THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh

Sunday, March 6, 2011

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

The grading is designed to encourage only the stronger students to attempt more than five problems. Each solution is graded out of 10. If the sum of the scores for the solutions to the five best problems does not exceed 30, this sum will be the final grade. If the sum of these scores does exceed 30, then all solutions will be graded for credit.

1. Let S be a nonvoid set of real numbers with the property that, for each real number x, there is a unique real number $f(x)$ belonging to S that is farthest from x, *i.e.*, for each y in S distinct from $f(x)$, $|x - f(x)| > |x - y|$. Prove that S must be a singleton.

2. Let u and v be positive reals. Minimize the larger of the two values

$$2u + \frac{1}{v^2} \qquad \text{and} \qquad 2v + \frac{1}{u^2} \ .$$

3. Suppose that S is a set of n nonzero real numbers such that exactly p of them are positive and exactly q are negative. Determine all the pairs (n, p) such that exactly half of the threefold products abc of distinct elements a, b, c of S are positive.

4. Let $\{b_n : n \geq 1\}$ be a sequence of positive real numbers such that

$$3b_{n+2} \geq b_{n+1} + 2b_n$$

for every positive integer n. Prove that either the sequence converges or that it diverges to infinity.

5. Solve the system

$$x + xy + xyz = 12$$
$$y + yz + yzx = 21$$
$$z + zx + zxy = 30 \ .$$

6. Two competitors play badminton. They play two games, each winning one of them. They then play a third game to determine the overall winner of the match. The winner of a game of badminton is the first player to score at least 21 points with a lead of at least 2 points over the other player.

   In this particular match, it is observed that the scores of each player listed in order of the games form an arithmetic progression with a nonzero common difference. What are the scores of the two players in the third game?

Please turn the page for further questions.

7. Suppose that there are 2011 students in a school and that each students has a certain number of friends among his schoolmates. It is assumed that if A is a friend of B, then B is a friend of A, and also that there may exist certain pairs that are not friends. Prove that there is a nonvoid subset S of these students for which every student in the school has an even number of friends in S.

8. The set of transpositions of the symmetric group S_5 on $\{1, 2, 3, 4, 5\}$ is

$$\{(12), (13), (14), (15), (23), (24), (25), (34), (35), (45)\}$$

where (ab) denotes the permutation that interchanges a and b and leaves every other element fixed. Determine a product of all transpositions, each occuring exactly once, that is equal to the identity permutation ϵ, which leaves every element fixed.

9. Suppose that A and B are two square matrices of the same order for which the indicated inverses exist. Prove that

$$(A + AB^{-1}A)^{-1} + (A + B)^{-1} = A^{-1} \ .$$

10. Suppose that p is an odd prime. Determine the number of subsets S contained in $\{1, 2, \cdots, 2p - 1, 2p\}$ for which (a) S has exactly p elements, and (b) the sum of the elements of S is a multiple of p.

Solutions

1. Let S be a nonvoid set of real numbers with the property that, for each real number x, there is a unique real number $f(x)$ belonging to S that is farthest from x, *i.e.*, for each y in S distinct from $f(x)$, $|x - f(x)| > |x - y|$. Prove that S must be a singleton.

Solution 1. Let a be an arbitrary real, and let b be the unique point of S that is farthest from a. Then every number in S lies on the same side of b as a. Let c be the unique number in S that is farthest from b. Then S must be in the closed interval bounded by b and c. It follows that both b and c are the farthest points in S from $\frac{1}{2}(b+c)$. By the unicity, we must have that $b = c$ and the result follows.

Solution 2. Since the condition applies to 0 in particular, $|x| \leq |f(0)|$ for all $x \in S$. Therefore S is bounded. Let $a = \inf S$ and $b = \sup S$. The $a \leq b$ and $S \subseteq [a, b]$. Since $a \leq x \leq f(a) \leq b$, so for all $x \in S$, it follows that $f(a) = b$. Similarly $f(b) = a$, and both a and b belong to S.

For each $x \in S$, $|x - \frac{1}{2}(a+b)| \leq |a - \frac{1}{2}(a+b)| = |b - \frac{1}{2}(a+b)|$, so that $a = f(\frac{1}{2}(a+b)) = b$. The desired result follows.

[Problem contributed by Bamdad Yahaghi.]

2. Let u and v be positive reals. Minimize the larger of the two values

$$2u + \frac{1}{v^2} \qquad \text{and} \qquad 2v + \frac{1}{u^2} \ .$$

Solution 1. Observe that

$$\begin{aligned}\left(2u + \frac{1}{v^2}\right) + \left(2v + \frac{1}{u^2}\right) &= \left(u + u + \frac{1}{u^2}\right) + \left(v + v + \frac{1}{v^2}\right) \\ &\geq 3\left(u \cdot u \cdot \frac{1}{u^2}\right)^{1/3} + 3\left(v \cdot v \cdot \frac{1}{v^2}\right)^{1/3} = 6\end{aligned}$$

by the Arithmetic-Geometric Means Inequality. It follow that one of $2u + v^{-2}$ and $2v + u^{-2}$ must be not less than 3. Since both assume the value 3 when $u = v = 1$, the required minimum value is 3.

Solution 2. Without loss of generality, we may assume that $u \geq v$, whereupon

$$2u + \frac{1}{v^2} \geq 2v + \frac{1}{v^2} \geq 2v + \frac{1}{u^2} \ ,$$

with equality if and only if $u = v = 1$. By checking the first derivative, it can be verified that $2v + v^{-2}$ achieves its minimum value of 3 when $v = 1$, so that $2u + v^{-2} \geq 3$ for all positive values of u and v, with equality if and only if $u = v = 1$.

3. Suppose that S is a set of n nonzero real numbers such that exactly p of them are positive and exactly q are negative. Determine all the pairs (n, p) such that exactly half of the threefold products abc of distinct elements a, b, c of S are positive.

Solution 1. A threefold product is positive if and only if all factors are positive or exactly one factor is positive; it is negative if and only if all factors or exactly one factor is negative. This leads to the equation

$$\binom{p}{3} + p\binom{q}{2} = \binom{q}{3} + q\binom{p}{2} \ .$$

This simplifies to

$$(p-q)[(p-q)^2 - 3(p+q) + 2] = 0 \ .$$

We have that $n = p + q$; let $m = |p - q|$. Then the condition is satisfied if and only if either $p = q$ and $n = 2p$ is even, or $n = \frac{1}{3}(m^2 + 2)$, m is not divisible by 3, and p and q are equal to $(m+1)(m+2)/6$ and $(m-1)(m-2)/6$ in some order.

Solution 2. Since half of the total number of products are to be positive, we obtain the condition

$$\binom{p}{3} + p\binom{n-p}{2} = \frac{1}{2}\binom{n}{3} .$$

This reduces to

$$2p(p-1)(p-2) + 6p(n-p)(n-p-1) = n(n-1)(n-2) ,$$

and ultimately to

$$\begin{aligned} 0 &= (8p^3 - 12p^2n + 6pn^2 - n^3) + (-6pn + 3n^2) + (4p - 2n) \\ &= (2p - n)[(2p - n)^2 - 3n + 2] . \end{aligned}$$

Therefore, we must have either that $2p = n$ or $n = \frac{1}{3}(m^2 + 2)$ where $m = |2p - n|$. The argument can be completed as in Solution 1.

Solution 3. There is no loss of generality in assuming that all the positive numbers are $+1$ and the negative numbers are -1. Then, when exactly half the threefold products are positive, the sum of all the threefold products vanishes. Let $m = p - q$, so that m is the sum of the numbers in S. Then

$$m^3 = \sum\{a^3 : a \in S\} + 3\sum\{a^2b : a, b \in S\} + 6\sum\{abc : a, b, c \in S\} = m + 3(n-1)m + 0$$

whence either $m = 0$ or $m^2 = 3n - 2$. This leads to the same conclusion as in Solution 1.

4. Let $\{b_n : n \geq 1\}$ be a sequence of positive real numbers such that

$$3b_{n+2} \geq b_{n+1} + 2b_n$$

for every positive integer n. Prove that either the sequence converges or that it diverges to infinity.

Solution 1. First, we establish that if $0 \leq u \leq v$, $b_n \geq v$ and $b_{n+1} \geq u$ for some n, then

$$b_{n+k} \geq u + \frac{2}{9}(v - u)$$

for $k \geq 2$. Note that

$$b_{n+2} \geq \frac{1}{3}u + \frac{2}{3}v = u + \frac{2}{3}(v - u) \geq u + \frac{2}{9}(v - u)$$

and

$$b_{n+3} \geq \frac{1}{3}\left[u + \frac{2}{3}(v - u)\right] + \frac{2}{3}u = u + \frac{2}{9}(v - u) .$$

Since each term in the sequence is not less than a weighted average of the previous two terms, it follows that

$$b_{n+k} \geq \frac{1}{3}\left[u + \frac{2}{3}(v - u)\right] + \frac{2}{3}u = u + \frac{2}{9}(v - u)$$

for $k \geq 2$.

Let $a = \liminf_{n\to\infty} b_n$. If $a = \infty$, then $\lim_{n\to\infty} b_n = \infty$. Otherwise, a is finite and, given $\epsilon > 0$, we can choose a positive index N for which $b_n > a - \epsilon$ for $n \geq N$. Let $n \geq N$ be arbitrary. Taking $v = b_n$ and $u = a - \epsilon$ in the foregoing result, we find that

$$b_{n+k} \geq a - \epsilon + \frac{2}{9}(b_n - a + \epsilon) > a - \epsilon + \frac{2}{9}(b_n - a)$$

so that

$$a = \liminf_{k\to\infty} b_{n+k} \geq a - \epsilon + \frac{2}{9}(b_n - a) \ .$$

Therefore $b_n - a \leq 9\epsilon/2$, whereupon $\limsup_{n\to\infty} b_n \leq a + 9\epsilon/2$. Since this holds for all $\epsilon > 0$, $\limsup b_n = a = \liminf b_n$ and the desired result follows.

Solution 2. For $n \geq 1$, let $c_n = \min(b_n, b_{n+1})$. Then $b_{n+1} \geq c_n$ and $b_{n+2} = \frac{1}{3}b_{n+1} + \frac{2}{3}b_n \geq c_n$, so that $c_{n+1} \geq c_n$ for $n \geq 1$.

Since $b_n \geq c_n$ for all n, $\{b_n\}$ diverges to infinity if $\{c_n\}$ does.

Otherwise, let $\{c_n\}$ be bounded. Since the sequence increases, it increases to a finite limit m. Let $\epsilon > 0$ be given. Suppose that N is selected so that $m - \frac{1}{2}\epsilon < c_n \leq m$ for $n \geq N$. If b_n is equal to either c_n or c_{n-1}, then $m - \epsilon < b_n < m$ for $n > N$. If b_n is equal to neither c_{n-1} nor c_n, then $c_{n-1} = b_{n-1}$ and $c_n = b_{n+1}$, whence

$$b_n \leq 3b_{n+1} - 2b_{n-1} \leq 3c_n - 2c_{n-1} \leq 3m - 2(m - \epsilon/2) \leq m + \epsilon \ .$$

Thus, $m - \epsilon < b_n < m + \epsilon$ for $n > N$. It follows that $\lim_{n\to\infty} = m$.

Solution 3. [A. Remorov] Let $u = \liminf_{n\to\infty} b_n$ and $v = \limsup_{n\to\infty} b_n$. If $\{b_n\}$ is unbounded, then, given $M > 0$, there exists r such that $b_r > 3M$. Then $b_{r+1} > \frac{1}{3}b_r > M$ and $b_{r+2} > \frac{2}{3}b_r > M$, whereupon it can be established by induction that $b_n > M$ for $n > r$. Hence $u \geq M$ for each $M > 0$ and $\lim_{n\to\infty} b_n = +\infty$.

Otherwise, $\{b_n\}$ is bounded and v is finite. Suppose, if possible, that $v > u$. Let s be selected so that $b_n > u - \frac{1}{5}(v - u)$ for $n \geq s$. If it happens that $b_{n+1} > u + \frac{1}{2}(v - u)$ for $n \geq s$, then

$$b_{n+2} > \frac{u}{3} + \frac{1}{2}\Big(v - u\Big) + \frac{2u}{3} - \frac{2}{5}\Big(v - u\Big) \ ,$$

from which it follows that $b_{n+k} > u + \frac{1}{10}(v - u)$ for all $k > 0$. But this would entail that $u \geq u + \frac{1}{10}(v - u)$, which is false. Therefore $b_{n+1} < u + \frac{1}{2}(v - u)$ for all $n \geq s$. Hence $v \leq u + \frac{1}{2}(v - u) < u + \frac{2}{3}(v - u)$, or $v - u < \frac{2}{3}(v - u)$, which is false. Therefore $v = u$ and the result follows.

5. Solve the system

$$x + xy + xyz = 12$$

$$y + yz + yzx = 21$$

$$z + zx + zxy = 30 \ .$$

Solution 1. Let $u = xyz$. Then $x + xy = 12 - u$ so that $z + z(12 - u) = 30$ and $z = 30/(13 - u)$. Similarly, $y = 21/(31 - u)$ and $x = 12/(22 - u)$. Plugging these expressions into any one of the three equations yields that

$$0 = u^3 - 65u^2 + 1306u - 7560 = (u - 10)(u - 27)(u - 28) \ .$$

We get the three solutions

$$(x, y, z) = (1, 1, 10), \Big(-\frac{12}{5}, \frac{21}{4}, -\frac{15}{7}\Big), (-2, 7, -2) \ ,$$

al of which satisfy the system.

Solution 2. Define u and obtain the expressions for x, y, z as in Solution 1. Substitute these into the equation $xyz = u$. This leads to the quartic equation

$$\begin{aligned} 0 &= u(u - 13)(u - 22)(u - 31) + (12)(21)(30) = u^4 - 66u^3 + 1371u^2 - 8866u + 7560 \\ &= (u - 1)(u^3 - 65u^2 + 1306u - 7560) \ . \end{aligned}$$

Apart from the three values of u already identified, we have $u = 1$. This leads to

$$(x, y, z) = \left(\frac{4}{7}, \frac{7}{10}, \frac{5}{2}\right) .$$

While, indeed, $xyz = 1$, we find that $x + xy + xyz = 69/35$, $y + yz + xyz = 69/20$, $z + zx + xyz = 69/14$, so the solution $u = 1$ is extraneous.

[This problem was posed by Stanley Rabinowitz in the Spring, 1982 issue of *AMATYC Review*.]

6. Two competitors play badminton. They play two games, each winning one of them. They then play a third game to determine the overall winner of the match. The winner of a game of badminton is the first player to score at least 21 points with a lead of at least 2 points over the other player.

   In this particular match, it is observed that the scores of each player listed in order of the games form an arithmetic progressions with a nonzero common difference. What are the scores of the two players in the third game?

Solution 1. [K. Ng] Let the scores for the three games be $a, a+u, a+2u$ for player A and $b, b+v, b+2v$ for player B. Suppose, wolog, that A wins the first game and that B wins the second. Then $a - b \geq 2$ and $(b+v) - (a+u) \geq 2$, whence $v - u \geq 2 + (a - b) \geq 4$. Hence

$$(b + 2v) - (a - 2v) = 2(v - u) - (a - b) \geq 6 .$$

Thus the third game is won by B by at least 6 points. Therefore, B must have exactly 21 points in the third game. It follows that B has more than 21 points in each of the previous games, as A in the first game.

Hence $a - b = 2$ and

$$2 = (b + v) - (a + u) = (v - u) - (a - b) = (v - u) - 2 ,$$

so that $v - u = 4$. Therefore, the difference in the scores of the third game is exactly $2(v - u) - (a - b) = 6$ and the final scores are $(15, 21)$.

Solution 2. There are two essentially different ways the score can present at the conclusion of a game: (a) The winner can have 21 points and the other no more than 19 points; (b) The winner can have at least 22 points and the loser two points fewer than the winner.

It is not possible for the scores in the first and third games to be of type (b). For the signed differences of the scores is also an arithmetic progression of three numbers, the first and third of which are both 2, both -2 or 2 and -2. In the first two cases, this would mean that the same person won all three games, and in the third case that the scores in the second game were the same and there would be no winner of the second game.

Therefore either the opening or closing game has one person with a score of 21 and the other with a score s less than 20. Let the scores of the first player be $(21, 21 + x, 21 + 2x)$ and of the second be $(s, s + y, s + 2y)$. We must have $y > x$, $s + y + 2 = 21 + x$ and $21 + 2x + 2 = s + 2y$. Therefor $s = 19 - (y - x) = 23 - 2(y - x)$, whence $s = 15$. Since the winner of the game with scores $(21, 15)$ is the same as that of the second game and different from that of the third, this game must be the final game of the match. Therefore, the scores of the final game are $(21, 15)$.

Note that $y - x = 4$, and that, whenever $x \geq 1$, sets of scores $(21, 15), (21 + x, 19 + x), (21 + 2x, 23 + 2x)$ all satisfy the conditions.

[This problem appeared in 2010 in the weekly journal *New Scientist*.]

7. Suppose that there are 2011 students in a school and that each students has a certain number of friends among his schoolmates. It is assumed that if A is a friend of B, then B is a friend of A, and also

that there may exist certain pairs that are not friends. Prove that there is a nonvoid subset S of these students for which every student in the school has an even number of friends in S.

Solution. Define a 2011×2011 matrix with entries a_{ij} in $\mathbf{Z}_2 \equiv \{0, 1\}$, the field of integers modulo 2, as follows: $a_{ij} = 1$ if and only if i and j are distinct and persons i and j are friends, and $a_{ij} = 0$ otherwise.

The determinant of this matrix is equal to

$$\sum a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{2011,\sigma_{2011}} ,$$

where the sum has $n!$ terms and is taken over all permutations σ of the symmetric group S_{2011}. Since 2011 is odd, any permutation of period 2, consisting of a product of independent transpositions, must leave at least one element fixed so that its contribution to the determinant sum is 0. The remaining permutation come in inverse pairs that yield the same product value in the sum. It follows that the determinant of the matrix has the value 0.

Therefore there is a linearly dependent subset of rows, and therefore a set of rows whose modulo 2 sum is 0. The portion of each column belonging to each row must contain an even number of 1's. This means that the person corresponding to that column must have evenly many friends among the persons corresponding to the rows.

[This problem was contributed by Ali Feiz Mohammadi.]

8. The set of transpositions of the symmetric group S_n on $\{1, 2, 3, 4, 5\}$ is

$$\{(12), (13), (14), (15), (23), (24), (25), (34), (35), (45)\}$$

where (ab) denotes the permutation that interchanges a and b and leaves every other element fixed. Determine a product of all transpositions, each occuring exactly once, that is equal to the identity permutation ϵ, which leaves every element fixed.

Solution. Multiplication proceeds from left to right. Observe that $(ab)(ac)(bc) = (ac)$ for any a and b. We observe that $(13)(23)$ carries 1 to 2 and that $(24)(14)$ carries 2 to 1, so that $(13)(24)(14)(23) = (12)(34)$. Therefore

$$(12)(13)(24)(14)(23)(34) = \epsilon .$$

Therefore

$$(15)(12)(25)(13)(24)(14)(23)(35)(34)(45) = \epsilon .$$

Comment. Other answers were

$$(25)(35)(34)(25)(14)(45)(15)(13)(23)(12) \quad \text{(K.Ng)},$$
$$(34)(45)(35)(23)(24)(15)(25)(14)(13)(12) \quad \text{(S.Sagatov)},$$
$$(12)(23)(34)(45)(14)(13)(15)(25)(24)(35) \quad \text{(S.E.Rumsey)}.$$

9. Suppose that A and B are two square matrices of the same order for which the indicated inverses exist. Prove that

$$(A + AB^{-1}A)^{-1} + (A + B)^{-1} = A^{-1} .$$

Solution 1. [V. Baydina]

$$\begin{aligned}(A + AB^{-1}A)^{-1} + (A + B)^{-1} &= [A(I + B^{-1}A)]^{-1} + (A + B)^{-1} \\ &= (B^{-1}B + B^{-1}A)^{-1}A^{-1} + (B + A)^{-1} \\ &= [B^{-1}(B + A)]^{-1}A^{-1} + (B + A)^{-1} \\ &= (B + A)^{-1}(BA^{-1} + I) \\ &= (B + A)^{-1}(BA^{-1} + AA^{-1}) \\ &= (B + A)^{-1}(B + A)A^{-1} = A^{-1} .\end{aligned}$$

Solution 2. [K. Ng]

$$[A+AB^{-1}A][A^{-1}-(A+B)^{-1}] = I - A(A+B)^{-1} + AB^{-1} - AB^{-1}A(A+B)^{-1}$$
$$I + AB^{-1} - A(I+B^{-1}A)(A+B)^{-1}$$
$$= I + AB^{-1} - A(I+B^{-1}A)[B(I+B^{-1}A)]^{-1}$$
$$= I + AB^{-1} - A(I+B^{-1}A)(I+B^{-1}A)^{-1}B^{-1}$$
$$= I + AB^{-1} - AB^{-1} = I \ .$$

Hence $A^{-1}-(A+B)^{-1} = (A+AB^{-1}A)^{-1}$ as desired.

Solution 3. Fix the matrix A and define a new multiplication of the set of square matrices by $X * Y = XA^{-1}Y$. This multplication is associative and distributive with respect to ordinary matrix addition. For any matrix X, we have that $X * A = X = A * X$ so that A is the new identity matrix; the inverse of a nonsingular matrix X is equal to $X^\circ \equiv AX^{-1}A$.

Observe that

$$\begin{aligned}(A+B)*(A+B^\circ) &= (A+B)A^{-1}(A+B^\circ) = (I+BA^{-1})(A+B^\circ) \\ &= (A+B) + (B^\circ + BA^{-1}B^\circ) = (A+B) + (B^\circ + A) \\ &= (A+B) + (A+B^\circ) \ .\end{aligned}$$

$*-$multiplying this equality on the left by $(A+B)^\circ$ and on the right by $(A+B^\circ)^\circ$ yields that

$$A = (A+B^\circ)^\circ + (A+B)^\circ = A(A+AB^{-1}A)^{-1}A + A(A+B)^{-1}A \ .$$

Multiplying this equation normally on the left and on the right by A^{-1} yields the desired reault.

10. Suppose that p is an odd prime. Determine the number of subsets S contained in $\{1, 2, \cdots, 2p-1, 2p\}$ for which (a) S has exactly p elements, and (b) the sum of the elements of S is a multiple of p.

Solution. Let ζ be a primitive pth root of unity, so that ζ is a power of $\cos(2\pi/p) + i\sin(2\pi/p)$ with the exponent not divisible by p. Let

$$f(x) = \prod_{i=1}^{2p}(x-\zeta^i) = (x^p-1)^2 = x^{2p} - 2x^p + 1 \ .$$

We can also write

$$\begin{aligned}f(x) &= x^{2p} + \cdots \left[\sum\{(-1)^p\zeta^{i_1+\cdots+i_p} : \{i_1, i_2, \cdots, i_p\} \subseteq \{1, 2, \cdots, 2p\}\right] x^p + \cdots + 1 \\ &= x^{2p} + \cdots - (a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a^{p-1}\zeta^{p-1})x^p + \cdots + 1 \ ,\end{aligned}$$

for some coefficients a_i. We are required to determine the value of a_0.

Let $g(x) = (a_0-2) + a_1x + a_2x^2 + \cdots + a_{p-1}x^{p-1}$. Observe that $g(x)$ vanishes when x is any primitive pth root of unity and that the sum defining the coefficient of x^p has $\binom{2p}{p}$ terms, so that $a_0 + a_1 + \cdots + a_{p-1} = \binom{2p}{p}$. Since $g(x)$ is a polynomial of degree $p-1$ whose zeros are the primitive pth roots of unity, $g(x)$ must be a constant multiple of $1 + x + \cdots + x^{p-1}$, so that

$$a_0 - 2 = a_1 = a_2 = \cdots = a_{p-1} \ .$$

Thus $a_0 + (p-1)(a_0-2) = \binom{2p}{p}$, so that

$$a_0 = \frac{1}{p}\left[\binom{2p}{p} - 2\right] + 2 \ .$$

[This problem was contributed by Ali Feiz Mohammadi.]