Department of Education, Ontario

Annual Examinations, 1946

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. (a) Recalling that the arithmetic mean of two positive unequal numbers is greater than the geometric mean, show that

$$\left(\frac{a_1+a_2+a_3+a_4}{4}\right)^4 > a_1a_2a_3a_4$$

where a_1, a_2, a_3, a_4 are positive and unequal. By setting $a_4 = \frac{1}{3}(a_1+a_2+a_3)$, deduce the inequality

$$\left(\frac{a_1+a_2+a_3}{3}\right)^3 > a_1a_2a_3 \ .$$

(b) Thence show that, if the perimeter $2s$ of a triangle is constant, its area represented by the function $\sqrt{s(s-a)(s-b)(s-c)}$ is maximum when the triangle is equilateral.

2. A correspondence is established between the values of x' and those of x by means of the equation

$$x' = f(x) = \frac{ax+b}{x+c} \ .$$

(a) Express x as a function of x'.

(b) What is the condition that $f(x') = x$ for arbitrary values of x?

(c) Assuming that the condition in (b) holds: (i) find the values x_1 and x_2 of x for which $f(x) = x$; (ii) if $x_1 + x_2 = 2x_0$, show that for arbitrary values of x

$$4(x-x_0)(x'-x_0) = (x_2-x_1)^2 \ .$$

3. Assume $1^5+2^5+\cdots+n^5 = P(n)$, where $P(n)$ is a polynomial in n with undetermined coefficients. Calculate three of the coefficients of $P(n)$.

4. In a game of chance five balls are thrown in succession. A score is made only when a ball rolls into one of three pockets marked 10, 20, 30, but the ball may miss the pockets entirely. If a ball rolls into a pocket, it is removed before the next ball is thrown. If the score for each throw is recorded separately, in how many ways can a total of at least 100 be obtained in five throws?

5. The three vertices of a triangle lie on an equilateral hyperbola. Prove that the altitudes of the triangle meet on the hyperbola.

6. Prove that the locus of a point which moves so that the tangents from it to the parabola $y^2 = 4x$ intersect at an angle of 45 degrees is an equilateral hyperbola with centre at $(-3, 0)$.

7. The angle between equal conjugate diameters of an ellipse is 60 degrees. Prove that the eccentricity of the ellipse is $\sqrt{6}/3$.

8. A variable chord of an ellipse subtends a right angle at the centre. Show that the chord always touches a fixed circle.

9. If d_1, d_2, d_3 are the distances of the incentre of a triangle ABC from the vertices, prove that
$$\frac{d_1 d_2 d_3}{abc} = \frac{r}{s}$$
where r is the radius of the inscribed circle and $2s = a + b + c$.

10. A uniform plank of length $2c$ rests with one end on a rough floor, the other end projecting over a smooth cylinder, of radius b, which is fastened to the floor. The axis of the cylinder is horizontal and is perpendicular to the direction of the plank. If the plank makes an angle θ with the floor and the angle of friction is α, prove that equilibrium is possible if
$$b \sin\alpha > c \tan\frac{\theta}{2} \cos\theta \sin(\theta + \alpha) \ .$$

11. Find all the values of x which satisfy the equation
$$\tan(x+\alpha)\tan(x+\beta) + \tan(x+\beta)\tan(x+\gamma) + \tan(x+\gamma)\tan(x+\alpha) = 1 \ .$$

12. An angle A (less than 180 degrees), a length a and the product k^2 of two lengths b and c are given. Show that these parts determine a triangle ABC is
$$a^2 \geq 4k^2 \sin^2\frac{A}{2} \ .$$