PUTNAM PROBLEMS

INEQUALITIES

2016-A-2. Given a positive integer n, let $M(n)$ be the largest integer m such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

Evaluate

$$\lim_{n\to\infty} \frac{M(n)}{n}.$$

2007-B-2. Suppose that $f : [0,1] \longrightarrow \mathbf{R}$ has a continuous derivative and that $\int_0^1 f(x)dx = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left| \int_0^\alpha f(x)dx \right| \le \frac{1}{8} \max_{0\le x\le 1} |f'(x)| \ .$$

2004-A-6. Suppose that $f(x,y)$ is a continuous real-valued function on the unit square $0 \le x \le 1$, $0 \le y \le 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y)dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)dy \right)^2 dx \le \left(\int_0^1 \int_0^1 f(x,y)dxdy \right)^2 + \int_0^1 \int_0^1 [f(x,y)]^2 dxdy \ .$$

2003-A-2. Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \le ((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n))^{1/n} \ .$$

2003-A-3. Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

2003-A-4. Suppose that a, b, c, A, B, C are real numbers, $a \ne 0$ and $A \ne 0$, such that

$$|ax^2 + bx + c| \le |Ax^2 + Bx + c|$$

for all real numbers x. Show that

$$|b^2 - 4ac| \le |B^2 - 4AC| \ .$$

2003-B-2. Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}$, form a new sequence of $n-1$ entries $3/4, 5/12, \cdots, (2n-1)/2n(n-1)$, by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbours of the second sequence to obtain a third sequence of $n-2$ entries and continue until the final sequence consists of a single number x_n. Show that $x_n < 2/n$.

2003-B-6. Let $f(x)$ be a continuous real-valued function defined on the interval $[0,1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| dxdy \ge \int_0^1 |f(x)| dx \ .$$

2002-B-3. Show that, for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne} \ .$$

1999-B-4. Let f be a real function with a continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ are positive for all x. Suppose that $f'''(x) \le f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

1998-B-1. Find the minimum value of

$$\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

1998-B-2. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the $x-$axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

1996-B-2. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}} \ .$$

1996-B-3. Given that $\{x_1, x_2, \cdots, x_n\} = \{1, 2, \cdots, n\}$, find, with proof, the largest possible value, as a function of n (with $n \ge 2$), of

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 \ .$$

1988-B-2. Prove or disprove: If x and y are real numbers with $y \ge 0$ and $y(y+1) \le (x+1)^2$, then $y(y-1) \le x^2$.