 3(x_1x_2 + x_2x_3 + x_3x_1)] = (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 \geq 0,$$

so the inequality holds with equality if and only if $x_1 = x_2 = x_3$.

For $n = 4$, we find that

$$(x_1 + x_2 + x_3 + x_4)^2 - 4(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1)$$
$$= x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_1x_3 + 2x_2x_4 - 2x_1x_2 - 2x_2x_3 - 2x_3x_4 - 2x_4x_1$$
$$= (x_1 - x_2)^2 + (x_3 - x_4)^2 + 2(x_1 - x_2)(x_3 - x_4) = (x_1 - x_2 + x_3 - x_4)^2 \geq 0,$$

so the inequality holds with equality if and only if $x_1 + x_3 = x_2 + x_4$.

Comment. Another case that might be tried is $x_i = i - 1$ for $1 \leq i \leq n$. Then $x_1 + \cdots + x_n = \frac{1}{2}n(n-1)$ and $x_1x_2 + \cdots + x_{n-1}x_n + x_nx_1 = \frac{1}{3}n(n-1)(n-2)$. The left side minus the right is equal to $\frac{1}{12}n^2(n-1)(5-n)$, which establishes that $n \leq 5$. In fact, equality occurs in the $n = 5$ case whenever the x_i form an arithmetic progression.

5. Let $f(x)$ be a real polynomial of degree 4 whose graph has two real inflection points. There are three regions bounded by the graph and the line passing through these inflection points. Prove that two of these regions have equal area and that the area of the third region is equal to the sum of the other two areas.

Solution. By scaling and translating, we may asume that the two inflection points are located at $x = -1$ and $x = 1$ and that $f(x)$ is monic. Since $f''(x)$ is a multiple of $(x+1)(x-1) = x^2 - 1$, we have that

$$f(x) = (x^4 - 6x^2 + 5) + (bx + c)$$

where $2b = f(1) - f(-1)$ and $2c = f(1) + f(-1)$. The line passing through the inflection points $(-1.f(-1)$ and $(1, f(1))$ is $y = g(x)$ with $g(x) = bx + c$. Since $f(x) - g(x) = (x^2 - 5)(x^2 - 1)$, the curves with equations $y = f(x)$ and $y = g(x)$ intersect when $x = \pm\sqrt{5}, \pm 1$. The three areas in question are given by

$$\left| \int_{-\sqrt{5}}^{-1} (x^4 - 6x^2 + 5) dx \right| = \left| \int_{1}^{\sqrt{5}} (x^4 - 6x^2 + 5) dx \right| = \left| \left[x^5/5 - 2x^3 + 5x \right]_1^{\sqrt{5}} \right| = \frac{16}{5}$$

and

$$\int_{-1}^{1} (x^4 - 6x^2 + 5) dx = \left[x^5/5 - 2x^3 + 5x \right]^{-1} = 32/5.$$

The result follows.

[This is Problem E817 from the *American Mathematics Monthly* February, 1949.]

6. Using the digits 1, 2, 3, 4, 5, 6, 7, 8, each exactly once, create two numbers and form their product. For example, $472 \times 83156 = 39249632$. What are the smallest and the largest values such a product can have?

Solution. Observe that, when $a > b > 0$, $c > 0$, then $(10b + c)a - (10a + c)b = c(a - b) > 0$. From this, we see that if the two numbers have unequal numbers of digits or the same number of digits, removing the last digit from the smaller and appending it to the larger, will result in a smaller product of the pair. Therefore, the smallest product will occur when one number has a single digit and the other seven digits. For both factors, the digits will appear in increasing order. It is straightforward to see that the smallest product is $1 \times 2345678 = 2345678$.

Similarly, if the two factors have unequal numbers of digits, removing the last digit from the larger factor and appending it to the smaller will result in a larger product for the pair. Therefore, the largest product will occur when both factors have four digits and the digits appear in decreasing order. Observe the $(a+c)(b+d) - (a+d)(b+c) = (a-b)(d-c)$. If $a > b$, this will be positive if and only if $c < d$. One number begins with 8 and the other with either 7 or 6. But since $87ab \times 6mcd < 8mab \times 67cd < 8mab \times 76cd$, one number begins with 8 and the other with 7. Applying the result further, we deduce that each digit after the first in the number beginning with 8 is less than the correspnding number after the first in the number beginning with 7. Therefore, we deduce that the largest product is $8531 \times 7642 = 65193902$.

7. Determine

$$\int_0^2 \frac{e^x dx}{e^{1-x} + e^{x-1}}.$$

Solution 1.

$$\int_0^2 \frac{e^x dx}{e^{1-x}+e^{x-1}} = e\int_{-1}^1 \frac{e^u du}{e^{-u}+e^u} = e\int_{-1}^1 \frac{e^{2u}du}{e^{2u}+1}$$
$$= \frac{e}{2}\left[\log(e^{2u}+1)\right]_{-1}^1 = \frac{e}{2}\left[\log\left(\frac{e^2+1}{e^{-2}+1}\right)\right]$$
$$= \frac{e}{2}\log e^2 = e.$$

Solution 2. Setting $u = e^x$, we find that the integral is equal to

$$\int_1^{e^2} \frac{du}{(e/u)+(u/e)} = e\int_1^{e^2} \frac{udu}{e^2+u^2} = \frac{e}{2}\left[\log(e^2+u^2)\right]_1^{e^2}$$
$$= \frac{e}{2}\log\left(\frac{e^2+e^4}{e^2+1}\right) = \frac{e}{2}\log e^2 = e.$$

Solution 3. We first establish a general result: *Suppose that f is continuous on the interval $[a,b]$. Then*

$$\int_a^b \frac{f(x-a)dx}{f(x-a)+f(b-x)} = \frac{1}{2}(b-a).$$

Making the substitution $x = a+b-u$, we see that the given integral is equal to

$$\int_a^b \frac{f(b-u)du}{f(u-a)+f(b-u)}$$

. Adding the two integrals together yields $\int_a^b dx = b-a$, from which the result is found.

Apply this result to $f(x) = e^x$, $a = 0$ and $b = 2$, to obtain

$$\int_0^2 \frac{e^x dx}{e^{1-x}+e^{x-1}} = e\int_0^2 \frac{e^x dx}{e^{2-x}+e^x} = e.$$

8. Let $\{a_n\}$ and $\{b_n\}$ be two *decreasing* positive real sequences for which

$$\sum_{n=1}^{\infty} a_n = \infty$$

and

$$\sum_{n=1}^{\infty} b_n = \infty.$$

Let I be a subset of the natural numbers, and define the sequence $\{c_n\}$ by

$$c_n = \begin{cases} a_n, & \text{if } n \in I \\ b_n, & \text{if } n \notin I. \end{cases}$$

Is it possible for $\sum_{n=1}^{\infty} c_n$ to converge?

Solution. The answer is *yes*. Let $s_{-1} = 0$, and for $n \geq 0$, let

$$s_n = \sum_{k=0}^{n} 2^{2^k}.$$

For $n \geq 0$, let I_n be the set of positive integers for which $s_{n-1} < k \leq s_n$. Define

$$a_k = 2^{-2^{2n}-n} \quad \text{and} \quad b_k = 2^{-2^{2n}} \quad \text{when} \quad k \in I_{2n};$$

$$a_k = 2^{-2^{2n+1}} \quad \text{and} \quad b_k = 2^{-2^{2n+1}-n} \quad \text{when} \quad k \in I_{2n+1}.$$

It can be verified that both $\{a_n\}$ and $\{b_n\}$ are decreasing.

Let

$$I = \bigcup_{n=0}^{\infty} I_{2n}.$$

Then, for each nonnegative n,

$$\sum_{k \in I_{2n+1}} a_k = \sum_{k \in I_{2n}} b_k = 1.$$

Therefore

$$\sum_{n \in I} a_n = 2;$$

$$\sum_{n \notin I} a_n = \infty;$$

$$\sum_{n \in I} b_n = \infty;$$

and

$$\sum_{n \notin I} b_n = 2.$$

[This problem was contributed by Franklin Vera Pacheco.]

9. What is the dimension of the vector subspace of $\mathbf{R}^n$ generated by the set of vectors

$$(\sigma(1), \sigma(2), \sigma(3), \cdots, \sigma(n))$$

where σ runs through all $n!$ of the permutations of the first n natural numbers.

Solution 1. (J. Love) The dimension cannot exceed n. Taking the difference of two permutations with identical outcomes except in the ith and nth positions where σ takes the values 1 and 2, we find that the vector space contains the vectors

$$(0, 0, \cdots, 0, 1, 0, \cdots, 0, -1) = (3, 4, \cdots, i+1, 2, i+2, \cdots, n, 1) - (3, 4, \cdots, i+1, 1, i+2, \cdots, n, 2).$$

This set S of $n-1$ vectors is linearly independent and generates the $(n-1)-$dimensional subpace $\{(x_1, x_2, \cdots, x_n) : x_1 + x_2 + \cdots + x_n = 0\}$. However, the sum of the entries of any element in the generating set is $\frac{1}{2}n(n+1)$, so that any generators along with S is a basis of n elements for the whole space. Thus the required dimension is n.

Solution 2. (C. Wang) As in the first solution, we can show that the span of the generating set contains all vectors of the form $(1, 0, \cdots, -1, \cdots, 0)$ and therefore the vector

$$\begin{aligned} \frac{1}{2}n(n+1)(1, 0, 0, \cdots, 0, 0) &= (1, 2, 3, \cdots, n-1, n) + (2, -2, 0, \cdots, 0, 0) + (3, 0, -3, \cdots, 0, 0) \\ &= (n-1, 0, 0, \cdots, -(n-1), 0) + (n, 0, 0, \cdots, 0, -n). \end{aligned}$$

Similarly, it can be shown that this span contains all of the basis vectors $(0, 0, \cdots, 1, \cdots, 0)$. Hence the required dimension is n.

Solution 3. The dimension cannot exceed n, and is in fact equal to n. We prove that $\{(1,2,3,4,\cdots,n-1,n),(2,3,4,5,\cdots,n,1),(3,4,5,6,\cdots,1,2),\cdots,(n,1,2,3,\cdots,n-2,n-1)\}$ is linearly independent by showing that the determinant of the matrix whose rows are these vectors is nonzero.

Using the fact that the absolute value of the determinant remains unchanged if one row is subtracted from another or if rows are interchanged, we have that

$$\begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & 5 & \cdots & n-1 & n & 1 \\ 3 & 4 & 5 & 6 & \cdots & n & 1 & 2 \\ & & \cdots & & \cdots & & \cdots & \\ n & 1 & 2 & 3 & \cdots & n-3 & n-2 & n-1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n-2 & n-1 & n \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & -(n-1) \\ 1 & 1 & 1 & 1 & \cdots & 1 & -(n-1) & 1 \\ 1 & 1 & 1 & 1 & \cdots & -(n-1) & 11 & \\ & & \cdots & & \cdots & & \cdots & \\ 1 & -(n-1) & 1 & 1 & \cdots & 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n-2 & n-1 & n \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & -(n-1) \\ 0 & 0 & 0 & 0 & \cdots & 0 & -n & n \\ 0 & 0 & 0 & 0 & \cdots & -n & 0 & n \\ & & \cdots & & \cdots & & \cdots & \\ 0 & -n & 0 & 0 & \cdots & 0 & 0 & n \end{vmatrix}$$

$$= \pm n^{n-2} \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & -(n-1) \\ 0 & 1 & 2 & 3 & \cdots & n-3 & n-2 & 2n-1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & -1 \\ & & \cdots & & \cdots & & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{vmatrix}$$

$$= \pm n^{n-2} \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & -(n-1) \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & -1 \\ & & \cdots & & \cdots & & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & (2n-1)-(n-2)(n-1)/2 \end{vmatrix} = \pm n^{n-2}(n^2-7n+4)/2.$$

10. (a) Let
$$g(x,y) = x^2y + xy^2 + xy + x + y + 1.$$

We form a sequence $\{x_0\}$ as follows: $x_0 = 0$. The next term x_1 is the unique root -1 of the linear equation $g(t,0) = 0$. For each $n \geq 2$, x_n is the root other than x_{n-2} of the equation $g(t, x_{n-1}) = 0$.

Let $\{f_n\}$ be the Fibonacci sequence determined by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Prove that, for any nonnegative integer k,

$$x_{2k} = \frac{f_k}{f_{k+1}} \qquad \text{and} \qquad x_{2k+1} = -\frac{f_{k+2}}{f_{k+1}}.$$

(b) Let
$$h(x,y) = x^2y + xy^2 + \beta xy + \gamma(x+y) + \delta$$

be a polynomial with real coefficients β, γ, δ. We form a bilateral sequence $\{x_n : n \in \mathbf{Z}\}$ as follows. Let $x_0 \neq 0$ be given arbitrarily. We select x_{-1} and x_1 to be the two roots of the quadratic equation

$h(t, x_0) = 0$ in either order. From here, we can define inductively the terms of the sequence for positive and negative values of the index so that x_{n-1} and x_{n+1} are the two roots of the equation $h(t, x_n) = 0$. We suppose that at each stage, neither of these roots is zero.

Prove that the sequence $\{x_n\}$ has period 5 (*i.e.* $x_{n+5} = x_n$ for each index n) if and only if $\gamma^3+\delta^2-\beta\gamma\delta = 0$.

(a) *Solution.* Observe that

$$g(x, y) = (xy + 1)(x + y + 1).$$

For each value of $y \neq 0$, the equation $g(x, y) = 0$ has two solutions: $y = -1/x$ and $y = -(x + 1)$. Observe that $g(x, y)$ is symmetrical in x and y, so that, for each consecutive pair x_n, x_{n+1} of terms in the sequence $g(x_n, x_{n+1}) = g(x_{n+1}, x_n) = 0$. Consider the equation $0 = g(t, x_1) = g(t, -1) = (-t + 1)t$. One of its solutions is $x_0 = 0$ and the other is $x_2 = 1$.

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For the equation $0 = g(t, x_2)$, we have that $g(x_1, x_2) = g(x_2, x_1) = 0$ and $x_1x_2 + 1 = 0$. Therefore $x_2 + x_3 + 1 = 0$, so that $x_3 = -(x_2 + 1)$. Continuing on in this way, we find that, for each positive integer k, $x_{2k-1}x_{2k} = -1$ and $x_{2k} + x_{2k+1} = -1$, whereupon

$$x_{2k+1} = -1 + \frac{1}{x_{2k-1}} = \frac{1 - x_{2k-1}}{x_{2k-1}}.$$

When $k = 1$, we find that $x_{2k-1} = x_1 = -1 = -f_2/f_1$. Suppose, for $k \geq 1$, we have that $x_{2k-1} = -f_{k+1}/f_k$. Then

$$x_{2k+1} = -1 - \frac{f_k}{f_{k+1}} = -\frac{f_{k+1} + f_k}{f_{k+1}} = -\frac{f_{k+2}}{f_{k+1}}.$$

By induction, we obtain the desired expression for x_{2k+1}. Also $x_{2k} = -1/x_{2k-1} = -f_k/f_{k+1}$.

(b) *Solution 1.* Observe that from the sum of the roots of $h(t, x_n) = 0$, we have that

$$x_{n-1} + x_{n+1} = -\left[\frac{x_n^2 + \beta x_n + \gamma}{x_n}\right],$$

or

$$x_{n-1} + x_n + x_{n+1} = -\beta - \frac{\gamma}{x_n}$$

for each n. The sequence will have period 5 if and only if the sum of any five consecutive terms is constant.

Since, for each integer n,

$$x_{n+2} + x_{n+1} + x_n = -\beta - \frac{\gamma}{x_{n+1}}$$

and

$$x_n + x_{n-1} + x_{n-2} = -\beta - \frac{\gamma}{x_{n-1}},$$

we have that

$$\begin{aligned}
x_{n+2} + x_{n+1} + x_n + x_{n-1} + x_{n-2} &= -2\beta - \gamma\left(\frac{x_{n+1} + x_{n-1}}{x_{n+1}x_{n-1}}\right) - x_n \\
&= -2\beta + \gamma\left(\frac{x_n^2 + \beta x_n + \gamma}{\gamma x_n + \delta}\right) - x_n \\
&= \frac{-(\beta\gamma + \delta)x_n + (\gamma^2 - 2\beta\delta)}{\gamma x_n + \delta} \\
&= -\left(\beta + \frac{\delta}{\gamma}\right) + \left(\frac{\gamma^3 + \delta^2 - \beta\gamma\delta}{\gamma^2 x_n + \gamma\delta}\right) \quad .
\end{aligned}$$

This sum is independent of n if and only if the term involving x_n vanishes identically, *i.e.*, if and only if the required condition holds.

Solution 2. From the formula for the product of the roots, we obtain that

$$x_{n-1}x_{n+1} = \frac{\gamma x_n + \delta}{x_n}$$

so that

$$x_{n-1}x_n x_{n+1} = \gamma x_n + \delta$$

for each index n. Therefor

$$\begin{aligned} x_{n+2}x_{n+1}x_n^2 x_{n-1}x_{n-2} &= (\gamma x_{n+1} + \delta)(\gamma x_{n-1} + \delta) \\ &= \gamma^2\left(\frac{\gamma x_n + \delta}{x_n}\right) - \gamma\delta\left(x_n + \beta + \frac{\gamma}{x_n}\right) + \delta^2 \\ &= (\gamma^3 + \delta^2 - \beta\gamma\delta) - \gamma\delta x_n \end{aligned}$$

whence

$$x_{n+2}x_{n+1}x_n x_{n-1}x_{n-2} = -\gamma\delta + \left(\frac{\gamma^3 + \delta^2 - \beta\gamma\delta}{x_n}\right) \quad .$$

The result again follows.

Comment. If $x_0 = 0$, then $h(t, x_0)$ is linear and there is a single root $-\delta/\gamma$. We can extend the sequence in only one direction, and it begins with the terms $0, -\delta/\gamma, (\gamma^3 + \delta^2 - \beta\gamma\delta)/(\gamma\delta), \cdots$.