THE UNIVERSITY OF TORONTO

UNDERGRADUATE MATHEMATICS COMPETITION

Sunday, March 18, 2001

Time: 3 hours

`No aids or calculators permitted.`

It is not necessary to complete the paper. Complete solutions to fewer problems will be worth more than incomplete solutions to many problems.

1. Let $a, b, c > 0$, $a < bc$ and $1 + a^3 = b^3 + c^3$. Prove that $1 + a < b + c$.

2. Let $O = (0, 0)$ and $Q = (1, 0)$. Find the point P on the line with equation $y = x + 1$ for which the angle OPQ is a maximum.

3. (a) Consider the infinite integer lattice in the plane (*i.e.*, the set of points with integer coordinates) as a graph, with the edges being the lines of unit length connecting nearby points. What is the minimum number of colours that can be used to colour all the vertices and edges of this graph, so that

   (i) each pair of adjacent vertices gets two distinct colours;

   (ii) each pair of edges that meet at a vertex gets two distinct colours; and

   (iii) an edge is coloured differently than either of the two vertices at the ends?

   (b) Extend this result to lattices in real $n-$dimensional space.

4. Let $\mathbf{V}$ be the vector space of all continuous real-valued functions defined on the open interval $(-\pi/2, \pi/2)$, with the sum of two functions and the product of a function and a real scalar defined in the usual way.

   (a) Prove that the set $\{\sin x, \cos x, \tan x, \sec x\}$ is linearly independent.

   (b) Let $\mathbf{W}$ be the linear space generated by the four trigonometric functions given in (a), and let T be the linear transformation determined on $\mathbf{W}$ into $\mathbf{V}$ by $T(\sin x) = \sin^2 x$, $T(\cos x) = \cos^2 x$, $T(\tan x) = \tan^2 x$ and $T(\sec x) = \sec^2 x$. Determine a basis for the kernel of T.

   Notes. A subset $\{v_1, v_2, \cdots, v_k\}$ of a vector space is *linearly independent* iff $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ for scalars c_i implies that $c_1 = c_2 = \cdots = c_k = 0$. The *kernel* of a linear transformation is the subspace that T maps to the zero vector. A *basis* for a vector space is a linearly independent set of vectors for which every element of the space is some linear combination.

5. Let n be a positive integer and x a real number not equal to a nonnegative integer. Prove that

$$\frac{n}{x} + \frac{n(n-1)}{x(x-1)} + \frac{n(n-1)(n-2)}{x(x-1)(x-2)} + \cdots + \frac{n(n-1)(n-2)\cdots 1}{x(x-1)(x-2)\cdots(x-n+1)} = \frac{n}{x-n+1} .$$

   [This was a problem given by Samuel Beatty on a regular problem assignment to first year honours mathematics students in the 1930s.]

6. Prove that, for each positive integer n, the series

$$\sum_{k=1}^{\infty} \frac{k^n}{2^k}$$

   converges to twice an odd integer not less than $(n+1)!$.

7. Suppose that $x \geq 1$ and that $x = \lfloor x \rfloor + \{x\}$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x and the fractional part $\{x\}$ satisfies $0 \leq x < 1$. Define

$$f(x) = \frac{\sqrt{\lfloor x \rfloor} + \sqrt{\{x\}}}{\sqrt{x}} \quad .$$

(a) Determine the supremum, *i.e.*, the least upper bound, of the values of $f(x)$ for $1 \leq x$.

(b) Let $x_0 \geq 1$ be given, and for $n \geq 1$, define $x_n = f(x_{n-1})$. Prove that $\lim_{n\to\infty} x_n$ exists.

8. A regular heptagon (polygon with seven equal sides and seven equal angles) has diagonals of two different lengths. Let a be the length of a side, b be the length of a shorter diagonal and c be the length of a longer diagonal of a regular heptagon (so that $a < b < c$). Prove ONE of the following relationships:

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = 6$$

or

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5 \ .$$

END

Solutions.

1. *Solution 1.* Since $(1+a)(1-a+a^2) = (b+c)(b^2-bc+c^2)$, and since $1-a+a^2$ and b^2-bc+c^2 are positive, we have that

$$1+a < b+c \Leftrightarrow 1-a+a^2 > b^2-bc+c^2 \ .$$

Suppose, if possible, that $1+a \geq b+c$. Then

$$\begin{aligned} b^2-bc+c^2 &\geq 1-a+a^2 \\ &\Rightarrow (b+c)^2 - 3bc \geq (1+a)^2 - 3a > (1+a)^2 - 3bc \\ &\Rightarrow (b+c)^2 > (1+a)^2 \Rightarrow b+c > 1+a \end{aligned}$$

which is a contradiction.

Solution 2. [J. Chui] Let $u = (1+a)-(b+c)$. Then

$$\begin{aligned} (1+a)^3 - (b+c)^3 &= u[(1+a)^2 + (1+a)(b+c) + (b+c)^2] \\ &= u[(1+a)^2 + (1+a)(b+c) + b^2 + 2bc + c^2] \ . \end{aligned}$$

But also

$$\begin{aligned} (1+a)^3 - (b+c)^3 &= (1+a^3) - (b^3+c^3) + 3a(1+a) - 3bc(b+c) \\ &= 0 + 3[a(1+a) - bc(b+c)] < 3bcu \ . \end{aligned}$$

It follows from these that

$$0 > u[(1+a)^2 + (1+a)(b+c) + b^2 - bc + c^2] = u[(1+a)^2 + (1+a)(b+c) + \frac{1}{2}(b-c)^2 + \frac{1}{2}(b^2+c^2)] \ .$$

Since the quantity in square brackets is positive, we must have that $u < 0$, as desired.

Solution 3. [A. Momin, N. Martin] Suppose, if possible, that $(1+a) \geq (b+c)$. Then

$$0 \leq (1+a)^2 - (b+c)^2 = (1+a^2) - (b^2+c^2) - 2(bc-a) < (1+a^2) - (b^2+c^2) \ .$$

Hence $1+a^2 > b^2+c^2$. It follows that

$$(1-a+a^2) - (b^2-bc+b^2) = (1+a^2) - (b^2+c^2) + (bc-a) > 0$$

so that

$$(1-a+a^2) > (b^2-bc+c^2) \ .$$

However

$$(1+a)(1-a+a^2) = 1+a^3 = b^3+c^3 + (b+c)(b^2-bc+c^2) \ ,$$

from which it follows that $1+a < b+c$, yielding a contradition. Hence, the desired result follows.

Solution 4. [P. Gyrya] Let $p(x) = x^3 - 3ax$. Checking the first derivative yields that $p(x)$ is strictly increasing for $x > \sqrt{a}$. Now $1+a \geq 2\sqrt{a} > \sqrt{a}$ and $b+c \geq 2\sqrt{bc} > 2\sqrt{a} > \sqrt{a}$, so both $1+a$ and $b+c$ lie in the part of the domain of $p(x)$ where it strictly increases. Now

$$p(1+a) = (1+a)^3 - 3a(1+a) = 1+a^3 = b^3+c^3 = (b+c)^3 - 3bc(b+c) < (b+c)^3 - 3a(b+c) = p(b+c)$$

from which it follows that $1+a < b+c$.

Solution 5. Consider the function $g(x) = x(1+a^3-x) = x(b^3+c^3-x)$. Then $g(1) = g(a^3) = a^3$ and $g(b^3) = g(c^3) = (bc)^3$. Since $a^3 < (bc)^3$ and the graph of $g(x)$ is a parabola opening down, it follows that b^3 and c^3 lie between 1 and a^3.

Now consider the function $h(x) = x^{1/3} + (b^3 + c^3 - x)^{1/3} = x^{1/3} + (1 + a^3 - x)^{1/3}$ for $0 \leq x \leq 1 + a^3$. Then $h(1) = h(a^3) = 1 + a$ and $h(b^3) = h(c^3) = b + c$. The graph of $h(x)$ resembles an inverted parabola, so since b^3 and c^3 lie between 1 and a^3, it follows that $1 + a < b + c$, as desired.

2. *Solution 1.* For the point P to maximize $\angle OPQ$, it must be a point of tangency of a circle with chord OQ and the line of equation $y = x + 1$. The general circle through O and Q has equation

$$\left(x - \frac{1}{2}\right)^2 + \left(y - k\right)^2 = \frac{1}{4} + k^2$$

or

$$x^2 - x + y^2 - 2ky = 0 \ .$$

Solving this with $y = x + 1$ yields

$$2x^2 + (1 - 2k)x + (1 - 2k) = 0 \ .$$

The roots of this quadratic equation are coincident if and only if $2k = 1$ or $1 - 2k = 8$. When $k = \frac{1}{2}$, we get the point of tangency $(0, 1)$. When $k = -\frac{7}{2}$, we get the point of tangency $(-2, -1)$.

The first solution corresponds to a circle with chord OQ touching the line of equation $y = x + 1$ at the point $P(0, 1)$. This circle has diameter PQ and the angle OPQ is equal to $45°$. This angle certainly exceeds angle OXQ, where X is any point other than P on the line in the upper half plane. The second solution corresponds to a circle touching the line of equation $y = x + 1$ below the $x-$axis at the point $R(-2, -1)$. The angle ORQ exceeds angle OYQ, where Y is any point other than R on the line in the lower half-plane. Now, the line $y = x + 1$ makes an angle of $45°$ with the line $y = -1$, and this angle exceeds $\angle ORQ$. Hence P is the desired point.

Solution 2. Let $(t, t + 1)$ be a typical point on the line $y = x + 1$. The slope of OP is $(t + 1)/t$ and of QP is $(t + 1)/(t - 1)$. Hence

$$\tan \angle OPQ = \frac{t + 1}{2t^2 + t + 1} \ .$$

Let $f(t) = (t + 1)/(2t^2 + t + 1)$, so that $(2t^2 + t + 1)f(t) = t + 1$. Then $(4t + 1)f(t) + (2t^2 + t + 1)f'(t) = 1$, whence

$$f'(t) = -\frac{2t(t + 2)}{(2t^2 + t + 1)^2} \ .$$

Now $f'(t) < 0$ when $t > 0$ and $t < -2$ while $f'(t) > 0$ when $-2 < t < 0$. Thus, $f(t)$ assumes its minimum value of $-1/7$ when $t = -2$ and its maximum value of 1 when $t = 0$. Hence $|f(t)| \leq 1$ with equality if and only if $t = 0$. Hence the desired point P is $(0, 1)$.

Comment. The workings can be made a little easier y considering $\cot \angle OPQ = 2t - 1 + 2(t + 1)^{-1}$. The derivative of this is $2[1 - (t + 1)^2]$, and this is positive if and only if $t > 0$ and $t < -2$.

Solution 3. Let $\theta = \angle OPQ$. From the Law of Cosines, we have that

$$1 = x^2 + (x + 1)^2 + (x - 1)^2 + (x + 1)^2 - 2\sqrt{(2x^2 + 2x + 1)(2x^2 + 2)} \cos \theta \ ,$$

whence

$$\cos \theta = \frac{2x^2 + x + 1}{\sqrt{(2x^2 + 2x + 1)(2x^2 + 2)}} \ .$$

Let

$$\begin{aligned} f(x) = \sec^2 \theta &= \frac{(2x^2 + 2x + 1)(2x^2 + 2)}{(2x^2 + x + 1)^2} \\ &= \frac{4x^4 + 4x^3 + 6x^2 + 4x + 2}{4x^2 + 4x^3 + 5x^2 + 2x + 1} \\ &= 1 + \frac{(x + 1)^2}{(2x^2 + x + 1)^2} = 1 + g(x)^2 \ , \end{aligned}$$

where $g(x) = (x+1)/(2x^2+x+1)$. Since $g'(x) = \frac{-2x(x+2)}{(2x^2+x+1)^2}$, we see that $g(x)$ has its maximum value when $x = 0$ and its minimum when $x = -2$, and vanishes when $x = -1$. Hence $g(x)^2$ and thus $\sec\theta$ assumes relative maxima when $x = 0$ and $x = -2$. A quick check reveals that $x = 0$ gives the overall maximum, and so that angle is maximized when P is located at $(0, 1)$.

3. Since each vertex and the four edges emanating from it must have different colours, at least five colours are needed. Here is a colouring that will work: Let the colours be numbered 0, 1, 2, 3, 4. Colour the point $(x, 0)$ with the colour $x \pmod 5$; colour the point $(0, y)$ with the colour $2y \pmod 5$; colour the points along each horizontal line parallel to the $x-$axis consecutively; colour the vertical edge whose lower vertex has colour $m \pmod 5$ with the colour $m+1 \pmod 5$; colour the horizontal edge whose left vertex has the colour $n \pmod 5$ with the colour $n+3 \pmod 5$.

This can be generalized to an $n-$dimensional lattice where $2n+1$ colours are needed by changing the strategy of colouring. The integer points on the line and the edges between them can be coloured $1 - -(3) - -2 - -(1) - -3 - -(2) - -1$ and so on, where the edge colouring is in parenthesis. Form a plane by stacking these lines unit distance apart, making sure that each vertex has a different coloured vertex above and below it; use colours 4 and 5 judiciously to colour the vertical edges. Now go to three dimensions; stack up planar lattices and struts unit distance apart, colouring each with the colours 1, 2, 3, 4, 5, while making sure that vertically adjacent vertices have separate colours, and use the colours 6 and 7 for vertical struts. Continue on.

4. *Solution 1.* (a) Suppose that $a\sin x + b\cos x + c\tan x + d\sec x = 0$ identically for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then evaluating this equation at $0, \pi/6, \pi/4, \pi/3$ leads to a system of four linear equations in a, b, c, d whose sole solution is given by $a = b = c = d = 0$.

(b) Since $\sin^2 x + \cos^2 x = 1$ and $1 + \tan^2 x = \sec^2 x$, it is clear from the linearity of T that $T(\sin x + \cos x + \tan x - \sec x) = 0$. On the other hand, by evaluating at three of the values of x used in (a), we can see that $\{\sin^2 x, \cos^2 x, \tan^2 x\}$ is linearly independent. Thus the nullity of T is at least 1 and the rank of T is at least 3. Hence the kernel of T has dimension 1 and must be the span of $\{\sin x + \cos x + \tan x - \sec x\}$.

Solution 2. (a)

$$a\sin x + b\cos x + c\tan x + d\sec x = 0 \qquad (\forall x)$$

$$\Leftrightarrow a\sin x\cos x + b\cos^2 x + c\sin x + d = 0 \qquad (\forall x)$$

$$\Leftrightarrow \sin x(a\cos x + c) = -(b\cos^2 x + d) \qquad (\forall x)$$

$$\Rightarrow (1-\cos^2 x)(a^2\cos^2 x + 2ac\cos x + c^2) = b^2\cos^4 x + 2bd\cos^2 x + d^2 \qquad (\forall x)$$

$$\Leftrightarrow (b^2+a^2)\cos^4 x + 2ac\cos^3 x + (c^2 - a^2 + 2bd)\cos^2 x - 2ac\cos x + (d^2 - c^2) = 0 \qquad (\forall x)$$

$$\Leftrightarrow a = b = c = d = 0 \ ,$$

from which the linear independence follows. (Note that the left expression is a polynomial in $\cos x$ which vanishes at infinitely many values.)

(b) $a\sin x + b\cos x + c\tan x + d\sec x$ belongs to the kernel of T

$$\Leftrightarrow a\sin^2 x + b\cos^2 x + c\tan^2 x + d\sec^2 x = 0 \qquad (\forall x)$$

$$\Leftrightarrow a + (b-a)\cos^2 x + (c+d)\sec^2 x - c = 0 \qquad (\forall x)$$

$$\Leftrightarrow (a-c) + (b-a)\cos^2 x + (c+d)\sec^2 x = 0 \qquad (\forall x)$$

$$\Leftrightarrow (b-a)\cos^4 x + (a-c)\cos^2 x + (c+d) = 0 \qquad (\forall x)$$

$$\Leftrightarrow b = a = c = -d \ ,$$

which occurs if and only if $a\sin x + b\cos x + c\tan x + d\sec x$ is a multiple of $\sin x + \cos x + \tan x - \sec x$. Hence the nullity of T is 1 and the rank of T is 3.

5. *Solution 1.* The result holds for $n = 1$ and $x \neq 0$. Suppose, as an induction hypothesis, the result holds for $n = k$ and x equal to any real number that is not a nonnegative integer; note that in this case, the left side has k terms. When $n = k + 1$,

$$\begin{aligned}
&\frac{k+1}{x} + \frac{(k+1)k}{x(x-1)} + \frac{(k+1)k(k-1)}{x(x-1)(x-2)} + \cdots + \frac{(k+1)k(k-1)\cdots 1}{x(x-1)(x-2)\cdots(x-k)} \\
&\quad = \frac{k+1}{x} + \frac{k+1}{x}\left[\frac{k}{x-1} + \frac{k(k-1)}{(x-1)(x-2)} + \cdots + \frac{k(k-1)\cdots 1}{(x-1)\cdots(x-k)}\right] \\
&\quad = \frac{k+1}{x} + \frac{k+1}{x}\left[\frac{k}{x-1-k+1}\right] = \frac{k+1}{x}\left[1 + \frac{k}{x-k}\right] \\
&\quad = \frac{k+1}{x}\left[\frac{x}{x-k}\right] = \frac{k+1}{x-(k+1)+1}
\end{aligned}$$

as desired. The result follows by induction.

Solution 2. [P. Gyrya] Let $\binom{a}{m}$ for real a and positive integer m denote $a(a-1)\cdots(a-m+1)/m!$ and $\binom{a}{0}$ be 1. It is straightforward to establish that, for each positive integer k:

$$\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}$$

and

$$\binom{x}{k} = \sum_{i=0}^{k} \binom{x-i-1}{k-i} .$$

The left side of the required equality is equal to

$$\frac{n!}{x\cdots(x-n+1)}\left[\binom{x-1}{n-1} + \binom{x-2}{n-2} + \cdots + \binom{x-n}{n-n}\right] = \frac{n!}{x\cdots(x-n+1)}\binom{x}{n-1} = \frac{n}{x-n+1}$$

as desired.

6. *Solution 1.* Convergence of the series results from either the ratio or the root test. For nonnegative integers n, let

$$S_n = \sum_{k=1}^{\infty} \frac{k^n}{2^k} .$$

Then $S_0 = 1$ and

$$\begin{aligned}
S_n - \frac{1}{2}S_n &= \sum_{k=1}^{\infty} \frac{k^n}{2^k} - \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}} \\
&= \sum_{k=1}^{\infty} \frac{k^n}{2^k} - \sum_{k=1}^{\infty} \frac{(k-1)^n}{2^k} \\
&= \sum_{k=1}^{\infty} \frac{k^n - (k-1)^n}{2^k} \\
&= \sum_{k=1}^{\infty} \left[\binom{n}{1}\frac{k^{n-1}}{2^k} - \binom{n}{2}\frac{k^{n-2}}{2^k} + \binom{n}{3}\frac{k^{n-3}}{2^k} - \cdots + (-1)^{n-1}\frac{1}{2^k}\right]
\end{aligned}$$

whence

$$S_n = 2\left[\binom{n}{1}S_{n-1} - \binom{n}{2}S_{n-2} + \binom{n}{3}S_{n-3} - \cdots + (-1)^{n-1}\right] .$$

An induction argument establishes that S_n is twice an odd integer.

Observe that $S_0 = 1$, $S_1 = 2$, $S_2 = 6$ and $S_3 = 26$. We prove by induction that, for each $n \geq 0$,

$$S_{n+1} \geq (n+2)S_n$$

from which the desired result will follow. Suppose that we have established this for $n = m - 1$. Now

$$S_{m+1} = 2\left[\binom{m+1}{1}S_m - \binom{m+1}{2}S_{m-1} + \binom{m+1}{3}S_{m-2} - \binom{m+1}{4}S_{m-3} + \cdots\right] .$$

For each positive integer r,

$$\begin{aligned}\binom{m+1}{2r-1}S_{m-2r+2} - \binom{m+1}{2r}S_{m-2r+1}\\ &\geq \left[\binom{m+1}{2r-1}(m-2r+3) - \binom{m+1}{2r}\right]S_{m-2r+1}\\ &= \binom{m+1}{2r-1}\left[(m-2r+3) - \left(\frac{m-2r+2}{2r}\right)\right]S_{m-2r+1} \geq 0 .\end{aligned}$$

When $r = 1$, we get inside the square brackets the quantity

$$(m+1) - \frac{m}{2} = \frac{m+2}{2}$$

while when $r > 1$, we get

$$(m-2r+3) - \left(\frac{m-2r+2}{2r}\right) > (m-2r+3) - (m-2r+2) = 1 .$$

Hence

$$\begin{aligned}S_{m+1} &\geq 2\left[\binom{m+1}{1}S_m - \binom{m+1}{2}S_{m-1}\right]\\ &\geq 2\left[(m+1)S_m - \frac{m(m+1)}{2}\cdot\frac{1}{m+1}S_m\right]\\ &= 2\left[m+1-\frac{m}{2}\right]s_m = (m+2)S_m .\end{aligned}$$

Solution 2. Define S_n as in the foregoing solution. Then, for $n \geq 1$,

$$\begin{aligned}S_n &= \frac{1}{2} + \sum_{k=2}^{\infty}\frac{k^n}{2_k}\\ &= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(k+1)^n}{2^k}\\ &= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{k^n + \binom{n}{1}k^{n-1} + \cdots + \binom{n}{n-1}k + 1}{2^k}\\ &= \frac{1}{2} + \frac{1}{2}\left[S_n + \binom{n}{1}S_{n-1} + \cdots + \binom{n}{n-1}S_1 + 1\right]\end{aligned}$$

whence

$$S_n = \binom{n}{1}S_{n-1} + \binom{n}{2}S_{n-2} + \cdots + \binom{n}{n-1}S_1 + 2 \ .$$

It is easily checked that $S_k \equiv 2 \pmod 4$ for $k = 0, 1$. As an induction hypothesis, suppose this holds for $1 \leq k \leq n-1$. Then, modulo 4, the right side is congruent to

$$2[\sum_{k=0}^{n} \binom{n}{k} - 2] + 2 = 2(2^n - 2) + 2 = 2^{n+1} - 2 \ ,$$

and the desired result follows.

For $n \geq 1$,

$$\begin{aligned}
\frac{S_{n+1}}{S_n} &= \frac{\binom{n+1}{1}S_n + \binom{n+1}{2}S_{n-1} + \cdots + \binom{n+1}{n}S_1 + 2}{S_n} \\
&= (n+1) + \frac{\binom{n+1}{2}S_{n-1} + \binom{n+1}{3}S_{n-2} + \cdots + (n+1)S_1 + 2}{\binom{n}{1}S_{n-1} + \binom{n}{2}S_{n-2} + \cdots + nS_1 + 2} \\
&\geq (n+1) + 1 = n + 2 \ ,
\end{aligned}$$

since each term in the numerator of the latter fraction exceeds each corresponding term in the denominator.

Solution 3. [of the first part using an idea of P. Gyrya] Let $f(x)$ be a differentiable function and let D be the differentiation operator. Define the operator L by

$$L(f)(x) = x \cdot D(f)(x) \ .$$

Suppose that $f(x) = (1-x)^{-1} = \sum_{k=0}^{\infty} x^k$. Then, it is standard that $L^n(f)(x)$ has a power series expansion obtained by term-by-term differentiation that converges absolutely for $|x| < 1$. By induction, it can be shown that the series given in the problem is, for each nonnegative integer n, $L^n(f)(1/2)$.

It is straightforward to verify that

$$L((1-x)^{-1}) = x(1-x)^{-2}$$

$$\begin{aligned}
L^2((1-x)^{-1}) &= x(1+x)(1-x)^{-3} \\
L^3((1-x)^{-1}) &= x(1+4x+x^2)(1-x)^{-4} \\
L^4((1-x)^{-1}) &= x(1+11x+11x^2+x^3)(1-x)^{-5} \ .
\end{aligned}$$

In general, a straightforward induction argument yields that for each positive integer n,

$$L^n(f)(x) = x(1 + a_{n,1}x + \cdots + a_{n,n-2}x^{n-2} + x^{n-1})(1-x)^{-(n+1)}$$

for some integers $a_{n,1}, \cdots, a_{n,n-2}$. Hence

$$L^n(f)(1/2) = 2(2^{n-1} + a_{n,1}2^{n-2} + \cdots + a_{n,n-2}2 + 1) \ ,$$

yielding the desired result.

7. (a) Let $x = y + z$, where $y = \lfloor x \rfloor$ and $z = \{x\}$. Then

$$f(x)^2 = 1 + \frac{2\sqrt{yz}}{y+z} \ ,$$

which is less than 2 because $\sqrt{yz} \leq \frac{1}{2}(y+z)$ by the arithmetic-geometric means inequality. Hence $0 \leq f(x) \leq \sqrt{2}$ for each value of x. Taking $y = 1$, we find that

$$\lim_{x\uparrow 2} f(x)^2 = \lim_{z\uparrow 1} \left(1 + \frac{2\sqrt{z}}{1+z}\right) = 2 \ ,$$

whence $\sup\{f(x) : x \geq 1\} = \sqrt{2}$.

(b) In determining the fate of $\{x_n\}$, note that after the first entry, the sequence lies in the interval $[1, 2)$. So, without loss of generality, we may assume that $1 \leq x_0 < 2$. If $x0n = 1$, then each $x_n = 1$ and the limit is 1. For the rest, note that $f(x)$ simplifies to $(1 + \sqrt{x-1})/\sqrt{x}$ on $(1, 2)$. The key point now is to observe that there is exactly one value v between 1 and 2 for which $f(v) = v$, $f(x) > x$ when $1 < x < v$ and $f(x) < x$ when $v < x < 2$. Assume these facts for a moment. A derivative check reveals that $f(x)$ is strictly increasing on $(1, 2)$, so that for $1 < x < v$, $x < f(x) < f(v) = v$, so that the iterates $\{x_n\}$ constitute a bounded, increasing sequence when $1 < x_0 < v$ which must have a limit. (In fact, this limit must be a fixed point of f and so must be v.) A similar argument shows that, if $v < x_0 < 2$, then the sequence of iterates constitute a decreasing convergent sequence (with limit v).

It remains to show that a unique fixed point v exists. Let $x = 1 + u$ with $u > 0$. Then it can be checked that $f(x) = x$ if and only if $1 + 2\sqrt{u} + u = 1 + 3u + 3u^2 + u^3$ or $u^5 + 6u^4 + 13u^3 + 12u^2 + 4u - 4 = 0$. Since the left side is strictly increasing in u, takes the value -4 when $u = 0$ and the value 32 when $u = 1$, the equation is satified for exactly one value of u in $(0, 1)$; now let $v = 1 + u$. The value of v turns out to be approximately 1.375.

8. *Solution 1.* Let A, B, C, D, E be consecutive vertices of the regular heptagon. Let AB, AC and AD have respective lengths a, b, c, and let $\angle BAC = \theta$. Then $\theta = \pi/7$, the length of BC, of CD and of DE is a, the length of AE is c, $\angle CAD = \angle DAE = \theta$, since the angles are subtended by equal chords of the circumcircle of the heptagon, $\angle ADC = 2\theta$, $\angle ADE = \angle AED = 3\theta$ and $\angle ACD = 4\theta$. Triangles ABC and ACD can be glued together along BC and DC (with C on C) to form a triangle similar to ΔABC, whence

$$\frac{a+c}{b} = \frac{b}{a} \ . \tag{1}$$

Triangles ACD and ADE can be glued together along CD and ED (with D on D) to form a triangle similar to ΔABC, whence

$$\frac{b+c}{c} = \frac{b}{a} \ . \tag{2}$$

Equation (2) can be rewritten as $\frac{1}{b} = \frac{1}{a} - \frac{1}{c}$. whence

$$b = \frac{ac}{c-a} \quad .$$

Substituting this into (1) yields

$$\frac{(c+a)(c-a)}{ac} = \frac{c}{c-a}$$

which simplifies to

$$a^3 - a^2c - 2ac^2 + c^3 = 0 \quad . \tag{3}$$

Note also from (1) that $b^2 = a^2 + ac$.

$$\begin{aligned}
\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} - 6 &= \frac{a^4c^2 + b^4a^2 + c^4b^2 - 6a^2b^2c^2}{a^2b^2c^2} \\
&= \frac{a^4c^2 + (a^4 + 2a^3c + a^2c^2)a^2 + c^4(a^2 + ac) - 6a^2c^2(a^2 + ac)}{a^2b^2c^2} \\
&= \frac{a^6 + 2a^5c - 4a^4c^2 - 6a^3c^3 + a^2c^4 + ac^5}{a^2b^2c^2} \\
&= \frac{a(a^2 + 3ac + c^2)(a^3 - a^2c - 2ac^2 + c^3)}{a^2b^2c^2} = 0 \ `.
\end{aligned}$$

$$\begin{aligned}\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}-5&=\frac{(a^4+2a^3c+a^2c^2)c^2+a^2c^4+a^4(a^2+ac)-5a^2c^2(a^2+ac)}{a^2b^2c^2}\\&=\frac{a^6+a^5c-4a^4c^2-3a^3c^3+2a^2c^4}{a^2b^2c^2}\\&=\frac{a^2(a+2c)(a^3-a^2c-2ac^2+c^3)}{a^2b^2c^2}=0\ .\end{aligned}$$

Solution 2. [of the first result by J. Chui] Let the heptagon be $ABCDEFG$ and $\theta=\pi/7$. Using the Law of Cosines in the indicated triangles ACD and ABC, we obtain the following:

$$\cos 2\theta=\frac{a^2+c^2-b^2}{2ac}=\frac{1}{2}\left(\frac{a}{c}+\frac{c}{a}-\frac{b^2}{ac}\right)$$

$$\cos 5\theta=\frac{2a^2-b^2}{2a^2}=1-\frac{1}{2}\left(\frac{b}{a}\right)^2$$

from which, since $\cos 2\theta=-\cos 5\theta$,

$$-1+\frac{1}{2}\left(\frac{b}{a}\right)^2=\frac{1}{2}\left(\frac{a}{c}+\frac{c}{a}-\frac{b^2}{ac}\right)$$

or

$$\frac{b^2}{a^2}=2+\frac{a}{c}+\frac{c}{a}-\frac{b^2}{ac}\ . \tag{1}$$

Examining triangles ABC and ADE, we find that $\cos\theta=b/2a$ and $\cos\theta=(2c^2-a^2)/(2c^2)=1-(a^2/2c^2)$, so that

$$\frac{a^2}{c^2}=2-\frac{b}{a}\ . \tag{2}$$

Examining triangles ADE and ACF, we find that $\cos 3\theta=a/2c$ and $\cos 3\theta=(2b^2-c^2)/(2b^2)$, so that

$$\frac{c^2}{b^2}=2-\frac{a}{c}\ . \tag{3}$$

Adding equations (1), (2), (3) yields

$$\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=6+\frac{c^2-bc-b^2}{ac}\ .$$

By Ptolemy's Theorem, the sum of the products of pairs of opposite sides of a concylic quadrilaterial is equal to the product of the diagonals. Applying this to the quadrilaterals $ABDE$ and $ABCD$, respectively, yields $c^2=a^2+bc$ and $b^2=ac+a^2$, whence $c^2-bc-b^2=a^2+bc-bc-ac-a^2=-ac$ and we find that

$$\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=6-1=5\ .$$

Solution 3. There is no loss of generality in assuming that the vertices of the heptagon are placed at the seventh roots of unity on the unit circle in the complex plane. Then $\zeta=\cos(2\pi/7)+i\sin(2\pi/7)$ be the fundamental seventh root of unity. Then $\zeta^7=1$, $1+\zeta+\zeta^2+\cdots+\zeta^6=0$ and (ζ,ζ^6), (ζ^2,ζ^5), (ζ^3,ζ^4) are pairs of complex conjugates. We have that

$$a=|\zeta-1|=|\zeta^6-1|$$

$$b=|\zeta^2-1|=|\zeta^9-1|$$

$$c = |\zeta^3 - 1| = |\zeta^4 - 1| \ .$$

It follows from this that

$$\frac{b}{a} = |\zeta + 1| \qquad \frac{c}{b} = |\zeta^2 + 1| \qquad \frac{a}{c} = |\zeta^3 + 1| \ ,$$

whence

$$\begin{aligned} \frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} &= (\zeta + 1)(\zeta^6 + 1) + (\zeta^2 + 1)(\zeta^5 + 1) + (\zeta^3 + 1)(\zeta^4 + 1) \\ &= 2 + \zeta + \zeta^6 + 2 + \zeta^2 + \zeta^5 + 2 + \zeta^3 + \zeta^4 \\ &= 6 + (\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6) = 6 - 1 = 5 \ . \end{aligned}$$

Also

$$\frac{a}{b} = |\zeta^4 + \zeta^2 + 1| \qquad \frac{b}{c} = |\zeta^6 + \zeta^3 + 1| \qquad \frac{c}{a} = |\zeta^2 + \zeta + 1| \ ,$$

whence

$$\begin{aligned} &\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = (\zeta^4 + \zeta^2 + 1)(\zeta^3 + \zeta^5 + 1) + (\zeta^6 + \zeta^3 + 1)(\zeta + \zeta^4 + 1) + (\zeta^2 + \zeta + 1)(\zeta^5 + \zeta^6 + 1) \\ &= (3 + 2\zeta^2 + \zeta^3 + \zeta^4 + 2\zeta^5) + (3 + \zeta + 2\zeta^3 + 2\zeta^4 + \zeta^6) + (3 + 2\zeta + \zeta^2 + \zeta^5 + 2\zeta^6) \\ &= 9 + 3(\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6) = 9 - 3 = 6 \ . \end{aligned}$$

Solution 4. Suppose that the circumradius of the heptagon is 1. By considering isosceles triangles with base equal to the sides or diagonals of the heptagon and apex at the centre of the circumcircle, we see that

$$\begin{aligned} a &= 2\sin\theta = 2\sin 6\theta = -2\sin 8\theta \\ b &= 2\sin 2\theta = -2\sin 9\theta \\ c &= 2\sin 3\theta = 2\sin 4\theta \end{aligned}$$

where $\theta = \pi/7$ is half the angle subtended at the circumcentre by each side of the heptagon. Observe that

$$\cos 2\theta = \frac{1}{2}(\zeta + \zeta^6) \qquad \cos 4\theta = \frac{1}{2}(\zeta^2 + \zeta^5) \qquad \cos 6\theta = \frac{1}{2}(\zeta^3 + \zeta^4)$$

where ζ is the fundamental primitive root of unity. We have that

$$\frac{b}{a} = 2\cos\theta = 2\cos 6\theta \qquad \frac{c}{b} = 2\cos 2\theta \qquad \frac{a}{c} = -2\cos 4\theta$$

whence

$$\begin{aligned} \frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} &= 4\cos^2 6\theta + 4\cos^2 2\theta + 4\cos^2 4\theta \\ &= (\zeta^3 + \zeta^4)^2 + (\zeta + \zeta^6)^2 + (\zeta^2 + \zeta^5)^2 \\ &= \zeta^6 + 2 + \zeta + \zeta^2 + 2 + \zeta^5 + \zeta^4 + 2 + \zeta^3 = 6 - 1 = 5 \ . \end{aligned}$$

Also

$$\begin{aligned} \frac{a}{b} &= \frac{\sin 6\theta}{\sin 2\theta} = 4\cos^2 2\theta - 1 = (\zeta + \zeta^6)^2 - 1 = 1 + \zeta^2 + \zeta^5 \\ -\frac{b}{c} &= \frac{\sin 9\theta}{\sin 3\theta} = 4\cos^2 3\theta - 1 = 4\cos^2 4\theta - 1 = (\zeta^2 + \zeta^5)^2 - 1 = 1 + \zeta^4 + \zeta^3 \\ \frac{c}{a} &= \frac{\sin 3\theta}{\sin\theta} = 4\cos^2 6\theta - 1 = (\zeta^3 + \zeta^4)^2 - 1 = 1 + \zeta^6 + \zeta \ , \end{aligned}$$

whence

$$\begin{aligned} \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} &= (3 + 2\zeta^2 + \zeta^3 + \zeta^4 + 2\zeta^5) + (3 + \zeta + 2\zeta^3 + 2\zeta^4 + \zeta^6) + (3 + 2\zeta + \zeta^2 + \zeta^5 + 2\zeta^6) \\ &= 9 + 3(\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6) = 9 - 3 = 6 \ . \end{aligned}$$