THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh

March 9, 2014

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

The grading is designed to encourage only the stronger students to attempt more than five problems. Each solution is graded out of 10. If the sum of the scores for the solutions to the five best problems does not exceed 30, this sum will be the final grade. If the sum of these scores does exceed 30, then all solutions will be graded for credit.

1. The *permanent*, per A, of a $n \times n$ matric $A = (a_{i,j})$, is equal to the sum of all possible products of the form $a_{1,\sigma(1)}a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$, where σ runs over all the permutations on the set $\{1, 2, \cdots, n\}$. (This is similar to the definition of determinant, but there is no sign factor.) Show that, for any $n \times n$ matrix $A = (a_{i,j})$ with positive real terms,

$$\text{per } A \geq n! \left(\prod_{1 \leq i,j \leq n} a_{i,j} \right)^{\frac{1}{n}} .$$

2. For a positive integer N written in base 10 numeration, N' denotes the integer with the digits of N written in reverse order. There are pairs of integers (A, B) for which A, A', B, B' are all distinct and $A \times B = B' \times A'$. For example,

$$3516 \times 8274 = 4728 \times 6153.$$

(a) Determine a pair (A, B) as described above for which both A and B have two digits, and all four digits involved are distinct.

(b) Are there any pairs (A, B) as described above for which A has two and B has three digits?

3. Let n be a positive integer. A finite sequence $\{a_1, a_2, \cdots, a_n\}$ of positive integers a_i is said to be *tight* if and only if $1 \leq a_1 < a_2 < \cdots < a_n$, all $\binom{n}{2}$ differences $a_j - a_i$ with $i < j$ are distinct, and a_n is as small as possible.

(a) Determine a tight sequence for $n = 5$.

(b) Prove that there is a polynomial $p(n)$ of degree not exceeding 3 such that $a_n \leq p(n)$ for every tight sequence $\{a_i\}$ with n entries.

4. Let $f(x)$ be a continuous realvalued function on $[0, 1]$ for which

$$\int_0^1 f(x)dx = 0 \qquad \text{and} \qquad \int_0^1 xf(x)dx = 1.$$

(a) Give an example of such a function.

(b) Prove that there is a nontrivial open interval I contained in $(0, 1)$ for which $|f(x)| > 4$ for $x \in I$.

Please turn over page for additional problems.

5. Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} \frac{1}{k\binom{n}{k}} = \sum_{k=1}^{n} \frac{1}{k2^{n-k}} = \frac{1}{2^{n-1}} \sum_{k=1}^{n} \frac{2^{k-1}}{k} = \frac{1}{2^{n-1}} \sum \left\{ \frac{\binom{n}{k}}{k} : k \text{ odd }, 1 \le k \le n \right\}).$$

6. Let $f(x) = x^6 - x^4 + 2x^3 - x^2 + 1$.

(a) Prove that $f(x)$ has no positive real roots.

(b) Determine a nonzero polynomial $g(x)$ of minimum degree for which all the coefficients of $f(x)g(x)$ are nonnegative rational numbers.

(c) Determine a polynomial $h(x)$ of minimum degree for which all the coefficients of $f(x)h(x)$ are positive rational numbers.

7. Suppose that $x_0, x_1, \cdots, x_n$ are real numbers. For $0 \le i \le n$, define

$$y_i = \max\,(x_0, x_1, \cdots, x_i).$$

Prove that

$$y_n^2 \le 4x_n^2 - 4\sum_{i=0}^{n-1} y_i(x_{i+1} - x_i).$$

When does equality occur?

8. The hyperbola with equation $x^2 - y^2 = 1$ has two branches, as does the hyperbola with equation $y^2 - x^2 = 1$. Choose one point from each of the four branches of the locus of $(x^2 - y^2)^2 = 1$ such that area of the quadrilateral with these four vertices is minimized.

9. Let $\{a_n\}$ and $\{b_n\}$ be positive real sequences such that

$$\lim_{n\to\infty} \frac{a_n}{n} = u > 0$$

and

$$\lim_{n\to\infty} \left(\frac{b_n}{a_n}\right)^n = v > 0.$$

Prove that

$$\lim_{n\to\infty} \left(\frac{b_n}{a_n}\right) = 1$$

and

$$\lim_{n\to\infty} (b_n - a_n) = u \log v.$$

10. Does there exist a continuous realvalued function defined on $\mathbf{R}$ for which $f(f(x)) = -x$ for all $x \in \mathbf{R}$?

Solutions

1. The *permanent*, per A, of a $n \times n$ matric $A = (a_{i,j})$, is equal to the sum of all possible products of the form $a_{1,\sigma(1)}a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$, where σ runs over all the permutations on the set $\{1, 2, \cdots, n\}$. (This is similar to the definition of determinant, but there is no sign factor.) Show that, for any $n \times n$ matrix $A = (a_{i,j})$ with positive real terms,

$$\text{per } A \geq n! \left(\prod_{1 \leq i,j \leq n} a_{i,j} \right)^{\frac{1}{n}} .$$

Solution. By the Arithmetic-Geometric Means Inequality, the sum defining the permanent (having $n!$ terms) is not less that $n!$ times the product of all these terms raised to the power $1/n!$. Each term in the definition of the permanent has n factors, so that product of all the terms has $n \cdot n!$ factors. There are n^2 entries $a_{i,j}$; each appears the same number $n \times n! \div n^2 = (n-1)!$ times. The result follows.

2. For positive integer N written in base 10 numeration, N' denotes the integer with the digits of N written in reverse order. There are pairs of integers (A, B) for which A, A', B, B' are all distinct and $A \times B = B' \times A'$. For example,

$$3516 \times 8274 = 4728 \times 6153.$$

(a) Determine a pair (A, B) as described above for which both A and B have two digits, and all four digits involved are distinct.

(b) Are there any pairs (A, B) as described above for which A has two and B has three digits?

Solution. (a) Let $A = 10a+b$ and $B = 10u+v$ where $1 \leq a, b, u, v \leq 9$. The condition $AB = B'A'$ leads to $99(au-bv) = 0$, so that a necessary condition for the property to hold is $au = bv$. There are many possibilities, including $(a, b; u, v) = (1, 3; 6, 2)$. Thus, for example, $13 \times 62 = 26 \times 31$. [The possibilities include $(A, B) = (12, 42), (12, 63), (13, 62), (12, 84), (14, 82), (23, 64), (24, 63), (24, 84), (23, 96), (26, 93), (34, 86), (46, 96), (48, 63)$.] ∎

(b) Let $A = 10a + b$ and $B = 100u + 10v + w$, where $abuw \neq 0$ and $0 \leq a, b, u, v, w \leq 9$. The condition $AB = B'A'$ leads to

$$111(au - bw) + 10[a(v - w) + b(u - v)] = 0.$$

Therefore $a(v - w) + b(u - v)$ must be a multiple of 111. If the two terms of this sum differ in sign, then the absolute value of the sum cannot exceed $9 \times 9 = 81$. If the two terms of the sum have the same sign, then the signs of $v - w$ and $u - v$ must be the same, and the absolute value of the sum cannot exceed

$$|a(v - w) + b(u - v)| \leq 9|(v - w) + (u - v)| = 9|u - w| = 81.$$

Therefore $a(v - w) + b(u - v) = 0$, so that $au = bw$. If $v - w$ and $u - v$ do not vanish, they must have opposite signs. These conditions are necessary and sufficient for an example. Trial and error leads to $(a, b; u, v, w) - (1, 2; 2, 3, 1)$. Thus, we obtain

$$12 \times 231 = 2772 = 231 \times 12.$$

In fact, we can replace these numbers by multiples that involve no carries to obtain the examples $(12s, 231t)$ where $1 \leq s \leq 4$ and $1 \leq t \leq 3$. Other examples of pairs are $(13, 682)$, $(28, 451)$.

Comment. At a higher level, we have that

$$992 \times 483 \times 156 = 651 \times 384 \times 299 = 2^7 \times 3^2 \times 7 \times 13 \times 23 \times 31.$$

3. Let n be a positive integer. A finite sequence $\{a_1, a_2, \cdots, a_n\}$ of positive integers a_i is said to be *tight* if and only if $1 \le a_1 < a_2 < \cdots < a_n$, all $\binom{n}{2}$ differences $a_j - a_i$ with $i < j$ are distinct, and a_n is as small as possible.

(a) Determine a tight sequence for $n = 5$.

(b) Prove that there is a polynomial $p(n)$ of degree not exceeding 3 such that $a_n \le p(n)$ for every tight sequence $\{a_i\}$ with n entries.

Solution. (a) A sequence having all differences different remains so if the same integer is added or subtracted from each term. Therefore $a_1 = 1$ for any tight sequence. Since there are $\binom{5}{2} = 10$ differences for a tight sequence with five entries, $a_5 \ge 11$. We show that $a_5 = 11$ cannot occur.

Suppose, if possible, that $a_1 = 1$ and $a_5 = 11$. Since each of the differences from 1 to 10, inclusive, must occur, and since the largest difference 10 is the sum of the four differences of adjacent entries, the differences of adjacent entries must be 1, 2, 3, 4 in some order. The difference 1 cannot be next to either of the differences 2 or 3, for then there would be a difference between nonadjacent entries would be 3 or 4, respectively. Thus 1 is the difference of either the first two or the last two entries and the adjacent difference would be 4. But then $a_3 - a_1 = a_5 - a_3 = 5$, which is not allowed. Since there is no other possibility, $a_5 \ge 12$.

Trying $a_5 = 12$, we find that the sequence $\{1, 2, 5, 10, 12\}$ is tight.

(b) The strategy is to construct a sequence that satisfies the difference condition such that the differences between pairs of terms have different congruence classes with respect to some modulus, dependent only on the number of terms between them. Thus, we make the difference between adjacent terms congruent to 1, between terms separated by one entry congruent to 2, *etc.*. To this end, let $a_1 = 1$ and

$$a_{i+1} = a_i + 1 + (i-1)(n-1)$$

for $1 \le i \le n-1$. Then,

$$a_i = i + \frac{1}{2}(i-1)(i-2)(n-1).$$

In particular,

$$a_n = n + \frac{1}{2}(n-1)^2(n-2) = \frac{1}{2}(n^3 - 4n^2 + 7n - 2).$$

When $1 \le i < j \le n$, we have that

$$a_j - a_i = (a_j - a_{j-1}) + (a_{j-1} - a_{j-2}) + \cdots + (a_{i+1} - a_i) \equiv j$$

modulo $(n-1)$. Furthermore,

$$(a_{j+1} - a_{i+1}) - (a_j - a_i) = (a_{j+1} - a_j) - (a_{i+1} - a_i) = (j-i)(n-1) > 0.$$

It follows that all the differences are distinct.

Any tight sequence with n terms must have its largest term no greater than $p(n) = \frac{1}{2}(n^3 - 4n^2 + 7n - 2)$.

Comment. This upper bound is undoubtedly too generous and it may be that there is an upper bound quadratic in n.

4. Let $f(x)$ be a continuous realvalued function on $[0, 1]$ for which

$$\int_0^1 f(x)dx = 0 \qquad \text{and} \qquad \int_0^1 xf(x)dx = 1.$$

(a) Give an example of such a function.

(b) Prove that there is a nontrivial open interval I contained in $(0, 1)$ for which $|f(x)| > 4$ for $x \in I$.

Solution. (a) Trying a function of the form $c(x-\frac{1}{2})$ yields the examples $f(x)=12x-6$. Other examples are $f(x)=-\frac{1}{2}\pi^2\cos(\pi x)$ and $f(x)=-2\pi\sin(2\pi x)$.

(b) If there existed no such interval I, then $|f(x)|\le 4$ for all $x\in[0,1]$. Then

$$\begin{aligned}1&=\int_0^1 f(x)(x-(1/2))dx=\int_0^{\frac{1}{2}}(-f(x))((1/2)-x)dx+\int_{\frac{1}{2}}^1 f(x)(x-(1/2))dx\\&<4\int_0^{\frac{1}{2}}((1/2)-x)dx+4\int_{\frac{1}{2}}^1(x-(1/2))dx=4\cdot(1/8)+4\cdot(1/8)=1.\end{aligned}$$

Note that the inequality is strict. Equality would require that $f(x)=-4$ on $(0,\frac{1}{2})$ and $f(x)=4$ on $(\frac{1}{2},1)$, an impossibility for a continuous function.

Note. B. Yaghani draws attention to Problem 25 of Chapter 11 appearing on page 184 of Michael Spivak, *Calculus* (W.A. Benjamin, 1967), where it is required to show that, if g is twice differentiable on $[0,1]$ with $g(0)=g'(0)=g'(1)=0$ and $g(1)=1$, then $|g''(c)|\ge 4$ for some $c\in[0,1]$. Setting $g(t)=\int_0^t(x-t)f(x)dx$, whereupon $g'(t)=-\int_0^t f(x)dx$ and $g''(t)=-f(t)$, yields the desired result.

[This is based on problem #4520 in the *American Mathematical Monthly.*]

5. Let n be a positive integer. Prove that

$$\sum_{k=1}^n\frac{1}{k\binom{n}{k}}=\sum_{k=1}^n\frac{1}{k2^{n-k}}=\frac{1}{2^{n-1}}\sum_{k=1}^n\frac{2^{k-1}}{k}=\frac{1}{2^{n-1}}\sum\left\{\frac{\binom{n}{k}}{k}:k\text{ odd },1\le k\le n\right\}.$$

Solution. The values of each of the sums for $1\le n\le 5$ are $1,1,5/6,2/3,8/15$.

The equality of the second and third sums is straightforward to establish. Let S_n denote the leftmost sum. Since $k\binom{n}{k}=n\binom{n-1}{k-1}$, we have that

$$S_n=\sum_{k=1}^n\left[n\binom{n-1}{k-1}\right]^{-1}.$$

Using the fact that

$$\left[(n+1)\binom{n}{k-1}\right]^{-1}+\left[(n+1)\binom{n}{k}\right]^{-1}=\left[n\binom{n-1}{k-1}\right]^{-1},$$

for $1\le k\le n$, it can be shown that $S_1=1$ and that

$$S_{n+1}=\frac{S_n}{2}+\frac{1}{n+1}$$

for $n\ge 1$.

Therefore

$$\begin{aligned}S_n&=\frac{1}{2^{n-1}}+\frac{1}{2\cdot 2^{n-2}}+\frac{1}{3\cdot 2^{n-3}}+\cdots+\frac{1}{k\cdot 2^{n-k}}+\cdots+\frac{1}{n}\\&=\sum_{k=1}^n\frac{1}{k\cdot 2^{n-k}}=\frac{1}{2^{n-1}}\sum_{k=1}^n\frac{2^{k-1}}{k}.\end{aligned}$$

For each positive integer n, let

$$T_n=\frac{1}{2^{n-1}}\sum_{k\text{ odd}}\frac{\binom{n}{k}}{k}.$$

Then $T_1 = 1$ and

$$2T_{n+1} - T_n = \frac{1}{2^{n-1}} \sum_{k \text{ odd}} \frac{\binom{n+1}{k} - \binom{n}{k}}{k} = \frac{1}{2^{n-1}} \sum_{k \text{ odd}} \frac{\binom{n}{k-1}}{k}$$
$$= \frac{1}{(n+1)2^{n-1}} \sum_{k \text{ odd}} \binom{n+1}{k} = \frac{2^n}{(n+1)2^{n-1}} = \frac{2}{n+1}.$$

Therefore the recursions for S_n and T_n agree when $n = 1$ and satisfy the same equations. Thus the result holds.

6. Let $f(x) = x^6 - x^4 + 2x^3 - x^2 + 1$.

(a) Prove that $f(x)$ has no positive real roots.

(b) Determine a nonzero polynomial $g(x)$ of minimum degree for which all the coefficients of $f(x)g(x)$ are nonnegative rational numbers.

(c) Determine a polynomial $h(x)$ of minimum degree for which all the coefficients of $f(x)h(x)$ are positive rational numbers.

Solution. (a) Note that

$$f(x) = (x^2 - 1)(x^4 - 1) + 2x^3 = (x^2 - 1)^2(x^2 + 1) + 2x^3$$

from which we see that $f(x) > 0$ for all $x > 0$. Alternatively,

$$f(x) = x^6 + x^3 + (x^3 - x^4) + (1 - x^2) = x^4(x^2 - 1) + x^2(x - 1) + x^3 + 1$$

from which we see that there are no roots in either of the intervals $[0, 1]$ and $[1, \infty)$.

(b) It is straightforward to see that multiplying $f(x)$ by a linear polynomial will not achieve the goal. We have that

$$f(x)(x^2 + bx + c) = x^8 + bx^7 + (c - 1)x^6 + (2 - b)x^5 + (2b - c - 1)x^4 + (2c - b)x^3 + (1 - c)x^2 + bx + c$$

from which we see that taking $c = 1$ and $1 \leq b \leq 2$ will yield the desired polynomial $g(x)$. Examples of suitable $g(x)$ are $x^2 + x + 1$ and $(x + 1)^2$.

(c) From (b), we note that no quadratic polynomial will serve. However, taking $h(x) = x^3 + 2x^2 + 2x + 1$, we find that

$$f(x)h(x) = x^9 + 2x^8 + x^7 + x^6 + x^5 + x^4 + x^2 + x^2 + 2x + 1.$$

Comment. It is known that for a given real polynomial $f(x)$, there exists a polynomial $g(x)$ for which all the coefficients of $f(x)g(x)$ are nonnegative (resp. positive) if and only if all the roots of $f(x)$ are positive (resp. nonnegative).

Another example is $f(x) = x^3 - x + 1$. Since $f(x) = x^3 + (1 - x) = x(x^2 - 1) + 1$, we see that there are no roots in $[0, 1]$ and $[1, \infty)$. No linear polynomial $x + c$ will do for $g(x)$. Since

$$f(x)(x^2 + bc + c) = x^5 + bx^4 + (c - 1)x^3 + (1 - b)x^2 + (b - c)x + c,$$

we require that $1 \leq c \leq b$ and $0 \leq b \leq 1$. The only possibility is that $g(x) = x^2 + x + 1$.

For strictly positive coefficients in the product, set $g(x) = ax^3 + bx^2 + cx + d$. Then

$$f(x)g(x) = ax^6 + bx^5 + (c - a)x^4 + (a + d - b)x^3 + (b - c)x^2 + (c - d)x + d.$$

For g to be suitable, we require that $c > a$, $a + d > b$ and $b > c > d$. We can take $g(x) = 3x^3 + 5x^2 + 4x + 3$.

[This example is due to Horst Brunotte in Düsseldorf, Germany.]

7. Suppose that $x_0, x_1, \cdots, x_n$ are real numbers. For $0 \le i \le n$, define

$$y_i = \max\ (x_0, x_1, \cdots, x_i).$$

Prove that

$$y_n^2 \le 4x_n^2 - 4\sum_{i=0}^{n-1} y_i(x_{i+1} - x_i).$$

When does equality occur?

Solution 1. We can simplify the problem. Suppose that for some $i < j$, $y_i = y_{i+1} = \cdots = y_j < y_{j+1}$. Then

$$y_i(x_{i+1} - x_i) + y_{i+1}(x_{i+2} - x_{i+1}) + \cdots + y_j(x_{j+1} - x_j) = y_i(x_{j+1} - x_i),$$

so that all of $x_{i+1}, \cdots, x_j$ do not figure in the terms of the right side. Therefore, with no loss of generality, we can assume that

$$x_0 < x_1 < \cdots < x_{n-1}$$

so that $y_i = x_i$ for $0 \le i \le n-1$, and y_n is equal to the maximum of x_{n-1} and x_n.

The right side of the inequality is equal to

$$\begin{aligned}
4x_n^2 - &4[x_0(x_1 - x_0) + x_1(x_2 - x_1) + \cdots + x_{n-1}(x_n - x_{n-1})] \\
&= 2x_0^2 + 2(x_1 - x_0)^2 + 2(x_2 - x_1)^2 + \cdots + 2(x_n - x_{n-1})^2 + 2x_n^2 \\
&= 2x_0^2 + 2(x_1 - x_0)^2 + 2(x_2 - x_1)^2 + \cdots + 2(x_{n-1} - x_{n-2})^2 + x_{n-1}^2 + (2x_n - x_{n-1})^2.
\end{aligned}$$

Therefore, the right side is greater than or equal to both of $2x_n^2$ and x_{n-1}^2. The desired result follows.

Equality occurs if and only if $x_0 = 0$ for each i.

Comment. It is natural to try a proof by induction. When $n = 1$, the right side is equal to

$$4x_1^2 - 4x_0x_1 + 4x_0^2 = 2x_1^2 + 2x_0^2 + 2(x_1 - x_0)^2$$

which is clearly greater than or equal to either of x_1^2 and x_0^2.

Suppose that the inequality holds for $n = m \ge 1$. Then

$$\begin{aligned}
4x_{m+1}^2 - 4\sum_{i=0}^{m} y_i(x_{i+1} - x_i) &= 4x_{m+1}^2 - 4x_m^2 + \left[4x_m^2 - 4\sum_{i=0}^{m-1} y_i(x_{i+1} - x_i)\right] - 4y_m(x_{m+1} - x_m) \\
&\ge 4x_{m+1}^2 - 4x_m^2 + y_m^2 - 4y_m(x_{m+1} - x_m) \equiv K
\end{aligned}$$

Since y_{m+1} is the greater of x_{m+1} and y_m, there are two cases to consider.

When $y_{m+1} = x_{m+1}$, then, since $x_m \le y_m \le y_{m+1}$ and $2y_{m+1} - y_m \ge y_m$,

$$K = (2y_{m+1} - y_m)^2 + 4x_m(y_m - x_m) \ge (2y_{m+1} - y_m)^2 \ge y_{m+1}^2.$$

However, when $y_{m+1} = y_m$, then

$$K = y_m^2 + 4(x_{m+1} - x_m)(x_{m+1} + x_m - y_m)$$

and it is not clear whether the second term is always nonnegative.

[This problem is due to Mathias Beiglböck of the University of Vienna, and was communicated to the contest by Florian Herzig of the University of Toronto.]

8. The hyperbola with equation $x^2 - y^2 = 1$ has two branches, as does the hyperbola with equation $y^2 - x^2 = 1$. Choose one point from each of the four branches of the locus of $(x^2 - y^2)^2 = 1$ such that area of the quadrilateral with these four vertices is minimized.

Solution. Let N, W, S, E be arbitrary points on the four branches that are symmetric respectively about the positive $y-$axis (northern), negative $x-$axis (western), negative $y-$axis (southern) and positive $x-$axis (eastern). Consider the diagonal NS of $NWSE$. If the secant through W that is parallel to NS passes through two distinct points of the western branch, then the area $[NWS]$ of triangle NSW can be made smaller by replacing W by the point of tangency to the western branch by a tangent parallel to NS. Similarly, the area $[NSE]$ can be made smaller when the tangent at E is parallel to NS. We can follow a similar argument for the diagonal WE. Thus, for any quadrilateral $NWSE$, we can find a quadrilateral of no larger area where the tangents to the hyberbolae at N and S are parallel to WE and the tangents to the hyperbolae at E and W are parallel to NS. Thus we may assume that $NWSE$ is a parallelogram.

Observe that the slopes of the diagonals WE as well as the tangents to the northern and southern branches range over the open interval $(-1, 1)$, and that the slopes of the diagonals NS and the tangents to the western and eastern branches are similarly related. Each allowable tangent slope occurs exactly twice, the tangents being related by a $180°$ rotation about the origin. Thus, in the minimization problem, we need consider only parallelograms centred at the origin with the foregoing tangent and diagonal relationship.

Let $N \sim (a, b)$ be any point on the northern branch, where with no loss of generality we may assume that $0 \leq a < b$ (and $b^2 - a^2 = 1$. The slope of the tangent at N is a/b, so the point E, located on the line with this slope through the origin must be at (b, a). Similarly, $S \sim (-a, -b)$ and $W \sim (-b, -a)$. Thus, $NWSE$ is in fact a rectangle whose sides are perpendicular to the asymptotes of the hyperbola and whose area is

$$|NE| \cdot |ES| = (\sqrt{2})(b - a))(\sqrt{2})(b + a) = 2(b^2 - a^2) = 2.$$

Therefore, the required minimum area is 2.

[This problem was contributed by Robert McCann.]

9. Let $\{a_n\}$ and $\{b_n\}$ be positive real sequences such that

$$\lim_{n\to\infty} \frac{a_n}{n} = u > 0$$

and

$$\lim_{n\to\infty} \left(\frac{b_n}{a_n}\right)^n = v > 0.$$

Prove that

$$\lim_{n\to\infty} \left(\frac{b_n}{a_n}\right) = 1$$

and

$$\lim_{n\to\infty} (b_n - a_n) = u \log v.$$

Solution. Observe that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \exp \frac{1}{n} \log \left(\frac{b_n}{a_n}\right)^n = \exp(0 \cdot v) = \exp 0 = 1.$$

Suppose that $v \neq 1$. Then for sufficiently large values of n, $b_n \neq a_n$ and we have that

$$(b_n - a_n) \left[\left(\frac{a_n}{b_n - a_n}\right) \log \left(1 + \frac{b_n - a_n}{a_n}\right)\right] = \frac{a_n}{n} \log \left(\frac{b_n}{a_n}\right)^n.$$

Note that

$$\lim_{n\to\infty} \frac{b_n - a_n}{a_n} = \lim_{n\to\infty} \left(\frac{b_n}{a_n} - 1 \right) = 0$$

and that $\lim_{t\to 0} t^{-1} \log(1+t) = 1$, so that

$$\left[\lim_{n\to\infty} (b_n - a_n) \right] \cdot 1 = u \log v$$

as desired.

If $v = 1$, we need to exercise more care, as b_n could equal a_n infinitely often. In this case, the indices can be partitioned into two sets A and B, where respectively $b_n = a_n$ for $n \in A$ and $b_n \neq a_n$ for $n \in B$; either of these sets could be infinite or finite. If infinite, taking the limit over $n \in A$ and over $n \in B$ yields the result $0 = u \log v$, and the problem is solved.

[This problem was contributed by the AN-anduud Problem Solving Group in Ulaanbataar, Mongolia.]

10. Does there exist a continuous realvalued function defined on $\mathbf{R}$ for which $f(f(x)) = -x$ for all $x \in \mathbf{R}$?

Solution 1. The answer is **no**. Since $f(f(x)) = -x$ defines a one-one function, it follows that $f(x)$ itself is one-one. Therefore $f(x)$ is either strictly increasing or strictly decreasing on $\mathbf{R}$. But this is impossible, since in either case, $f(f(x))$ would be increasing.

Solution 2. The answer is **no**. Suppose that f is such a function and that $f(0) = u$. Then $f(u) = f(f(0)) = 0$ and so $u = f(0) = f(f(u)) = -u$. Hence $u = 0$ and $f(0) = 0$.

Suppose that $f(v) = 0$. Then $f(0) = f(f(v)) = -v$. Therefore $f(v) = 0$ if and only if $v = 0$.

Let $a \neq 0$ and $b = f(a)$. Then $f(b) = -a$ and $f(-a) = -b$. Consider the four numbers $a, b, -a, -b$ repeated cyclically. The alternate ones have opposite signs, so there must be two numbers of the same sign followed by one of the opposite sign. With no loss of generality, we can assume that a and b are positive while $-a$ and $-b$ are negative. Then $f(a)$ and $f(b)$ have opposite signs, so by the Intermediate Value Theorem, there must exist c between a and b for which $f(c) = 0$. But this is a contradiction. Therefore, there is no continuous function with the desired property.