Department of Education, Ontario

Annual Examinations, 1950

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. Let x be any number between -1 and $+1$. Noting that $n^2 = n(n+1) - n$, find a formula for the sum of the infinite series

$$1 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1} + \cdots.$$

2. A man and his wife invite four other couples to dinner. They sit at a circular table with all chairs of the same type, the host and hostess at fixed diametrically opposite places. In how many ways can the guests be seated so that no man sits beside another man or beside his own wife?

3. Find a cube root of the complex number $2 + 11\sqrt{-1}$, given that this number has a square root of the form $a + b\sqrt{-1}$, where a and b are integers.

4. Find an integer between 0 and 385 such that the remainders when it is divided by 5, 7, 11 are 3, 4, 6, respectively.

5. Tangents are drawn to a hyperbola from any point on one of two branches of the conjugate hyperbola. Show that their chord of contact will touch the other branch of the hyperbola.

6. Prove that on the axis of any parabola there is a certain point K which has the property that, if a chord PQ of the parabola is drawn through it, then

$$\frac{1}{PK^2} + \frac{1}{QK^2}$$

is constant for all positions of the chord.

7. At each of three pointson a parabola a tangent and a normal are drawn. Show that the orthocentre of the triangle formed by the three tangents and that of the triangle formed by the normals are equidistant from the axis of the parabola.

8. Any point on an ellipse is joined to the extremities of the major axis. Prove that the portion of a directrix intercepted by the joining lines subtends a right angle at the corresponding focus.

9. A uniform beam of weight W and length L can move freely in a vertical plane about a hinge at one end A. To the other end of the beam a rope is fastened which passes over a small smooth pulley fixed vertically above A. A weight w, attached to the free end of the rope, maintains the beam in equilibrium. If θ is the angle of inclination of the beam with the horizontal and h is the height of the pulley above A, prove that

$$\sin\theta = \frac{h}{2L}\left(1 - \frac{4w^2}{W^2}\right) + \frac{L}{2h} \ .$$

10. Two regular polygons of m and n sides are inscribed in two concentric circles of radii a and b, respectively. Prove that the sum of the squares of all the line-segments which join the vertices of one polygon to the vertices of the other is

$$mn(a^2 + b^2) \ .$$

11. If $1 + x$, $1 - y$, x, y, is a sequence of numbers in harmonic progression, show that all the different values of $x + y$ are given by

$$\cos\left[2(3n+1)\frac{\pi}{9}\right] \ ,$$

where $n = 0, 1, 2$.

12. Two triangles are determined by the given parts a, b, A. If P_1 and P_2 are the circumcentres of these two triangles and O_1 and O_2 are their orthocentres, prove that

$$O_1O_2 = 2P_1P_2\cos A \ .$$