PUTNAM PROBLEMS

FINITE MATHEMATICS, COMBINATORICS

2018-B-6. Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$

2017-A-6. The 30 edges of a regular icosahedron are distinguished by labelling them $1, 2, \ldots, 30$. How many different ways are there to paint each edge red, white, or blue such that each of the 20 triangular faces of the icosahedron has two edges of the same colour and a third edge of a different colour?

2015-B-5. Let P_n be the number of permutations π of $\{1, 2, \ldots, n\}$ such that

$$|i - j| = 1 \qquad \text{implies} \qquad |\pi(i) - \pi(j)| \leq 2$$

for all i, j in $\{1, 2, \ldots, n\}$. Show that for $n \geq 2$, the quantity

$$P_{n+5} - P_{n+4} - P_{n+3} + P_n$$

does not depend on n, and find its value.

2014-B-5. In the 75th Annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible $n \times n$ matrices with entries in the field $\mathbf{Z}/p\mathbf{Z}$ of integers modulo p, where n is a fixed positive integer and p is a fixed prime number. The rules of the game are:

1. A player cannot choose an element that has been chosen by either player on any previous term.
2. A player can only choose an element that commutes with all previously chosen elements.
3. A player who cannot choose an element on his/her turn loses the game.

Patniss takes the first turn. Which player has a winning strategy? (Your answer may depend on n and p.)

2013-A-1. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

2013-A-4. A finite collectionof digits 0 and 1 is written around a circle. An *arc* of length $L \geq 0$ consists of L consecutive digits around the circle. For each arc w, let $Z(w)$ and $N(w)$ denote the number of 0's in w and the number of 1's in w, respectively. Assume that $|Z(w) - Z(w')| \leq 1$ for any two arcs w, w' of the same length. Suppose that some arcs $w_1, \ldots, w_k$ have the property that

$$Z = \frac{1}{k}\sum_{j=1}^{k} Z(w_j) \qquad \text{and} \qquad N = \frac{1}{k}\sum_{j=1}^{k} N(w_j)$$

are both integers. prove that there exists an arc w with $Z(w) = Z$ and $N(w) = N$.

2013-A-6. Define a function $w : \mathbf{Z} \times \mathbf{Z} \longrightarrow \mathbf{Z}$ as follows. For $|a|, |b| \leq 2$, let $w(a, b)$ be as in the table shown; otherwise, let $w(a,b) = 0$.

$w(a,b)$				b	
a	-2	-1	0	1	2
-2	-1	-2	2	-2	-1
-1	-2	4	-4	4	-2
0	2	-4	12	-4	2
1	-2	4	-4	4	-2
2	-1	-2	2	-2	-1

For every subset of $\mathbf{Z} \times \mathbf{Z}$, define

$$A(S) = \sum_{(\mathbf{s},\mathbf{s}') \in S \times S} w(\mathbf{s} - \mathbf{s}').$$

Prove that if S is any finite nonempty subset of $\mathbf{Z}\times\mathbf{Z}$, then $A(S) > 0$. (For example, if $S = \{(0,1),(0,2),(2,0),(3,1)\}$, then the terms of $A(S)$ are 12, 12, 12, 12, 4, 4, 0, 0, 0, 0, -1, -1, -2, -2 -4, -4.)

2013-B-3. Let P be a nonempty collection of subsets of $\{1,\dots,n\}$ such that:
(i) if $S, S' \in P$, then $S \cup S' \in P$ and $S \cap S' \in P$, and
(ii) if $S \in P$ and $S \neq \emptyset$, then there is a subset $T \subset S$ such that $T \in P$ and T contains exactly one fewer element that S.

Suppose that $f : P \to \mathbf{R}$ is a function such that $f(\emptyset) = 0$ and

$$f(S \cup S') = f(S) + f(S') - f(S \cap S')$$

for all $S, S' \in P$. Must there exist real numbers $f_1, \dots, f_n$ such that

$$f(S) = \sum_{i \in S} f_i$$

for every $S \in P$?

2013-B-5. Let $X = \{1, 2, \dots, n\}$, and let $k \in X$. Show that there are exactly $k \cdot n^{n-1}$ functions $f : X \to X$ such that for every $x \in X$ there is a $j \geq 0$ such that $f^{(j)}(x) \leq k$. [Here $f^{(j)}$ denotes the j^{th} iterate of f, so that $f^{(0)}(x) = x$ and $f^{(j+1)}(x) = f(f^{(j)}(x))$.]

2013-B-6. Let $n \geq 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of n spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space s, places a stone in the nearest empty space to the left of s (if such a space exists), and places a stone in the nearest empty space to the right of s (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?

2012-B-3. A round-robin tournament among $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

2011-B-4. In a tournament, 2011 players meet 2011 times to play a multiplayer game. Each game is played by all 2011 players together and each ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $\mathbf{T} = (T_{hk})$ and $\mathbf{W} = (W_{hk})$. Initially, $\mathbf{T} = \mathbf{W} = \mathbf{0}$. After every game, for every (h, k) (including $h = k$), if players h and k tied (that is, both won or both lost), then the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

Prove that, at the end of the tournament, $\det(\mathbf{T} + i\mathbf{W})$ is a non-negative integer divisible by 2^{2010}.

2010-B-3. There are 2010 boxes labeled $B_1, B_2, \dots, B_{2010}$ and $2010n$ balls have been distributed among them for some positive integer n. Youmay redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

2008-B-6. Let n and k be positive integers. Say that a permutation σ of $\{1, 2, \dots, n\}$ is $k-$*limited* if $|\sigma(i) - i| \leq k$ for all i. Prove that the number of $k-$limited permutations of $\{1, 2, \dots, n\}$ is odd if and only if $n \equiv 0$ or $1 \pmod{2k+1}$.

2007-B-6. For each positive integer n, let $f(n)$ be the number of ways to make $n!$ cents using an unordered collection of coins, each worth $k!$ cents for some k, $1 \leq k \leq n$. Prove that for some constant C, independent of n,

$$n^{n^2/2 - Cn} e^{-n^2/4} \leq f(n) \leq n^{n^2/2 + Cn} e^{-n^2/4} \ .$$

2006-A-2. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

2006-A-4. Let $S = \{1, 2, \cdots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

(i) $\pi(k) > \pi(k+1)$ for $k = 1$;

(ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;

(iii) $\pi(k-1) < \pi(k)$ for $k = n$.

(For example, if $n = 5$ and π takes values at 1, 2, 3, 4, 5 of 2, 1, 4, 5, 3, then π has a local maximum of 2 at $k = 1$ and a local maximum of 5 at $k = 4$.)

What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

2005-A-2. Let $S = \{(a, b) : a = 1, 2, \cdots, n, b = 1, 2, 3\}$. A *rook tour* of S is a polygonal path made up of line segments connecting points $p_1, p_2, \cdots, p_{3n}$ in sequence such that (i) $p_i \in S$, (ii) p_i and p_{i+1} are a unit distance apart, for $1 \leq i < 3n$, (iii) for each $p \in S$, there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$.

(Here is an example of a rook tour for $n = 5$:

$$(1, 1), (2, 1), (2, 2), (2, 1), (3, 1), (3, 2), (3, 3), (3, 2)$$

$$(3, 1), (4, 1), (4, 2), (4, 3), (5, 3), (5, 2), (5, 1) \ .)$$

2005-B-4. For positive integers m and n, let $f(m, n)$ denote the number of $n-$tuples $(x_1, x_2, \cdots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

2005-B-6. Let S_n denote the set of all permutations of the numbers $1, 2, \cdots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi)+1} = (-1)^{n+1} \frac{n}{n+1} \ .$$

2003-A-5. A Dyck $n-$path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the $x-$axis. A return is a maximal sequence of contiguous downsteps that terminates on the $x-$axis. For example, the Dyck 5-path (up, up, down, up, up, down, down, down, up, down) has two returns of lengths 3 and 1 respectively. Show that there is a one-to-one correspondence between the Dyck $n-$paths with no return of even length and the Dyck $(n-1)-$paths.

2002-A-4. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and Player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

2002-B-2. Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

Each player, in turn, signs his name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

2002-B-4. An integer n, unknown to you, has been randomly chosen in the intercal $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than 2/3.

2000-B-5. Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \cdots,$ of positive integers as follows:

Integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n.

Show that there exists infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.

1997-A-2. Players $1, 2, 3, \cdots, n$ are seated around a table and each has a single penny. Player 1 passes a penny to Player 2, who then passes two pennies to Player 3. Player 3 then passes one penny to Player 4, who then passes two players to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

1996-A-3. Suppose that each of twenty students has made a choice of anywhere from zero to six courses from a total of six courses offered. Prove or disprove: There are five students and two courses such that all five have chosen both courses or all five have chosen neither.

1996-B-1. Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with a proof, the number of subsets of $\{1, 2, \cdots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets if selfish.

1996-B-5. Given a finite string S of symbol X and O, we write $\Delta(S)$ for the number of X's in S minus the number of O's. For example, $\Delta(XOOXOOX) = -1$. We call a string S **balanced** if every substring T of (consecutive symbols of) S has $-2 \leq \Delta(T) \leq 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length n.

1995-A-4. Suppose we have a necklace of n beads. Each bead is labelled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \cdots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k-1 \qquad \text{for} \quad k = 1, 2, \cdots, n \ .$$

1995-B-1. For a partition π of $\{1,2,3,4,5,6,7,8,9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1,2,3,4,5,6,7,8,9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

1995-B-5. A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking **either**

a. one bean from a heap, provided at least two beans are left behing in that heap, **or**

b. a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

1994-A-3. Show that if the points of an isosceles right triangle of side length 1 are earch colored with one of four colors, then there must be two points of the same color which are at least a distance $2-\sqrt{2}$ apart.

1994-A-6. Let f_1, f_2, $\cdots$, f_{10} be bijections of the set of integers such that for each integer n, there is some composition $f_{1_1} \circ f_{1_2} \circ \cdots \circ f_{i_m}$ of these functions (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions

$$\mathfrak{F} = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}} : e_i = 0 \ \text{ or } \ 1 \ \text{ for } \ 1 \leq i \leq 10\}$$

(f_i^0 is the identity function and $f_i^1 = f_i$). Show that if A is any nonempty finite set of integers, then at most 512 of the functions in $\mathfrak{F}$ map A to itself.

1993-A-3. Let $\mathfrak{P}_\mathfrak{n}$ be the set of subsets of $\{1, 2, \cdots, n\}$. Let $c(n,m)$ be the number of functions $f : \mathfrak{P}_n \to \{1, 2, \cdots .m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n,m) = \sum_{j=1}^{m} j^n \ .$$

1992-B-1. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S?