**PUTNAM PROBLEMS**

**ALGEBRA**

*Putnam problems*

**2018-A-3.** Determine the greatest possible value of $\sum_{i=0}^{10} \cos(3x_i)$ for real numbers $x_1, x_2, \ldots, x_{10}$ satisfying $\sum_{i=1}^{10}(x_i) = 0$.

**2018-B-2.** Let $n$ be a positive integer, and let

$$f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}.$$

Prove that $f_n$ has no roots in the closed unit disc $\{z \in \mathbf{C} : |z| \le 1\}$.

**2017-A-2.** Let $Q_0(x) = 1$, $Q_1(x) = x$ and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \ge 2$. Show that, whenever $n$ is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

**2015-B-1.** Let $f$ be a three times differentiable function (defined on $\mathbf{R}$ and real-valued) such that $f$ has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

**2014-A-5.** Let

$$P_n(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1}.$$

Prove that the polynomials $P_j(x)$ and $P_k(x)$ are relatively prime for all positive integers $j$ and $k$ with $j \neq k$.

**2014-B-4.** Show that for each positive integer $n$, all the roots of the polynomial

$$\sum_{k=0}^{n} 2^{k(n-k)} x^k$$

are real numbers.

**2013-A-3.** Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and $x$, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^n} = 0.$$

Prove that there exists a real number $y$ with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

**2010-B-4.** Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

**2009-A-1.** Let $f$ be a real-valued function in the plane such that for every square $ABCD$ in the plane,

$$f(A) + f(B) + f(C) + f(D) = 0\ .$$

Does it follow that $f(P) = 0$ for every point $P$ in the plane?

**2008-A-1.** Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers $x$, $y$, and $z$. Prove that there exists a function $g : \mathbf{R} \to \mathbf{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers $x$ and $y$.

**2008-A-5.** Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1))$, $(f(2), g(2))$, . . ., $(f(n), g(n))$ in $\mathbf{R}^2$ are the vertices of a regular $n-$gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n - 1$.

**2007-A-1.** Find all values of $\alpha$ for which the curves $y = \alpha x^2 + \alpha x + 1/24$ and $x = \alpha y^2 + \alpha y + 1/24$ are tangent to each other.

**2007-A-2.** A *repunit* is a positive integer whose digits in base 10 are all ones. Find all polynomials $f$ with real coefficients such that if $n$ is a repunit, then so is $f(n)$.

**2007-B-1.** Let $f$ be a polynomial with positive integer coefficients. Prove that if $n$ is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

**2007-B-4.** Let $n$ be a positive integer. Find the number of pairs $P, Q$ of polynomials with real coefficients such that

$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$

and deg $P >$ deg $Q$.

**2007-B-5.** Let $k$ be a positive integer. Prove that there exist polynomials $P_0(n)$, $P_1(n)$, $\cdots$, $P_{k-1}(n)$ (which may depend on $k$) such that, for any integer $n$,

$$\left\lfloor \frac{n}{k} \right\rfloor = P_0(n) + P_1(n) \left\lfloor \frac{n}{k} \right\rfloor + \cdots + P_{k-1}(n) \left\lfloor \frac{n}{k} \right\rfloor^{k-1} .$$

**2006-B-1.** Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points $A, B$, and $C$, which are the vertices of an equilateral triangle, and find its area.

**2005-A-3.** Let $p(z)$ be a polynomial of degree $n$, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

**2005-B-1.** Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers $a$. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to $\nu$.)

**2005-B-5.** Let $P(x_1, \cdots, x_n)$ denote a polynomial with real coefficients in the variables $x_1, \cdots, x_n$, and suppose that

$$\text{(a)} \qquad \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) P(x_1, \cdots, x_n) = 0 \qquad \text{(identically)}$$

and that

$$\text{(b)} \qquad x_1^2 + \cdots + x_n^2 \ \text{ divides } \ P(x_1, \cdots, x_n) .$$

Show that $P = 0$ identically.

**2004-A-4.** Show that for any positive integer $n$ there is an integer $N$ such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n)^n$$

where the $c_i$ are rational numbers and each $a_{ij}$ is one of the numbers, $-1, 0, 1$.

**2004-B-1.** Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that $r$ is a rational number such that $P(r) = 0$. Show that the $n$ numbers

$$c_n r, c_n r^2 + c_{n-1} r, c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \cdots, c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

**2003-A-4.** Suppose that $a, b, c, A, B, C$ are real numbers, $a \neq 0$ and $A \neq 0$, such that

$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$$

for all real numbers $x$. Show that

$$|b^2 - 4ac| \leq |B^2 - 4AC| \ .$$

**2003-B-1.** Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

**2003-B-4.** Let

$$f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$$

where $a, b, c, d, e$ are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number, and if $r_1 + r_2 \neq r_3 + r - 4$, then $r_1 r_2$ is a rational number.

**2002-A-1.** Let $k$ be a positive integer. The $n$th derivative of $1/(x^k - 1)$ has the form $P_n(x)/(x^k - 1)^{n+1}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

**2002-B-6.** Let $p$ be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo $p$ to a product of polynomials in the form $ax + by + cz$, where $a, b, c$ are integers. (We say two integer polynomials are congruent modulo $p$ if corresponding coefficients are congruent modulo $p$.)

**2001-A-3.** For each integer $m$, consider the polynomial

$$P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2 \ .$$

For what values of $m$ is $P_m(x)$ the product of two nonconstant polynomials with integer coefficients?

**2001-B-2.** Find all pairs of real numbers $(x, y)$ satisfying the system of equations

$$\frac{1}{x} + \frac{1}{2y} = (x^2 + 3y^2)(3x^2 + y^2)$$

$$\frac{1}{x} - \frac{1}{2y} = 2(y^4 - x^4) \ .$$

**2000-A-6.** Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \cdots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for $n \geq 0$. Prove that if there exists a positive integer $m$ for which $a_m = 0$, then either $a_1 = 0$ or $a_2 = 0$.

**1999-A-1.** Find polynomials $f(x)$, $g(x)$ and $h(x)$, if they exist, such that, for all $x$,

$$|f(x)| - |g(x)| + |h(x)| = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

**1999-A-2.** Let $p(x)$ be a polynomial that is non-negative for all $x$. Prove that, for some $k$, there are polynomials $f_1(x), \cdots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2 \ .$$

**1999-B-2.** Let $P(x)$ be a polynomial of degree $n$ such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$.Show that if $P(x)$ has at least two distinct roots then it must have $n$ distinct roots. [The roots may be either real or complex.]

**1997-B-4.** Let $a_{m,n}$ denote the coefficient of $x^n$ in the expansion of $(1 + x + x^2)^m$. Prove that for all $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1 \ >$$

**1995-B-4.** Evaluate

$$\sqrt[8]{2207 - \cfrac{1}{2207 - \frac{1}{2207 - \cdots}}} \ .$$

Express your answer in the form $(a + b\sqrt{c})/d$, where $a, b, c, d$ are integers.

**1993-B-2.** For nonnegative integers $n$ and $k$, define $Q(n,k)$ to be the coefficient of $x^k$ in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{n} \binom{n}{j} \binom{n}{k - 2j} \ ,$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers $a$ and $b$ with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$ and $\binom{a}{b} = 0$ otherwise.)