Department of Education, Ontario

Annual Examinations, 1952

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. The equations of the sides of a parallelogram are $2x - y = 7$, $x + 3y = 5$, $2x - y = 2$, and $x + 3y = -1$. Find the equations of the diagonals without solving for the vertices.

2. Find the equations of those conjugate diameters of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ which are of equal length.

3. A tangent is drawn to the circle $x^2 + y^2 = a^2$ and a perpendicular tangent is drawn to the circle $x^2 + y^2 = b^2$. Find the equation of the locus of the intersection of these tangents.

4. A tangent is drawn to the parabola $y^2 = 4px$ at the point $P(x_1, y_1)$, and Q is any other point on this tangent. Perpendiculars drawn from Q cut the directrix at V and the focal line, FP, at W. Prove that the line segments VQ and FW have the same length.

5. Given that d, e, f represent the distances from the vertices of a triangle ABC to the centre of its inscribed circle, prove that

$$\frac{abc}{def} = \frac{s}{r}$$

where r is the radius of the inscribed circle and $2s = a + b + c$.

6. A rod AB of negligible weight and of length $2a$, hangs from a point O to which it is attached by light strings AO and BO, each of length $b > a$. Weights W and $2W$ are attached to the ends A and B, respectively. Prove that, in equilibrium the rod is inclined at an angle θ to the horizontal , where

$$\tan\theta = \frac{a}{3\sqrt{b^2 - a^2}}\ .$$

7. Two objects, A and B, are due west of an observer. After walking a units due north, the observer notes that the line segment AB subtends an angle α at his eye. After he has walked an additional b units due north, the segment AB subtends an angle β at his eye. Prove that the distance AB is given by the formula

$$\frac{b(2a+b)}{(a+b)\cot\beta - a\cot\alpha} \ .$$

8. Without using tables, prove that

$$\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14} = \frac{1}{8} \ .$$

9. Show that, when a polynomial $P(x)$ is divided by $(x-a)(x-b)$, the remainder is

$$P(a)\frac{x-b}{a-b} + P(b)\frac{x-a}{b-a} \ .$$

10. Show that any root of

$$(x+a+b)(x^{-1}+a^{-1}+b^{-1}) = 1$$

is also a root of

$$(x^n+a^n+b^n)(x^{-n}+a^{-n}+b^{-n}) = 1 \ ,$$

where n is any odd integer and where a and b are different than zero.

11. Show that, if n is a positive integer greater than 1,

$$\frac{n}{1-x} - \frac{1-x^n}{(1-x)^2}$$

is a polynomial of degree $n-2$, and find its coefficients

(a) when it is arranged in powers of x,

(b) when it is arranged in powers of $(x-1)$.

12. Eighteen people are to be paired off to form nine pairs, which pairs are afterwards to be divided into two groups of five pairs and four pairs. In how many ways can this be done?