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Please send your solutions to

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individually as you solve the problems. Electronic files can be sent to *barbeau@math.utoronto.ca*. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

640. Suppose that $n \geq 2$ and that, for $1 \leq i \leq n$, we have that $x_i \geq -2$ and all the x_i are nonzero with the same sign. Prove that

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) > 1 + x_1 + x_2 + \cdots + x_n \quad ,$$

641. Observe that $x^2 + 5x + 6 = (x + 2)(x + 3)$ while $x^2 + 5x - 6 = (x + 6)(x - 1)$. Determine infinitely many coprime pairs (m, n) of positive integers for which both $x^2 + mx + n$ and $x^2 + mx - n$ can be factored as a product of linear polynomials with integer coefficients.

642. In a convex polyhedron, each vertex is the endpoint of exactly three edges and each face is a concyclic polygon. Prove that the polyhedron can be inscribed in a sphere.

643. Let n^2 distinct integers be arranged in an $n \times n$ square array ($n \geq 2$). Show that it is possible to select n numbers, one from each row and column, such that if the number selected from any row is greater than another number in this row, then this latter number is less than the number selected from its column.

644. Given a point P, a line $\mathfrak{L}$ and a circle $\mathfrak{C}$, construct with straightedge and compasses an equilateral triangle PQR with one vertex at P, another vertex Q on $\mathfrak{L}$ and the third vertex R on $\mathfrak{C}$.

645. Let $n \geq 3$ be a positive integer. Are there n positive integers $a_1, a_2, \cdots, a_n$ not all the same such that for each i with $3 \leq i \leq n$ we have

$$a_i + S_i = (a_i, S_i) + [a_i, S_i] \quad .$$

where $S_i = a_1 + a_2 + \cdots + a_i$, and where $(\cdot, \cdot)$ and $[\cdot, \cdot]$ represent the greatest common divisor and least common multiple respectively?

646. Let ABC be a triangle with incentre I. Let AI meet BC at L, and let X be the contact point of the incircle with the line BC. If D is the reflection of L in X on line BC , we construct B' and C' as the reflections of D with respect to the lines BI and CI, respectively. Show that the quadrailateral $BCC'B'$ is cyclic.

Solutions

626. Let ABC be an isosceles triangle with $AB = AC$, and suppose that D is a point on the side BC with $BC > BD > DC$. Let BE and CF be diameters of the respective circumcircles of triangles ABD and ADC, and let P be the foot of the altitude from A to BC. Prove that $PD : AP = EF : BC$.

Solution 1. Since angles BDE and CDF are both right, E and F both lie on the perpendicular to BC through D. Since $ABDE$ and $ADCF$ are concyclic,

$$\angle AEF = \angle ABD = \angle ABC = \angle ACB = \angle ACD = \angle AFD = \angle AFE \ .$$

Therefore triangles AEF and ABC are similar. Thus AEF is isosceles and its altitude through A is perpendicular to DEF and parallel to BC, so that it is equal to PD. Therefore, from the similarity, $PD : AP = EF : BC$, as desired.

Solution 2. Since the chord AD subtends the same angle ($\angle ABC = \angle ACB$) in circles ABD and ACD, these circles must have equal diameters. The rotation with centre A that takes B to C takes the circle ABD to a circle with chord AC of equal diameter. The angle subtended at D by AB on the circumcircle of ABD is the supplement of the angle subtended at D by AC on the circumcircle of ACD. Therefore, this image circle must be the circle ACD. Therefore the diameter BE is carried to the diameter CF, and E is carried to F. Hence $AE = AF$ and $\angle BAC = \angle EAF$. Thus, triangles ABC and AEF are similar.

Now consider the composite of a rotation about A through a right angle followed by a dilatation of factor $|AE|/|AB|$. This transformation take B to E and C to F, and therefore the altitude AP to the altitude AM of triangle AEF which is therefore parallel to BC. Since D lies on the circumcircle of ABD with diameter BE, $\angle BDE = 90°$. Similarly, $\angle CDF = 90°$. Hence $AMDP$ is a rectangle and $AM = PD$. The result follows from the similarity of triangles ABC and AEF.

627. Let

$$f(x,y,z) = 2x^2 + 2y^2 - 2z^2 + \frac{7}{xy} + \frac{1}{z} \ .$$

There are three pairwise distinct numbers a, b, c for which

$$f(a,b,c) = f(b,c,a) = f(c,a,b) \ .$$

Determine $f(a,b,c)$. Determine three such numbers a, b, c.

Solution. Suppose that a, b, c are pairwise distinct and $f(a,b,c) = f(b,c,a) = f(c,a,b)$. Then

$$2a^2 + 2b^2 - 2c^2 + \frac{7}{ab} + \frac{1}{c} = 2b^2 + 2c^2 - 2a^2 + \frac{7}{bc} + \frac{1}{a}$$

so that

$$4(a^2 - c^2) = \left(\frac{1}{a} - \frac{1}{c}\right)\left(1 - \frac{7}{b}\right) = \frac{1}{abc}(c-a)(b-7) \ .$$

Therefore $4abc(a+c) = 7 - b$. Similarly, $4abc(b+a) = 7 - c$. Subtracting these equations yields that $4abc(c-b) = c - b$ so that $4abc = 1$. It follows that $a + b + c = 7$.

Therefore

$$\begin{aligned} f(a,b,c) &= 2(a^2+b^2) - 2c^2 + 28c + 4ab \\ &= 2(a+b)^2 - 2c^2 + 28c = 2(7-c)^2 - 2c^2 + 28c \\ &= 98 - 28c + 2c^2 - 2c^2 + 28c = 98 \ . \end{aligned}$$

We can find such triples by picking any nonzero value of c and solving the quadratic equation $t^2 - (7 - c)t + (1/4c) = 0$ for a and b. For example, taking $c = 1$ yields the triple

$$(a,b,c) = \left(\frac{6+\sqrt{35}}{2}, \frac{6-\sqrt{35}}{2}, 1\right) \ .$$

628. Suppose that AP, BQ and CR are the altitudes of the acute triangle ABC, and that

$$9\overrightarrow{AP} + 4\overrightarrow{BQ} + 7\overrightarrow{CR} = \overrightarrow{O} \ .$$

Prove that one of the angles of triangle ABC is equal to $60°$.

Solution 1. [H. Spink] Since the sum of the three vectors $9\overrightarrow{AP}$, $4\overrightarrow{BQ}$, $7\overrightarrow{CR}$ is zero, there is a triangle whose sides have lengths $9|AP|$, $4|BQ|$, $7|CR|$ and are parallel to the corresponding vectors.

Where H is the orthocentre, we have that

$$\angle BHP = 90° - \angle QBC = \angle ACB$$

so that the angle between the vectors $\overrightarrow{AP}$ and $\overrightarrow{BQ}$ is equal to angle ACB. Similarly, the angle between vectors $\overrightarrow{BQ}$ and $\overrightarrow{CR}$ is equal to angle BAC. It follows that the triangle formed by the vectors is similar to triangle ABC and

$$|AB| : 7|CR| = |BC| : 9|AP| = |CA| : 4|BQ| \ .$$

Since twice the area of the triangle ABC is equal to

$$|AB| \times |CR| = |BC| \times |AP| = |CA| \times |BQ| \ ,$$

we have that (with conventional notation for side lengths)

$$\frac{c^2}{7} = \frac{a^2}{9} = \frac{b^2}{4}$$

so that $a : b : c = 3 : 2 : \sqrt{7}$.

If one angle of the triangle is equal to $60°$ we would expect it to be neither the largest nor the smallest. Accordingly, we compute the cosine of angle ACB, namely

$$\frac{a^2 + b^2 - c^2}{2ab} = \frac{9 + 4 - 7}{2 \times 3 \times 2} = \frac{6}{12} = \frac{1}{2} \ .$$

Therefore $\angle ACB = 60°$.

Solution 2. Let the angles of the triangle be $\alpha = \angle BAC$, $\beta = \angle CBA$ and $\gamma = \angle ACB$; let p, q, r be the respective magnitudes of vectors $\overrightarrow{AP}$, $\overrightarrow{BQ}$, $\overrightarrow{CR}$. Taking the dot product of the vector equation with $\overrightarrow{BC}$ and noting that $\angle QBC = 90 - \gamma$ and $\angle BCR = 90 - \beta$, we find that $4q \sin \gamma = 7r \sin \beta$. Similarly, $9p \sin \gamma = 7r \sin \alpha$ and $9p \sin \beta = 4q \sin \alpha$. Using the conventional notation for the sides of the triangle, we have that

$$a : b : c = \sin \alpha : \sin \beta : \sin \gamma = 9p : 4q : 7r \ .$$

However, we also have that twice the area of triangle ABC is equal to $ap = bq = cr$, so that $a : b : c = (1/p) : (1/q) : (1/r)$. Therefore $9p^2 = 4q^2 = 7r^2 = k$, for some constant k. Therefore

$$\begin{aligned} \cos \angle ACB &= \frac{a^2 + b^2 - c^2}{2ab} = \frac{81p^2 + 16q^2 - 49r^2}{72pq} \\ &= \frac{9k + 4k - 7k}{12k} = \frac{1}{2} \ , \end{aligned}$$

from which it follows that $\angle C = 60°$.

Solution 3. [C. Deng] Observe that

$$|BQ| = |BC| \cos \angle QBC = |BC| \angle \sin ACB \ ,$$

$$|CR| = |BC| \cos \angle RCB = |BC| \sin \angle ABC \ .$$

Resolving in the direction of $\overrightarrow{BC}$, we find from the given equation that

$$4|BC| \cos^2 \angle QBC = 4|BQ| \cos \angle QBC = 7|CR| \cos \angle RCB = 7|BC| \cos^2 \angle RCB$$

$$\Longrightarrow 4 \sin^2 \angle ACB = 7 \sin^2 \angle ABC \ .$$

By the Law of Sines, $AC : AB = \sin \angle ABC : \sin \angle ACB = 2 : \sqrt{7}$. Similarly $AC : BC = 2 : 3$, so that $CA : AB : BC = 2 : \sqrt{7} : 3$. The cosine of angle ACB is equal to $(4+9-7)/12 = 1/2$, so that $\angle ACB = 60°$.

629. (a) Let $a > b > c > d > 0$ and $a + d = b + c$. Show that $ad < bc$.

(b) Let a, b, p, q, r, s be positive integers for which

$$\frac{p}{q} < \frac{a}{b} < \frac{r}{s}$$

and $qr - ps = 1$. Prove that $b \geq q + s$.

(a) *Solution 1.* Since $c = a + d - b$, we have that

$$bc - ad = b(a + d - b) - ad = (a - b)b - (a - b)d = (a - b)(b - d) > 0 \ .$$

Solution 2. Let $a + d = b + c = u$. Then

$$bc - ad = b(u - b) - (u - d)d = u(b - d) - (b^2 - d^2) = (b - d)(u - b - d) \ .$$

Now $u = b + c > b + d$, so that $u - b - d > 0$ as well as $b - d > 0$. Hence $bc - ad > 0$ as desired.

Solution 3. Let $x = a - b > 0$. Since $a - b = c - d$, we have that $a = b + x$ and $d = c - x$. Hence

$$bc - ad = bc - (b + x)(c - x) = bx - cx + x^2 = x^2 + x(b - c) > 0 \ .$$

Solution 4. Since $\sqrt{a} > \sqrt{b} > \sqrt{c} > \sqrt{d}$, $\sqrt{a} - \sqrt{d} > \sqrt{b} - \sqrt{c}$. Squaring and using $a + d = b + c$ yields $2\sqrt{bc} > 2\sqrt{ad}$, whence the result.

(b) *Solution.* Since all variables represent integers,

$$aq - bp > 0, br - as > 0 \Longrightarrow aq - bp \geq 1, br - as \geq 1 \ .$$

Therefore

$$b = b(qr - ps) = q(br - as) + s(aq - bp) \geq q + s \ .$$

630. (a) Show that, if

$$\frac{\cos \alpha}{\cos \beta} + \frac{\sin \alpha}{\sin \beta} = -1 \quad ,$$

then

$$\frac{\cos^3 \beta}{\cos \alpha} + \frac{\sin^3 \beta}{\sin \alpha} = 1 \quad .$$

(b) Give an example of numbers α and β that satisfy the condition in (a) and check that both equations hold.

(a) *Solution 1.* Let

$$\lambda = \frac{\cos \beta}{\cos \alpha} \qquad \text{and} \qquad \mu = \frac{\sin \beta}{\sin \alpha} \ .$$

Since $\lambda^{-1} + \mu^{-1} = -1$, we have that $\lambda + \mu = -\lambda\mu$. Now

$$1 = \cos^2\beta + \sin^2\beta = \lambda^2\cos^2\alpha + \mu^2\sin^2\alpha = \lambda^2 + (\mu^2 - \lambda^2)\sin^2\alpha = \lambda^2 - (\mu - \lambda)\lambda\mu\sin^2\alpha \ .$$

Hence

$$\begin{aligned}
\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \lambda^3\cos^2\alpha + \mu^3\sin^2\alpha \\
&= \lambda(\lambda^2\cos^2\alpha + \mu^2\sin^2\alpha) + (\mu - \lambda)\mu^2\sin^2\alpha \\
&= \lambda + (\mu - \lambda)\mu^2\sin^2\alpha \\
&= \frac{1}{\lambda}[\lambda^2 + (\lambda^2 - 1)\mu] \\
&= \frac{1}{\lambda}[\lambda^2 + \lambda^2\mu + \lambda + \lambda\mu \\
&= \lambda + \lambda\mu + 1 + \mu = 1 \quad .
\end{aligned}$$

Solution 2. [M. Boase]

$$\begin{aligned}
\frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} = -1 \Longrightarrow \\
\sin(\alpha + \beta) + \sin\beta\cos\beta = 0 \ .
\end{aligned} \tag{$*$}$$

Therefore

$$\begin{aligned}
\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \frac{\cos\beta(1 - \sin^2\beta)}{\cos\alpha} + \frac{\sin\beta(1 - \cos^2\beta)}{\sin\alpha} \\
&= \frac{\cos\beta}{\cos\alpha} + \frac{\sin\beta}{\sin\alpha} - \sin\beta\cos\beta\left(\frac{\sin\beta}{\cos\alpha} + \frac{\cos\beta}{\sin\alpha}\right) \\
&= \frac{\sin(\alpha + \beta)}{\cos\alpha\sin\alpha} - \frac{\cos\beta\sin\beta(\cos(\alpha - \beta))}{\cos\alpha\sin\alpha} \\
&= \frac{-2\sin\beta\cos\beta + 2\sin(\alpha + \beta)\cos(\alpha - \beta)}{2\sin\alpha\cos\alpha} \qquad \text{using} \quad (*) \\
&= \frac{-2\sin\beta\cos\beta + [\sin 2\alpha + \sin 2\beta]}{\sin 2\alpha} = 1
\end{aligned}$$

since $2\sin\beta\cos\beta = \sin 2\beta$.

Solution 3. [A. Birka] Let $\cos\alpha = x$ and $\cos\beta = y$. Then

$$\frac{\sin\alpha}{\sin\beta} = \pm\sqrt{\frac{1 - x^2}{1 - y^2}} \quad .$$

Since

$$\frac{x}{y} + 1 = \mp\sqrt{\frac{1 - x^2}{1 - y^2}} \quad .$$

then

$$(x^2 + 2xy + y^2)(1 - y^2) = y^2(1 - x^2) \quad ,$$

whence

$$x^2 + 2xy = 2xy^3 + y^4 \quad .$$

Thus,

$$\begin{aligned}
\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \frac{y^3}{x} \pm (1 - y^2)\sqrt{\frac{1 - y^2}{1 - x^2}} \\
&= \frac{y^3}{x} - \frac{(1 - y^2)y}{x + y} = \frac{y^4 + 2xy^3 - xy}{x(x + y)} \\
&= \frac{x^2 + xy}{x(x + y)} = 1 \quad .
\end{aligned}$$

Solution 4. [J. Chui] Note that the given equation implies that $\sin 2\beta = -2\sin(\alpha+\beta)$ and that the numerator of

$$\frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} + \frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha}$$

is one quarter of

$$\begin{aligned}
&4[\cos^2\alpha\sin\alpha\sin\beta + \sin^2\alpha\cos\alpha\cos\beta + \cos^4\beta\sin\alpha\sin\beta + \sin^4\beta\cos\alpha\cos\beta] \\
&\quad = 4[\cos^2\alpha\sin\alpha\sin\beta + \sin^2\alpha\cos\alpha\cos\beta + (\cos^2\beta - \cos^2\beta\sin^2\beta)\sin\alpha\sin\beta \\
&\qquad + (\sin^2\beta - \sin^2\beta\cos^2\beta)\cos\alpha\cos\beta] \\
&\quad = (4\cos^2\alpha + 4\cos^2\beta - \sin^2 2\beta)\sin\alpha\sin\beta + (4\sin^2\alpha + 4\sin^2\beta - \sin^2 2\beta)\cos\alpha\cos\beta \\
&\quad = 2\sin 2\alpha\cos\alpha\sin\beta + 2\sin 2\beta\cos\beta\sin\alpha + 2\sin 2\alpha\sin\alpha\cos\beta + 2\sin 2\beta\cos\alpha\sin\beta \\
&\qquad - \sin^2 2\beta(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\
&\quad = 2(\sin 2\alpha + \sin 2\beta)\sin(\alpha+\beta) - \sin^2 2\beta\cos(\alpha-\beta) \\
&\quad = 2\sin(\alpha+\beta)[\sin 2\alpha + \sin 2\beta - 2\sin(\alpha+\beta)\cos(\alpha-\beta)] = 0 \quad ,
\end{aligned}$$

since

$$\sin 2\alpha + \sin 2\beta = \sin(\overline{\alpha+\beta} + \overline{\alpha-\beta}) + \sin(\overline{\alpha+\beta} - \overline{\alpha-\beta}) \ .$$

Solution 5. [A. Tang] From the given equation, we have that

$$\sin(\alpha+\beta) = -\sin\beta\cos\beta \quad ,$$

$$\frac{\cos\beta}{\cos\alpha} = \frac{-\sin\beta}{\sin\alpha+\sin\beta} \quad ,$$

and

$$\frac{\sin\beta}{\sin\alpha} = \frac{-\cos\beta}{\cos\alpha+\cos\beta} \quad .$$

Hence

$$\begin{aligned}
\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \cos^2\beta\left[\frac{-\sin\beta}{\sin\alpha+\sin\beta}\right] + \sin^2\beta\left[\frac{-\cos\beta}{\cos\alpha+\cos\beta}\right] \\
&= -\frac{\sin\beta\cos\beta[\cos\alpha\cos\beta + \sin\alpha\sin\beta + 1]}{4\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)} \\
&= \frac{\sin(\alpha+\beta)[\cos(\alpha-\beta)+1]}{[2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta)][2\cos^2\frac{1}{2}(\alpha-\beta)]} = 1 \quad .
\end{aligned}$$

Solution 6. [D. Arthur] The given equations yield $2\sin(\alpha+\beta) = -\sin 2\beta$, $\cos\alpha\sin\beta = -\cos\beta(\sin\alpha + \sin\beta)$ and $\sin\alpha\cos\beta = -\sin\beta(\cos\alpha+\cos\beta)$. Hence

$$\begin{aligned}
\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \frac{\cos^2\beta(\cos\beta\sin\alpha) + \sin^2\beta(\sin\beta\cos\alpha)}{\cos\alpha\sin\alpha} \\
&= \frac{-\cos^2\beta\sin\beta(\cos\alpha+\cos\beta) - \sin^2\beta\cos\beta(\sin\alpha+\sin\beta)}{\cos\alpha\sin\alpha} \\
&= \frac{-\cos\beta\sin\beta(\cos\alpha\cos\beta + \cos^2\beta + \sin\alpha\sin\beta + \sin^2\beta)}{\cos\alpha\sin\alpha} \\
&= \frac{-\sin 2\beta(1+\cos(\alpha-\beta))}{\sin 2\alpha} \\
&= \frac{-\sin 2\beta + 2\sin(\alpha+\beta)\cos(\alpha-\beta)}{\sin 2\alpha} \\
&= \frac{-\sin 2\beta + \sin 2\alpha + \sin 2\beta}{\sin 2\alpha} = 1 \quad .
\end{aligned}$$

Solution 7. [C. Deng] Let $\sin\beta = x$, $\cos\beta = y$, and $(\sin\alpha)/(\sin\beta) = c$. Thus, $(\cos\alpha)/(\cos\beta) = -1 - c$. We have that

$$x^2 + y^2 = 1$$

and

$$(cx)^2 + (-1-c)y)^2 = 1 \ .$$

Solving the system yields that

$$x^2 = \frac{c^2+2c}{1+2c} \ , \quad y^2 = \frac{1-c^2}{1+2c} \ .$$

Therefore,

$$\begin{aligned}\frac{\sin^3\beta}{\sin\alpha} + \frac{\cos^3\beta}{\cos\alpha} &= \frac{x^2}{c} + \frac{y^2}{-1-c} = \frac{c^2+2c}{c(2c+1)} + \frac{1-c^2}{(-c-1)(2c+1)} \\ &= \frac{c+2}{2c+1} + \frac{c-1}{2c+1} = 1 \ .\end{aligned}$$

(b) Solution. The given equation is equivalent to $2\sin(\alpha+\beta) + \sin 2\beta = 0$. Try $\beta = -45°$ so that $\sin(\alpha - 45°) = \frac{1}{2}$. We take $\alpha = 75°$. Now

$$\sin 75° = \sin(45° + 30°) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}+1}{2}\right)$$

and

$$\cos 75° = \cos(45° + 30°) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}-1}{2}\right) \ .$$

It is straightforward to check that both equations hold.

631. The sequence of functions $\{P_n\}$ satisfies the following relations:

$$P_1(x) = x \ , \qquad P_2(x) = x^3 \ ,$$

$$P_{n+1}(x) = \frac{P_n^3(x) - P_{n-1}(x)}{1 + P_n(x)P_{n-1}(x)} \ , \qquad n = 1, 2, 3, \cdots .$$

Prove that all functions P_n are polynomials.

Solution 1. Taking $x = 1, 2, 3, \cdots$ yields the respective sequences

$$\{1, 1, 0, -1, -1, 0, \cdots\} \ , \quad \{2, 8, 30, 112, 418, 1560, \cdots\} \ , \quad \{3, 27, 240, 2133, \cdots\} \ .$$

In each case, we find that

$$P_{n+1}(x) = x^2 P_n(x) - P_{n-1}(x) \tag{1}$$

for $n = 2, 3, \cdots$. If we can establish (1) in general, it will follow that all the functions P_n are polynomials.

From the definition of P_n, we find that

$$P_{n+1} + P_{n-1} = P_n(P_n^2 - P_{n+1}P_{n-1}) \ .$$

Therefore, it suffices to establish that $P_n^2 - P_{n+1}P_{n-1} = x^2$ for each n. Now, for $n \geq 2$,

$$\begin{aligned}[P_{n+1}^2 - P_{n+2}P_n] - [P_n^2 - P_{n+1}P_{n-1}] &= P_{n+1}(P_{n+1} + P_{n-1}) - P_n(P_{n+2} + P_n) \\ &= P_{n+1}P_n(P_n^2 - P_{n+1}P_{n-1}) - P_nP_{n+1}(P_{n+1}^2 - P_{n+2}P_n) \\ &= -P_{n+1}P_n[(P_{n+1}^2 - P_{n+2}P_n) - (P_n^2 - P_{n+1}P_{n-1})] \ ,\end{aligned}$$

so that either $P_{n+1}(x)P_n(x)+1 \equiv 0$ or $P_{n+1}^2 - P_{n+2}P_n = P_n^2 - P_{n+1}P_{n-1}$. The first identity is precluded by the case $x = 1$, where it is false. Hence

$$P_{n+1}^2 - P_{n+2}P_n = P_n^2 - P_{n+1}P_{n-1}$$

for $n = 2, 3, \cdots$. Since $P_2^2(x) - P_3(x)P_1(x) = x^2$, the result follows.

Solution 2. [By inspection, we make the conjecture that $P_n(x) = x^2 P_{n-1}(x) - P_{n-2}$. Rather than prove this directly from the rather awkward condition on P_n, we go through the back door.] Define the sequence $\{Q_n\}$ for $n = 0, 1, 2, \cdots$ by

$$Q_0(x) = 0 \ , \qquad Q_1(x) = x \ , \qquad Q_{n+1} = x^2 Q_n(x) - Q_{n-1}(x)$$

for $n \geq 1$. It is clear that $Q_n(x)$ is a polynomial of degree $2n-1$ for $n = 1, 2, \cdots$. We show that $P_n(x) = Q_n(x)$ for each n.

Lemma: $Q_n^2(x) - Q_{n+1}Q_{n-1} = x^2$ for $n \geq 1$.

Proof: This result holds for $n = 1$. Assume that it holds for $n = k - 1 \geq 1$. Then

$$\begin{aligned} Q_k^2(x) - Q_{k+1}(x)Q_{k-1}(x) &= Q_k^2(x) - (x^2 Q_k(x) - Q_{k-1}(x))Q_{k-1}(x) \\ &= Q_k(x)(Q_k(x) - x^2 Q_{k-1}(x)) + Q_{k-1}^2(x) \\ &= -Q_k(x)Q_{k-2}(x) + Q_{k-1}^2(x) = x^2 \ . \quad \spadesuit \end{aligned}$$

From the lemma, we find that

$$\begin{aligned} &Q_{n+1}(x) + Q_{n-1}(x) + Q_{n+1}(x)Q_n(x)Q_{n-1}(x) \\ &\quad = x^2 Q_n(x) + Q_{n+1}(x)Q_n(x)Q_{n-1}(x) = Q_n(x)(x^2 + Q_{n+1}(x)Q_{n-1}(x)) = Q_n^3(x) \end{aligned}$$

$$\Longrightarrow Q_{n+1}(x) = \frac{Q_n^3(x) - Q_{n-1}(x)}{1 + Q_n(x)Q_{n-1}(x)} \qquad (n = 1, 2, \cdots) \ .$$

We know that $Q_1(x) = P_1(x)$ and $Q_2(x) = P_2(x)$. Suppose that $Q_n(x) = P_n(x)$ for $n = 1, 2, \cdots, k$. Then

$$Q_{k+1}(x) = \frac{Q_k^3(x) - Q_{k-1}(x)}{1 + Q_k(x)Q_{k-1}(x)} = \frac{P_k^3(x) - P_{k-1}(x)}{1 + P_k(x)P_{k-1}(x)} = P_{k+1}(x)$$

from the definition of P_{k+1}. The result follows.

Comment: It can also be established that $P_{n+1}^2 + P_n^2 = (1 + P_n P_{n+1})x^2$ for each $n \geq 0$.

Solution 3. [I. Panayotov] First note that the sequence $\{P_n(x)\}$ is defined for all values of x, *i.e.*, the denominator $1 + P_{n-1}(x)P_n(x)$ never vanishes for n and x. Suppose otherwise, and let n be the least number for which there exists u for which $1 + P_{n-1}(u)P_n(u) = 0$. Then $n \geq 3$ and

$$-1 = P_{n-1}(u)P_n(u) = \frac{P_{n-1}(u)^4 - P_{n-1}(u)P_{n-2}(u)}{1 + P_{n-1}(u)P_{n-2}(u)}$$

which implies that $P_{n-1}(u)^4 = -1$, a contradiction.

We now prove by induction that $P_{n+1} = x^2 P_n - P_{n-1}$. Suppose that $P_k = x^2 P_{k-1} - P_{k-2}$ for $3 \leq k \leq n$, so that in particular we know that P_k is a polynomial for $1 \leq k \leq n$. Substituting for P_k yields

$$P_{k-1}^3(x) = P_{k-1}(x)[x^2 + x^2 P_{k-1}(x)P_{k-2}(x) - P_{k-2}^2(x)]$$

for all x. If $P_{k-1}(x) \neq 0$, then

$$P_{k-1}^2(x) = x^2 + x^2 P_{k-1}(x)P_{k-2}(x) - P_{k-2}^2(x) \ .$$

Both sides of this equation are polynomials and so continuous functions of x. Since the roots of P_{k-1} constitute a finite discreet set, this equation holds when x is one of the roots as well. Now

$$\begin{aligned}
P_{n+1} &= \frac{P_n^3 - P_{n-1}}{1 + P_n P_{n-1}} = \frac{P_n(x^2 P_{n-1} - P_{n-2})^2 - P_{n-1}}{1 + P_n P_{n-1}} \\
&= \frac{P_n(x^4 P_{n-1}^2 - x^2 P_{n-1} P_{n-2} + x^2 - P_{n-1}^2) - P_{n-1}}{1 + P_n P_{n-1}} \\
&= \frac{P_n(x^2 P_n P_{n-1} + x^2 - P_{n-1}^2) - P_{n-1}}{1 + P_n P_{n-1}} \qquad \text{since} \quad x^2 P_{n-1} - P_{n-2} = P_n \\
&= \frac{(x^2 P_n - P_{n-1})(1 + P_n P_{n-1})}{1 + P_n P_{n-1}} = x^2 P_n - P_{n-1} \quad .
\end{aligned}$$

The result follows.

632. Let a, b, c, x, y, z be positive real numbers for which $a \leq b \leq c$, $x \leq y \leq z$, $a + b + c = x + y + z$, $abc = xyz$, and $c \leq z$, Prove that $a \leq x$.

Solution. Let

$$p(t) = (t - a)(t - b)(t - c) = t^3 - (a + b + c)t^2 + (ab + bc + ca)t - abc$$

and

$$q(t) = (t - x)(t - y)(t - z) = t^3 - (x + y + z)t^2 + (xy + yz + zx)t - xyz \ .$$

Then $p(t) - q(t) = (ab + bc + ca - xy - yz - zx)t$ never changes sign for positive values of t. Since $p(t) > 0$ for $t > c$, we have that $p(z) - q(z) = p(z) \geq 0$, so that $p(t) \geq q(t)$ for all $t > 0$.

Hence, for $0 < t < a$, we have that $q(t) \leq p(t) < 0$, from which it follows that $q(t)$ has no root less than a. Hence $x \geq a$ as desired.