THE QUADRATIC

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§1. Exercises on Basic Properties

1.1. Let $f(x)$ be a quadratic polynomial and suppose that we divide it by $x-u$ and obtain a remainder r, where u and r are constants: $f(x)=(x-u)g(x)+r$.

(a) Explain why the degree of $g(x)$ is 1.

(b) Prove that $r=f(u)$.

(c) Prove that u is a root of the equation $f(x)=0$ if and only if there is a linear polynomial $g(x)$ for which $f(x)=(x-u)g(x)$.

1.2. (a) How many different quadratic polynomials $f(x)$ can you find for which $f(0)=5$, $f(1)=3$ and $f(2)=-7$? Determine all of them.

(b) Determine all of the polynomials $g(x)$ of degree not exceeding 2 for which $g(-3)=2$, $g(-1)=-1$ and $g(4)=0$.

1.3. (a) Suppose that $f(x)$ and $g(x)$ are two polynomials of degree not exceeding 2 for which $f(u)=g(u)$, $f(v)=g(v)$ and $f(w)=g(w)$ for three distinct numbers u, v and w. Prove that $f(x)$ and $g(x)$ must be the same polynomial.

(b) Suppose that $h(x)$ is a quadratic polynomial that vanishes at the two distinct numbers u and v, *i.e.*, $h(u)=h(v)=0$. Prove that $h(x)$ must be a constant multiple of $(x-u)(x-v)$.

(c) Let a, b, c be three distinct numbers. Determine a quadratic polynomial $h(x)$ for which $h(a)=h(b)=0$ and $h(c)=1$.

1.4. (a) Suppose that a, b and c are three distinct numbers and that $f(x)$, $g(x)$ and $h(x)$ are quadratic polynomials for which

$$h(a)=h(b)=g(a)=g(c)=f(b)=f(c)=0$$

and

$$f(a)=g(b)=h(c)=1 \quad .$$

Let $p(x)=uf(x)+vg(x)+wh(x)$ for some constants u, v and w. Determine the values of $p(a)$, $p(b)$ and $p(c)$.

(b) Suppose that $p(x)$ is a polynomial of degree less than three for which $p(a)$, $p(b)$ and $p(c)$ are specified. Prove that, for every x,

$$p(x)=p(a)\frac{(x-b)(x-c)}{(a-b)(a-c)}+p(b)\frac{(x-a)(x-c)}{(b-a)(b-c)}+p(c)\frac{(x-a)(x-b}{(c-a)(c-b)} \quad .$$

(c) Use the format of (b) to determine the polynomials $f(x)$ and $g(x)$ asked for in Exercise 1.2. Check that you get the same answer as you did before.

(d) Use (b) to give a necessary and sufficient condition involving an arbitrary set $\{a,b,c\}$ of numbers that a polynomial p has degree strictly less than 2. [Hint: Look at the coefficient of x^2 in (b).]

1.5. (a) For each of the quadratic polynomials

$$x^2, \quad 1 - x^2, \quad \frac{1}{2}x(x+1), \quad x^2 + 3x + 3$$

construct a table listing in order nonnegative integral values of x, the corresponding values of the polynomial and the difference between the values of the polynomials at consecutive integers. What do you notice about the sequence of differences? If you take differences of consecutive differences, what happens?

(b) Let $p(x) = ax^2 + bx + c$ be a general quadratic polynomial. Verify that, if $q(x) = p(x+1) - p(x)$ and $r(x) = q(x+1) - q(x)$, then $q(x)$ is a linear polynomial and $r(x)$ is a constant polynomial.

(c) It is given that $f(x)$ is a quadratic polynomial. Fill in the missing entries in the following table:

x	$f(x)$	$g(x) = f(x+1) - f(x)$	$h(x) = g(x+1) - g(x)$
0	5		
		4	
1	?		-1
		?	
2	?		?
		?	
3	?		?
		?	
4	?		?

What do you think $f(x)$ is?

1.6. Let $p(x)$, $q(x)$, $r(x)$ be as given in Exercise 1.5.(b). Suppose that $p(0)$, $q(0)$ and $r(0)$ are given.

(a) Prove that, for each positive integer n,

$$p(n) = p(0) + q(0) + q(1) + \cdots + q(n-1)$$

$$q(n) = q(0) + r(0) + r(1) + \cdots + r(n-1) = q(0) + (n-1)r(0) \ .$$

(b) Determine a formula for $p(n)$ in terms of $p(0)$.

1.7. (a) Show that for every quadratic equation $(x-p)(x-q) = 0$, there exist constants a, b, c with $c \neq 0$ such that $(x-a)(b-x) = c$ is equivalent to given equation and the faulty reasoning "either $x - a$ or $b - x$ must equal c" yields the correct answer "$x = p$ or $x = q$".

(b) Determine constants a, b, c with $c \neq 0$ so that the equation $(x-19)(x-97) = 0$ can be "solved" in such a manner.

(*International Mathematical Talent Search, Round 25.*)

1.8. (a) Write down some values of the polynomial $x^2 + x + 1$ for $x = 0, 1, 2, 3, \cdots$. Observe that the product of two consecutive values in the list occur elsewhere in the list. Formulate and prove a general result.

(b) Answer (a) for the polynomial $x^2 + x = x(x+1)$.

(c) Any integer that is the product of two consecutive integers is called *oblong*. Part (b) can be used to show that there are infinitely many triples (a, b, c) of oblong numbers for which $c = ab$. Investigate the existence of triples of oblong numbers no two of which are consecutive but for which the product of two of them is equal to the third.

1.9. Let $p(x)$ be a *monic* quadratic polynomial. (This means that the *leading coefficient* is 1, so that it has the form $p(x) = x^2 + bx + c$.) Suppose also that its coefficients are integers.

(a) Prove that there exists an integer k such that $p(0)p(1) = p(k)$. How many possible such values of k are there?

(b) More generally, prove that for each integer n, there is at least one integer m for which $p(n)p(n+1) = p(m)$.

(c) Are there any values of n for which the value of m determined in (b) is unique?

1.10. Does there exist a quadratic polynomial $f(x)$ with integer coefficients and the unusual property that, whenever x is a positive integer which consists only of 1's, then $f(x)$ is also a positive integer consisting only of 1's (where the representation is to base 10)?

§2. Exercises on Completing the Square and Transformations

2.1. Let $f(x)$ be a real-valued function.

(a) Compare the graphs of the equations $y = f(x)$ and $y = f(2x)$.

(b) Compare the graphs of the equation $y = f(x)$ and $y = 4f(x)$.

(c) Corroborate your answers to parts (a) and (b) with the example $f(x) = x^2$.

2.2. Let $ax^2 + bx + c$ be a quadratic polynomial with real coefficients.

(a) Verify that it equals

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{1}{4a}(b^2 - 4ac)$$

.

(b) From (a), argue that, when $a > 0$, the quadratic assumes its minimum value when $x = -b/2a$, while if $a < 0$, it assumes its maximum value when $x = -b/2a$.

(c) Use this to describe a transformation in the plane that takes the graph of the equation

$$y = ax^2 + bx + c$$

to the graph of the equation $y = x^2$.

2.3. Is it true that all parabolas are the same shape? Explain.

2.4. One geometric definition of a *parabola* is that it is the locus of points whose distance from a fixed point (the focus) is equal to its distance from a fixed line (the directrix). It is asserted that the graph of the equation $y = ax^2 + bx + c$ is a parabola. Is this consistent with the geometric definition? If so, what is the focus? What is the directrix? Look first at some special cases, such as $y = x^2$, $y = x^2 + c$, $y = ax^2$, $y = x^2 - 3x + 2$.

§3. Exercises on Solutions of Quadratics

3.1. Let m and n be the solutions of the quadratic equation $x^2 + bx + c = 0$. Show that b and c are the solutions of the quadratic equation $x^2 + (m + n - mn)x - mn(m + n) = 0$.

3.2. Suppose that $a \neq c$ and that $x = (b - d)/(a - c)$ satisfies one of the equations $x^2 - ax + b = 0$ and $x^2 - cx + d = 0$. Prove that this value of x satisfies the other.

3.3. Let $p(x)$ and $q(x)$ be two quadratic polynomials with integer coeffients. Suppose that there is an *irrational* number c for which $p(c) = q(c) = 0$. Prove that one of the polynomials $p(x)$ and $q(x)$ is a constant multiple of the other.

3.4. Let $p(x) = x^2 + bx + c$. Suppose that $p(0)$ and $p(1)$ are solutions of the quadratic equation $p(x) = 0$. What are the possible values of the pair (b, c)?

3.5. (a) Show that, for every quadratic equation $(x - p)(x - q) = 0$, there exist constants a, b, c with $c \neq 0$ such that $(x - b)(b - x) = c$ is equivalent to the original equation and the following reasoning "either $x - a$ or $b - x$ must equal to c" yields the correct answers "$x = p$ or $x = q$".

(b) Determine constants a, b, c with $c \neq 0$ so that the equation $(x - 19)(97 - x) = 0$ can be "solved" in this manner.

[Round 25 of the International Mathematical Talent Search.]

3.6. Suppose x and y are integers. Solve the equation

$$x^2y^2 - 7x^2y + 12x^2 - 21xy - 4y^2 + 63x + 70y - 174 = 0 \ .$$

[Problem 2332 from *Crux Mathematicorum*.]

3.7. Two nested concentric rectangles, are given, with corresponding sides parallel and each side of the inner rectangle the same distance from the corresponding side of the outer.

(a) Prove that, if the area of the inner rectangle is exactly half that of the outer rectangle, then the perimeter of the inner rectangle is equal to the sum of the lengths of the diagonals of the outer rectangle.

(b) Verify the result in (a) when the outer rectangle has dimensions 3×4, 8×15 and more generally $(m^2 - n^2) \times 2mn$ where m and n are positive integers.

§4. Exercises on Inequalities

4.1. Let a and b be positive real numbers. Using the fact that the quadratic equation $0 = (x - a)(x - b) = x^2 - (a + b)x + ab$ has real roots and the discriminant condition, verify the Arithmetic-Geometric Means Inequality

$$\sqrt{ab} \leq \frac{a+b}{2} \ .$$

When does equality occur?

4.2. (a) Suppose that a, b, c, u, v, w are real numbers. Using the fact that the quadratic polynomial

$$(ax + u)^2 + (bx + v)^2 + (cx + w)^2 = (a^2 + b^2 + c^2)x^2 + 2(au + bv + cw)x + (u^2 + v^2 + w^2)$$

is always nonnegative, argue that it has either coincident real roots or nonreal roots. Use the discriminant condition for this to obtain the Cauchy-Schwarz Inequality

$$au + bv + cw \leq (a^2 + b^2 + c^2)^{\frac{1}{2}}(u^2 + v^2 + w^2)^{\frac{1}{2}} \ .$$

When does equality hold?

(b) Generalize (a) to obtain an inequality for $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$.

§5. Exercises on Sum and Product of Roots

5.1. The roots of the quadratic equation $x^2 + bx + c = 0$ are m and n. Verify that b and c satisfy the quadratic equation $x^2 + (m + n - mn)x - mn(m + n) = 0$.

5.2. (a) Determine any solution in positive integers to the diophantine equation

$$x^2 + y^2 + z^2 + w^2 = xyzw \ .$$

(b) It is possible to show that the diophantine equation in (a) has infinitely many solutions in positive integers by the following argument. Suppose that we have found a solution $(x, y, z, w) = (a, b, c, d)$. Consider the quadratic equation

$$x^2 - bcdx + (b^2 + c^2 + d^2) = 0 \quad .$$

One root of this equation is the integer a. Argue that there is a second root a' which is also an integer. Show that $(x, y, z, w) = (a', b, c, d)$ is another solution of the equation in (a). Use this strategy, starting with the solution you found in (a), to obtain a sequence of different solutions to the equation.

§6. Exercises on Polynomials of Higher Degree

6.1. Consider the equation $x^4 - 2x^3 - x^2 - 2x + 1 = 0$. Since the coefficients $(1, -1, -2, -1, 1)$ are symmetric about the middle one, it turns out that there is a special method for solving such an equation which reduces to the solution of quadratic equations.

(a) Prove that, if the equation has a nonzero solution $x = u$, then $x = 1/u$ also satisfies the equation.

(b) Verify that $x = 0$ does not satisfy the equation. Deduce that the equation is equivalent to

$$\left(x^2 + \frac{1}{x^2}\right) - 2\left(x + \frac{1}{x}\right) - 1 = 0 \ .$$

(c) Set $t = x + \frac{1}{x}$ and verify that $x^2 + \frac{1}{x^2} = t^2 - 2$. Verify that the equation, with this substitution, becomes $t^2 - 2t - 3 = 0$. Solve for t. $[t = -1, 3.]$

(d) Solve the equations $x + \frac{1}{x} = -1$ and $x + \frac{1}{x} = 3$, and argue that the solutions to these two equations satisfy the original equation.

(e) Use the result in (c) to obtain a factorization of $x^4 - 2x^3 - x^2 - 2x + 1$ as a product of two quadratic polynomials.

6.2. (a) Write down several examples of products of four consecutive integers, such as $3 \times 4 \times 5 \times 6 = 360$.

(b) Observe that in each case the result is not a square. Why do you think this is?

Extending the observation in (b), it appears on the basis of numerical evidence that the product of four consecutive integers is 1 less than a perfect square. This suggests that we might introduce variables to check the truth of this in general. What is the general form for the product of consecutive integers?

(d) Consider $f(x) = x(x + 1)(x + 2)(x + 3)$. Rewriting the terms (think why one might want to do this) thus, $f(x) = [x(x + 3)][(x + 1)(x + 2)]$, verify that $(x + 1)(x + 2) = x(x + 3) + 2$ and so

$$f(x) = [x(x + 3)]^2 + 2[x(x + 3)]$$

and use this to show that $f(x) + 1$ is the square of a quadratic polynomial. What is this quadratic polynomial?

(e) Some might prefer to represent the product of four consecutive integers are $g(x) = (x - 1)x(x + 1)(x + 2)$. Is this equally valid? Why might one choose this form? Prove that $g(x) + 1$ is equal to the square of a quadratic polynomial.

(f) By considering the equation

$$1 = a^2 - b^2 = (a - b)(a + b) \quad ,$$

give a rigorous argument that two positive squares cannot differ by 1. Deduce that the functions $f(x)$ and $g(x)$ can never be a positive square when x is an integer.

6.3. In his paper, *Recherches sur les racines imaginaires des équations*, published in *Mem. de l'academie des sciences de Berlin* (5) (1749), 1751, 222-288 = *Opera omnia* (1) 6, 78-141, Leonard Euler (1707-1783) presents what turns out to be a subtly incorrect proof of a version of the Fundamental Theorem of Algebra, that each polynomial with real coefficients can be written as a product of linear and quadratic polynomials with real coefficients. However, his argument works in the case of a quartic polynomial $h(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$ with $A \neq 0$.

(a) Prove that $h(x)$ can be a factored as a product of quadratic polynomials if and only if $h(kx)$ and $h(x - k)$ can be so factored for any nonzero constant k. [Hint: If $h(x) = f(x)g(x)$ is an identity if x, what happens if you replace x by kx and $x - k$?]

(b) Let $k = -B/4A$. Verify that the coefficient of x^3 in $h(x + k)$ is 0.

(c) From (a) and (b), argue that, without loss of generality, it is enough to prove that any polynomial of the form

$$h(x) = x^4 + ax^2 + bx + c$$

can be factored as a product of real quadratics.

Henceforth, we will suppose that $h(x)$ has this form.

(d) Suppose that $b = 0$ so that $h(x) = x^4 + ax^2 + c$. Let $a^2 \geq 4c$. Use the theory of the quadratic to prove that $h(x)$ can be written as a product of the form $(x^2 - r)(x^2 - s)$ for real values of r and s.

(e) Suppose that $b = 0$ and that $a^2 < 4c$. Verify that $c > 0$ and that

$$x^4 + ax^2 + c = (x^2 + \sqrt{c})^2 - (2\sqrt{c} - a)x^2$$

so that $h(x)$ can be factored as a difference of squares.

(f) We now turn to the case

$$h(x) = x^4 + ax^2 + bx + c \quad ,$$

where $b \neq 0$. The polynomial $h(x)$ can be factored as a product of quadratics if and only if real numbers u, v, w can be found for which

$$x^4 + ax^2 + bx + c = (x^2 + ux + v)(x^2 - ux + w) \quad .$$

By expanding the right side and comparing coefficients on the two sides of the equation, obtain the set of conditions

$$\begin{aligned} a &= v + w - u^2 \\ b &= u(w - v) \\ c &= vw \ , \end{aligned}$$

which, in turn, are equivalent to

$$\begin{aligned} w + v &= a + u^2 \\ w - v &= \frac{b}{u} \\ 4vw &= 4c \ . \end{aligned}$$

Thus, if we can find a suitable real value of u, then the real values of v and w can be obtained from the first two of these equations and we can write out the desired factorization. Verify that

$$\begin{aligned} 2w &= u^2 + a + \frac{b}{u} \\ 2v &= u^2 + a - \frac{b}{u} \end{aligned}$$

and thus show that u must satisfy

$$u^6 + 2au^4 + (a^2 - 4c)u^2 - b^2 = 0 \quad .$$

(g) In (f), it suffices to show that the sextic equation is satisfied by some real value of u. One way to do this is through a result called the *Intermediate Value Theorem* for continuous functions, which applies in particular to polynomials. Let

$$f(x) = x^6 + 2ax^4 + (a^2 - 4c)x^2 - b^2 = x^6(1 + 2ax^{-2} + (a^2 - 4c)x^{-4} - b^2x^{-6}) \ .$$

Verify that $f(0) < 0$ and that $f(x)$ is positive for some very large values of x. The graph of $f(x)$ is a continuous curve which lies below the $x-$axis when $x = 0$ but lies above the axis when x is large. Deduce that it must cross the axis somewhere, and so there must be a real number u such that $f(u) = 0$.

(h) Here is an erroneous argument to show that $f(0) < 0$. (It was this approach that got Euler into trouble with polynomials of higher degree.) Can you spot the difficulty? As above, try

$$x^4 + ax^2 + bx + c = (x^2 + ux + v)(x^2 - ux + w) \ .$$

Each of the quadratics can be factored as a product of linear polynomials, so that

$$x^4 + ax^2 + bx + c = (x - \alpha)(x - \beta)(x - \gamma)(x - \delta) \ .$$

By comparing coefficients, verify that $\alpha + \beta + \gamma + \delta = 0$, and that u is the sum of two of the roots. There are six ways of pairing the roots and the correspond to six possible values of u:

$$\alpha + \beta \ , \quad \alpha + \gamma \ , \quad \alpha + \delta \ ,$$

$$\beta + \gamma \ , \quad \beta + \delta \ , \quad \gamma + \delta \ .$$

Observe that, for any possible value of u, its negative is also a possible value of u, so that the sextic equation $f(x) = 0$ satisfied by u has the form

$$f(u) \equiv (u^2 - \lambda^2)(u^2 - \mu^2)(u^2 - \nu^2) = 0 \ .$$

The left side has an odd number ($\frac{1}{2}\binom{4}{2} = 3$) of terms, and so its constant coefficient, $f(0)$, being the product of three squares, must be negative.

§7. Rational functions.

7.1. Let $f(x) = (x^2 + 2x + 2)/(x + 1)$ be defined for all real values of x not equal to -1.
(a) By considering the solvability of the quadratic equation $x^2 + 2x + 2 = k(x + 1)$, prove that $f(x)$ cannot assume any value strictly between -2 and 2 but that it can assume all other real values.

(b) By considering the signs of the expressions $f(x) - 2$ and $f(x) + 2$, corroborate the result of (a).

(c) Use a calculator to obtain the graph of $y = f(x)$. Does this validate (a) and (b)?

(d) Verify that $f(x) = x + 1 + \frac{1}{x+1}$. Use this representation to obtain a rough sketch of the graph of $y = f(x)$. Does this agree with (c)? Describe the asymptotes of the graph.

7.2. Use the techniques of Exercise 7.1 to analyze the range of values and the graphs of the following rational functions:

(a)

$$\frac{x^2 + x + 4}{x + 1}$$

(b)

$$\frac{x^2 + 4x - 4}{x + 2} \ .$$

7.3. Let a, b, c be parameters and let

$$f(x) = \frac{x^2 + bx + c}{x + a} \ .$$

We are concerned with conditions on a, b, c under which each real number can be written in the form $f(x)$ for some real x, *i.e.*, for each real k, $f(x) = k$ is solvable. Three approaches will be followed in this and the next two problems.

(a) Verify that $f(x) = k$ is equivalent to

$$x^2 + (b - k)x + (c - ak) = 0 \ .$$

(b) Verify that the discriminant of the quadratic equation in (a) is

$$k^2 - 2(b - 2a)k + (b^2 - 4c) = [k - (b - 2a)]^2 - 4(a^2 - ab + c) \ .$$

(c) Using (b), prove that $f(x) = k$ is solvable for *each* real value of k if and only if $a^2 - ab + c < 0$.

(d) Prove that, if $f(x) = k$ is solvable for each real value of k, then $x^2 + bx + c = 0$ must have real roots. Is the converse of this result true?

7.4. Let $f(x)$ be the function of Exercise 7.3.

(a) Verify that

$$f(x) = \frac{x^2 + bx + c}{x + a} = x + (b - a) + \frac{c + a^2 - ab}{x + a}$$
$$= (x + a) + \frac{c + a^2 - ab}{x + a} + (b - 2a) \ .$$

(b) Suppose that $c + a^2 - ab > 0$ and that $x > -a$. Verify that

$$f(x) \geq 2\sqrt{c + a^2 - ab} + (b - 2a)$$

with equality if and only if $x + a = \sqrt{c + a^2 - ab}$.

(c) Suppose that $c + a^2 - ab > 0$ and that $x < -a$. Verify that

$$f(x) \leq -2\sqrt{c + a^2 - ab} + (b - 2a)$$

with equality if and only if $x + a = -\sqrt{c + a^2 - ab}$.

(d) Deduce from (f) and (g) that $f(x) = k$ is not solvable when $c + a^2 - ab > 0$ and

$$b - 2a - 2\sqrt{c + a^2 - ab} < k < b - 2a + 2\sqrt{c + a^2 - ab} \ .$$

(e) Suppose that $c + a^2 - ab < 0$. Argue that, as x increases from $-a$, then $f(x)$ passes through all real values. Similarly argue that as x decreases from $-a$, then $f(x)$ passes through all real values. Observe that this result, along with (h), corroborates the result of Exercise 7.3.(c).

7.5. Let $f(x)$ be the function of Exercise 7.3.

(a) Suppose the roots r, s of $x^2 + bx + c = 0$ are both real with $r \leq s$. Observe that $x^2 + bx + c < 0$ if and only if $r < x < s$.

(b) Suppose that r and s lie on the same side of $-a$. Without loss of generality, let $-a < r \leq s$. Argue that, when $x > -a$, $f(x)$ must assume a minimum value, say m_2 when $x = x_2$, while if $x < -a$, then $f(x)$ must assume a maximum value, say m_1 when $x = x_1$, where $m_1 < 0$. We will argue that $m_1 < m_2$

so that $f(x)$ cannot assume any value between m_1 and m_2. Consider $f(x) - m_2$; this is a quadratic polynomial that vanishes when $x = x_2$ and is nonnegative when x when $x > -a$. Deduce that

$$f(x) - m_2 = \frac{(x - x_2)^2}{x + a}$$

so that $f(x)$ assumes the value m_2 only when $x = x_2$, and so never when $x < -a$. Conclude that $f(x)$ cannot assume any value between m_1 and m_2.

(c) Suppose that r and s lie on opposite sides of $-a$, so that $r < -a < s$. Prove that $f(x) > 0$ for $r < x < -a$ and that $f(x) < 0$ for $-a < x < s$. Indeed, show that $f(x)$ passes through all real values as x increases from or decreases from $-a$.

(d) Deduce from (k) and (l) that $f(x) = k$ us solvable for all real k if and only if $r < -a < s$. Using the fact that $r = \frac{1}{2}(-b - \sqrt{b^2 - 4c})$ and $s = \frac{1}{2}(-b + \sqrt{b^2 - 4c})$, show that this is equivalent to $a^2 - ab + c < 0$.

(e) Suppose that the equation $x^2 + bx + c = 0$ has nonreal roots. Observe that this has two consequences: $4c > b^2$, and $f(x)$ never assumes the value 0. Deduce that

$$a^2 - ba + c > (a - \frac{b}{2})^2 \geq 0 \quad .$$

(f) Using the results of parts (a) to (e), prove that $f(x) = k$ is solvable for each real value of k if and only if $a^2 - ba + c < 0$.

§8. Second order recursions.

8.1. Suppose that $x = u$ and $x = v$ satisfy the quadratic equation $x^2 = px + q$. Define

$$w_0 = 2, \quad w_1 = u + v, \quad w_2 = u^2 + v^2, \quad w_3 = u^3 + v^3, \quad \cdots \quad , w_n = u^n + v^n, \quad \cdots \quad .$$

(a) Prove that, when $n \geq 2$, then $w_n = pw_{n-1} + qw_{n-2}$.

(b) Check (a) when u and v are the solutions of the equations (i) $x^2 = 3x - 2$ and (ii) $x^2 = 3x + 2$.

(c) Suppose that $x_n = 7u^n - 5v^n$, where u and v are as defined above. Is it true that $x_n = px_{n-1} + qx_{n-2}$ for $n \geq 2$?

8.2. Let x_n be a sequence satisfying a *second order recursion*. This means that two consecutive terms, say x_0 and x_1 can be chosen arbitrarily, and that there are fixed multipliers p and q such that for all values of n, $x_n = px_{n-1} + qx_{n-2}$.

(a) Write out the first few terms of the following sequences satisfying a second order recursion in each of the following cases:

(i) $x_0 = 0$, $x_1 = 1$, $p = q = 1$ (Fibonacci sequence);
(ii) $x_0 = 0$, $x_1 = 1$, $p = 2$, $q = -1$;
(iii) $x_0 = 1$, $x_1 = 1$, $p = 2$, $q = 1$;
(iv) $x_0 = 3$, $x_1 = -2$, $p = 1$, $q = -2$;
(v) $x_0 = 3$, $x_1 = 2$, $p = 1$, $q = -1$.

(b) Verify that a geometric progression $\{a, ar, ar^2, ar^3, \cdots\}$ satisfies the recursion $x_n = px_{n-1} + qx_{n-2}$ if and only if r is a solution of the quadratic equation $x^2 = px + q$.

(c) Suppose that the equation $x^2 = px + q$ has two distinct solutions $x = r$ and $x = s$. Let x_0 and x_1 be any two numbers. Solve the system of equations

$$y + z = x_0$$
$$ry + sz = x_1$$

for y and z. Prove that, if $\{x_n\}$ satisfies the second order recursion $x_n = px_{n-1} + qx_{n-2}$, then $x_n = yr^n + zs^n$ for each value of n.

(d) Use (c) to obtain the general term of the sequences in (a). Are there any situations in which the method does not work? Why?

8.3. In this exercise, we examine the situation of a second order recursion as described in Exercise 14 in which the associated quadratic equation $x^2 = px + q$ has a double solution.

(a) Prove that $x^2 = px + q$ has a double solution if and only if $q = -p^2/4$.

In the notation introduced above, we thus are interested in investigating sequences satisfying a recursion of the type

$$x_n = px_{n-1} - \frac{p^2}{4}x_{n-2} \ . \qquad (*)$$

We cannot proceed as in Exercise 14 to get the general solution of the recursion as we have only a single solution of the related quadratic to manipulate. Suppose that r is this root.

(b) Verify that $r = p/2$.

(c) Define y_n by $x_n = r^n y_n$. By substituting into (*), show that $2(y_n - y_{n-1}) = y_{n-1} - y_{n-2}$ and deduce that $y_n - y_{n-1} = y_{n-1} - y_{n-2}$ for each value of n. What does this tell us about the nature of the sequence $\{y_n\}$?

(d) Verify that $x_n = (\alpha n + \beta)r^n$ satisfies (*).

(e) Imitate the strategy of Exercise 14 to show that every solution to the recursion (*) is in the form given by (c).

(f) Solve the recursion $x_n = 6x_{n-1} - 9x_{n-2}$ $(n \geq 2)$ where $x_0 = -2$ and $x_1 = -3$.

8.4. *DeMoivre's Formula* (a) Verify that the sequence defined by $x_n = \cos n\theta$ satisfies the recursion

$$x_{n+2} = (2\cos\theta)x_{n+1} - x_n$$

for $n \geq 0$ with initial conditions $x_0 = 1$ and $x_1 = \cos\theta$.

(b) Verify that the sequence defined by $x_n = \sin n\theta$ satisfies the recursion

$$x_{n+2} = (2\cos\theta)x_{n+1} - x_n$$

for $n \geq 0$ with initial conditions $x_0 = 0$ and $x_1 = \sin\theta$.

(c) Verify that the solutions of the quadratic equation $t^2 - (2\cos\theta)t + 1 = 0$ are $\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$.

(d) Solve the recursions in (a) and (b) to obtain

$$\cos n\theta = \frac{1}{2}(\cos\theta + i\sin\theta)^n + \frac{1}{2}(\cos\theta - i\sin\theta)^n$$

and

$$\sin n\theta = \frac{1}{2i}(\cos\theta + i\sin\theta)^n - \frac{1}{2i}(\cos\theta - i\sin\theta)^n \ .$$

Deduce from this, De Moivre's Rule:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \ .$$

This idea is from the note

V.J. Matsko, *De Moivre's Rule, Recurion Relations, and Number Theory* Mathematics and Informatics Quarterly 8 (1998), 12-14.

§9. Geometry and Trigonometry

9.1. (a) Sketch the parabola with equation $y^2 = 4x$. Consider the family of parallel chords with equation $y = mx + b$, where m is a fixed parameter and k is allowed to vary. Argue that the midpoint of the chord of equation $y = mx + b$ is given by (X, Y) where $X = \frac{1}{2}(x_1 + x_2)$ and $Y = mX = b$, with x_1 and x_2 the two solutions of the quadratic equation

$$(mx + b)^2 = 4x \qquad \text{or} \qquad m^2x^2 + (2bm - 4)x + b^2 = 0 \ \ .$$

(b) Without solving the quadratic equation in (a), use the relationship between the coefficients and roots to obtain an expression for X. Show that Y does not depend on b. What does this tell you about the locus of (X, Y)?

(c) Redo parts (a) and (b) by setting up an equation in y rather than x and computing $Y = \frac{1}{2}(y_1 + y_2)$ directly.

9.2. A *diameter* of a conic section is the locus of the midpoints of a family of parallel chors.

(a) Sketch the ellipse with equation $(x^2/9) + (y^2/4) = 1$ along with some chords in the family $y = x + k$ where k is a parameter. (This could be done with a calculator or with some geometric computer software. In the latter case, try to trace the midpoints of the chords.)

(b) Follow the strategy used in Exercise 16 to show that the locus of the midpoints of the chords is a straight line.

(c) Generalize to the general conic section of equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ \ .$$

Corroborate your findings by taking particular choices of coefficients and chord slopes and graphing them with a computer or calculator.

9.3. Suppose that, in a triangle ABC, one angle B and two sidelengths $a = |BC|$ and $b = |AC|$ are known. What is the length of the remaining side? One way to obtain this is to use the Law of Cosines $b^2 = a^2 + c^2 - 2ac\cos B$ to obtain $c = |AB|$. Let us rewrite this third sidelength as a variable x and arrange the equation to

$$x^2 - (2a\cos B)x + (a^2 - b^2) = 0 \ \ . \qquad (*)$$

This is a quadratic equation, and so will have two solutions, which could be real or nonreal, positive or negative, or coincident. In this exercise, we will see how this relates to the geometry of the situation.

(a) Verify that the discriminant D of the quadratic in (*) is $4(b^2 - a^2\sin^2 B)$. Explain why D is nonnegative if and only if a, b and B correspond to data for a feasible triangle. What happens if $D = 0$? Explain how the geometry supports the fact that (*) has a single solution is this case.

(b) Suppose that a, B and B are data for a feasible triangle. By considering the sum of the roots, explain why (*) has at least one positive solution.

(c) Determine conditions of a and b that (*) has (i) exactly one, (ii) exactly two, positive solutions. Relate this to the geometric possibilities for the triangle. In the case where there is a negative solution, explain how it might be interpreted.

9.4. Let a, b, c be real numbers. We consider solutions of the quadratic equation $az^2 + bz + c = 0$ where $z = x + yi$ is a complex number.

(a) Show that the complex equation $az^2 + bz + c = 0$ is equivalent to the system of real equations:

$$a(x^2 - y^2) + bx + c = 0 \qquad (1)$$
$$axy + by = 0 \qquad (2)$$

(b) Considering (2) in the form $y(ax + b) = 0$, describe its locus.

(c) Show that (1) can be written in the form

$$\left(x + \frac{b}{2a}\right)^2 - y^2 = \left(\frac{1}{2a}\right)^2 (b^2 - 4ac) \ .$$

Describe the locus of this equation in the three cases: (i) $b^2 = 4ac$; (ii) $b^2 > 4ac$; (iii) $b^2 < 4ac$.

(d) The solutions of the system (1) and (2) are represented in the plane by points (x, y) that lie on the intersection of the loci of (1) and (2). When $b^2 = 4ac$, show that there is a single such point and that it lies on the real axis. When $b^2 > 4ac$, show that there are two points on the real axis, each a reflection of the other in the line Re $z = -b/2a$. When $b^2 < 4ac$, show that there are two points not on the real axis that are mirror images of each other with respect to the real axis. Explain how this is consistent with what you already now about real and imaginary roots of a quadratic.

§10. Approximation.

10.1. Let c be a positive real number. A standard way to approximate the square root of c is to begin with a positive guess u and then proceed to a new guess $v = \frac{1}{2}(u + c/u)$. This is repeated over and over until the desired degree of approximation is reached.

(a) Verify that if $c = 2$ and the first guess is 1, then this process yields the sequence of approximants: 1, $\frac{3}{2} = 1.5$, $\frac{17}{12} = 1.416667$, $\frac{577}{408} = 1.414216$ (where the decimals forms are not exact).

(b) Use the process to approximate $\sqrt{3}$.

(c) Show that $u < \sqrt{c}$ if and only if $c/u > \sqrt{c}$ and that $u > \sqrt{c}$ if and only if $c/u < \sqrt{c}$. Noting that v is the average of u and c/u, explain why it is reasonable to expect that v might be a better approximation that u.

(d) Verify that

$$v - \sqrt{c} = \frac{1}{2u}(u - \sqrt{c})^2 \ .$$

Deduce that every approximation beyond the first exceeds $\sqrt{c}$, and prove that from this point on the sequences decreases. Why does the sequence tend towards $\sqrt{c}$?

10.2. We look at the geometry of the situation of Exercise 10.1. As before, we have that $c > 0$.

(a) Let $x > 0$. Use the Arithmetic-Geometric Means Inequality (Exercise 4.1) to prove that $\frac{1}{2}(x+c/x) \geq \sqrt{c}$. with equality if and only if $x = \sqrt{c}$.

(b) Verify that

$$\frac{1}{2}\left(x_1 + \frac{c}{x_1}\right) - \frac{1}{2}\left(x_2 + \frac{c}{x_2}\right) = \frac{1}{2}(x_1 - x_2)\left(1 - \frac{c}{x_1 x_2}\right) \quad .$$

Use this to argue that $\frac{1}{2}(x+c/x)$ is a decreasing function of x for $0 < x < \sqrt{c}$ and an increasing function of x for $\sqrt{c} < x$.

(c) With the same axes, sketch the graphs of both of the curves $y = x$ and $y = \frac{1}{2}(x + c/x)$ for $x > 0$. Where do these curves intersect? What are the asymptotes of the second curve?

(d) Using the graphs in (c), we can illustrate the behaviour of the approximating sequence for $\sqrt{c}$ described in Exercise 10.1. Let $u_1 > 0$ be the first approximant. Locate on your sketch a possible position of $(u_1, 0)$. Let $u_2 = \frac{1}{2}(u_1 + c/u_1)$. Locate (u_1, u_2), (u_2, u_2) and $(u_2, 0)$. These three points will be on the respective curves $y = \frac{1}{2}(x + c/x)$, $y = x$ and $y = 0$. We continue on in this way. Suppose that u_n has been found. Let

$$u_{n+1} = \frac{1}{2}\left(u_n + \frac{c}{u_n}\right) \quad .$$

Locate $(u_n, 0)$, (u_n, u_{n+1}), (u_{n+1}, u_{n+1}) and $(u_{n+1}, 0)$. Describe from your diagram what eventually happens to the terms of the sequence $\{u_n\}$.

10.3. The recursion of Exercise 10.1 can be defined when c is negative, even through c does not have a real square root in this case. What will happen? To focus the discussion, consider the case $c = -1$.

(a) Sketch the curve

$$y = \frac{1}{2}\left(x - \frac{1}{x}\right)$$

for real nonzero x, and attempt an anlysis as in Exercise 10.2.(d), using various starting points. In this case, you may find it helpful to use a calculator or computer to generate the terms of the sequence of "approximants", or even to use the computer to draw the whole situation for you.

(b) To get a handle on the situation, we note that any real number can be written in the form $x = \cot\theta$ for some number θ lying strictly between 0 and π. Consider the transformation

$$T : x \longrightarrow \frac{1}{2}\left(x - \frac{1}{x}\right) \quad .$$

If $x = \cot\theta$, show that the image of x under this transformation is $\cot 2\theta$. Thus, in terms of θ the mapping is *conjugate* (essentially the same in its mathematical structure) to U: $\theta \rightarrow 2\theta$ (modulo π) (this simply means that if you add, subtract two angles or multiply by a constant, you add an integral multiple of π to put the result of the operation in the interval $(0, \pi)$ using a kind of "clock arithmetic").

(c) Does the transformation T have any *fixed points*? (These are points x for which $T(x) = x$. You can answer this question directly, but also by looking at the mapping U and reinterpreting what you find in terms of T.)

(d) Let $U^2(\theta) = U(U(\theta)$ and for $n \geq 3$, let $U^n(\theta) = U(U^{n-1}(\theta))$. Determine a simple expression for $U^n(\theta)$.

(e) Does the transformation T have any points of *period* 2? (This asks whether there are any numbers u for which $T(u) = v$ for some number v and $T(v) = u$, so that two applications of the mapping T take the point back to itself.) Answer this question directly by looking at the equation

$$T(T(x)) = x \quad .$$

Now answer it by working through the operator U. For what values of θ does $U(U(\theta) = 4\theta$ differ from θ by a multiple of π. Are your results consistent?

(f) A point p is a point of *period* k for T if and only if $T^k(p) = p$, where $T^1(x) = T(x)$ and $T^k(x) = T(T^{k-1}(x))$ for $k \geq 2$. Either directly or working through the operator U, determine if T has points of period k for k is a positive integer exceeding 1. Use a calculator to work out the approximate values of such points and check the result by applying the operator T.

§11. The logistic dynamical system.

We suppose that k is a positive parameter and define the function $p_k(x) = kx(1-x)$ for $0 \leq x \leq 1$. We can use p_k to define a *dynamical system* as follows:

Begin with any point x_0 in the closed interval $[0, 1] \equiv x : 0 \leq x \leq 1$. For each nonnegative integer n, define $x_{n+1} = p_k(x_n)$.

11.1. One can use graphical methods in helping us visualize how the sequence defined for the dynamical system behaves. Suppose that we have a sketch of the curves with equations

$$y = p_k(x)$$

and

$$y = x \ .$$

For each nonnegative integer n, plot the points $(x_n, 0)$ and $(x_n, p_k(x_n)) = (x_n, x_{n+1})$. By drawing lines parallel to the axes and making use of the line $y = x$, indicate geometrically how the point $(x_{n+1}, 0)$ can be found. Thus, we can indicate on the $x-$axis the progress of the sequence $\{x_n\}$.

11.2. Consider the case $0 < k < 1$. Sketch the curves as indicated in (a) and use your diagram to argue that $\lim_{n\to\infty} x_n = 0$. Verify this analytically, by first verifying that $0 < x_{n+1} < kx_N$ whenever $0 < x_0 < 1$.

11.3. Suppose that $k > 1$. Determine a number u for which $0 < u < 1$ and $p_k(u) = u$.

11.4. Consider the case $1 < k < 2$. Sketch the curves as in (a), being careful to indicate on which side of the line $x = \frac{1}{2}$ the curves intersect. Analyze the types of behaviour of the sequence for values of x_0 in the closed interval $[0, 1]$.

11.5. Consider the case $2 < k < 3$. Sketch the curves as in (a) and analyze the behaviour or sequences $\{x_n\}$. Verify that

$$x_{n+1} - u = k(x_n - u)(1 - u - x_n)$$

and use this to check that, when $x_n > 1 - u$, $x_{n+1} - u$ and $x_n - u$ have opposite signs and $|x_{n+1} - u| < |x_n - u|$. Analyze the behaviour of the sequence $\{x_n\}$ for various cases of x_0 in $[0, 1]$.

11.6. Let $k > 1$ and let u be as defined in part (c). Determine $p'_k(u)$ in terms of k, where p'_k denotes the derivative of p_k. Prove that $|p'_k(u)| < 1$ if and only if $1 < k < 2$. What effect do you think that the value of the derivative of p_k at u has on the behaviour of sequences $\{x_n\}$ that start off with a value x_0 close to u?

11.7. We study the possibility of sequences $\{x_n\}$ of *period 2*, *i.e.*, there are two distinct values v and w for which $x_n = v$ when n is even and $x_n = w$ when n is odd, so that the sequence proceeds $\{u, v, u, v, \cdots\}$. To do this, we define the *second iterate* of p_k:

$$q_k(x) = p_k(p_k(x)) = kp_k(x)(1 - p_k(x)) \ .$$

Determine the polynomial q_k and specify its degree. Prove that if $p_k(v) = w$ and $p_k(w) = v$, then $q_k(v) = v$ and $q_k(w) = w$.

11.8. To solve the equation $x = q_k(x)$, we can write it in the form

$$x - q_k(x) = 0 \ .$$

Explain why $x - p_k(x)$ is a factor of the left side, and use this fact to write the left side as a product of quadratics. Thus determine v and w.

11.9. For the cases $1 < k < 2$, $2 < k < 3$, $k = 3$ and $k < 3$, show on a graph the location of v and w.

11.10. Investigate the behaviour of the sequence $\{x_n\}$ when $k > 3$. You may find a pocket calculator of some use in this enterprise.

§12. Miscellaneous

12.1. Let $ABCD$ be a cyclic quadrilateral with side AD of length d, with d the diameter of the corcumcircle of $ABCD$. Suppose that AB and BC both have length a while CD has length b. We are given that a, b and d are three positive integers.

(a) Prove that d cannot be a prime number, nor twice an odd prime number.

(b) What is the minimum integral value of d that admits the given configuration?