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Please send your solutions to

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no later than February 5, 2009.

Electronic files can be sent to *valeria.pandelieva@sympatico.ca*. However, if you respond electronically, please do not scan a handwritten solution. The attachment uses an inordinate amount of space and often causes difficulties in downloading. Also, the handwriting is frequently indistinct or splotchy. Please type the solution using a word-processing package (TeX is good and can be sent in a pdf file). If your file is not downloadable, then you will be asked to send your solutions by mail; do not use anything fancy or exotic.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

Problems for January

591. The point O is arbitrarily selected from the interior of the angle KAM. A line g is constructed through the point O, intersecting the ray AK at the point B and the ray AM at the point C. Prove that the value of the expression
$$\frac{1}{[AOB]} + \frac{1}{[AOC]}$$
does not depend on the choice of the line g. [*Note:* $[MNP]$ denotes the area of triangle MNP.]

592. The incircle of the triangle ABC is tangent to the sides BC, CA and AB at the respective points D, E and F. Points K from the line DF and L from the line EF are such that $AK \| BL \| DE$. Prove that:

   (a) the points A, E, F and K are concyclic, and the points B, D, F and L are concyclic;

   (b) the points C, K and L are collinear.

593. Consider all natural numbers M with the following properties:

   (i) the four rightmost digits of M are 2008;

   (ii) for some natural numbers $p > 1$ and $n > 1$, $M = p^n$.

   Determine all numbers n for which such numbers M exist.

594. For each natural number N, denote by $S(N)$ the sum of the digits of N. Are there natural numbers N which satisfy the condition severally:

   (a) $S(N) + S(N^2) = 2008$;

   (b) $S(N) + S(N^2) = 2009$?

595. What are the dimensions of the greatest $n \times n$ square chessboard for which it is possible to arrange 111 coins on its cells so that the numbers of coins on any two adjacent cells (*i.e.* that share a side) differ by 1?

596. A 12×12 square array is composed of unit squares. Three squares are removed from one of its major diagonals. Is it possible to cover completely the remaining part of the array by 47 rectangular tiles of size 1×3 without overlapping any of them?

597. Find all pairs of natural numbers (x, y) that satisfy the equation

$$2x(xy - 2y - 3) = (x + y)(3x + y) .$$

Solutions.

577. $ABCDEF$ is a regular hexagon of area 1. Determine the area of the region inside the hexagon thst belongs to none of the triangles ABC, BCD, CDE, DEF, EFA and FAB.

Solution 1. Let O be the centre of the hexagon. The hexagon is the union of three nonoverlapping congruent rhombi, $ABCO$, $CDEO$, $EFAO$, each of area $\frac{1}{3}$. Each rhombus is the union of two congruent triangles, each of area $\frac{1}{6}$. In particular, $[ABC] = \frac{1}{2}$.

Let BD and AC intersect at P, and BF and AC at Q. By reflection about BE, we see that $BP = BQ$, triangle BPQ is equilateral and $\angle BPQ = 60°$. Since triangle BPC is isosceles (use symmetry) and $\angle BPC = 120°$, $CP = PB = PQ = BQ = QA$. Therefore $[BPC] = [BPQ] = [BQA] = \frac{1}{3}[ABC] = \frac{1}{18}$.

The union of triangles ABC, BCD, CDE, DEF, EFA, FAB is comprised of twelve nonoverlapping triangles congruent to either of the triangles BPC or BPQ, as so has area $\frac{2}{3}$. Therefore the area of the prescribed region inside the hexagon is $\frac{1}{3}$.

Solution 2. Let O be the centre of the hexagon. Since triangle ACE is the union of triangles OAC, OCE, OEA, and since $[OAC] = [BAC]$, $[OCE] = [DCE]$, $[OEA] = [FEA]$, it follows that $[ACE] = \frac{1}{2}[ABCDEF] = \frac{1}{2}$. As in Solution 1, we determine that $[BPQ] = [DUT] = [FRS] = \frac{1}{18}$, where $U = BD \cap CE$, $T = CE \cap DF$, $S = DF \cap EA$, $R = AE \cap BF$. Hence the area of the inner region is $\frac{1}{2} - (3 \times \frac{1}{18}) = \frac{1}{3}$.

Solution 3. [T. Bappi] The inner figure is a regular hexagon. To see this, consider a rotation of $60°$ about the centre of the given hexagon that takes

$$A \rightarrow F \rightarrow E \rightarrow D \rightarrow C \rightarrow B \rightarrow A .$$

Then

$$AE \rightarrow FD \rightarrow EC \rightarrow DB \rightarrow CA \rightarrow BF \rightarrow AE ,$$

so that the rotation takes the intersections of each adjacent pair of these chords to the intersection of the next adjacent pair, and thus each vertex of the inner figure to the next.

Let AC intersect BD and BF at P and Q respectively. Then PQ is a side of the inner hexagon and triangle BPQ is equilateral. By the Law of Sines,

$$PQ : BC = BP : BC = \sin\angle 30° : \sin\angle 120° = 1/2 : \sqrt{3}/2 ,$$

so that the inner hexagon is similar to the given hexagon with factor $1/\sqrt{3}$. Hence its area must be $1/3$.

Solution 4. [D. Hidru] Let BD and BF intersect AC at P and Q respectively. Using a symmetry argument, we can establish that $CP = BP = BQ = QA$ and that triangle BDF is equilateral. [Do this.] Since $\angle PBQ = 60°$, triangle BPQ is equilateral and $CP = PQ = QA$. Therefore, $[BCP] = [BPQ] = [BQA]$. The diagonals of hexagon $ABCDEF$ trisect the sides of triangle BDF, so that this triangle can be partitioned into nine equilateral triangles congruent to triangle BPQ, six of which make up the inner region whose area is to be found.

The hexagon $ABCDEF$ is the nonoverlapping union of 18 triangles of equal area, six of which are congruent to triangle BCP and twelve of which are congruent to triangle BPQ. It follows that the inner region has area $1/3$.

Solution 5. [J. Zung] By symmetry, the inner region is a regular hexagon. Let R and r be the respective inradii of the given and inner hexagons, and let O be their common centre. Triangle OAB and OBC are equilateral and $OB \perp AC$; hence OB and AC right bisect each other. Therefore, OAB is an equilateral triangle with side length equal to $2r$ and altitude R, so that $R/2r = \sqrt{3}/2$ and $r/R = 1/\sqrt{3}$. Thus, the inner and outer regular hexagons are similar with factor $1/\sqrt{3}$ and the area of the inner figure is $1/3$.

Comment. In the original statement of the problem, triangle DEF was omitted by mistake from the statement. In this case, the region whose area was to be found is the union of $PQRSTU$ and one of the twelve small triangles; the answer is $1/3 + 1/18 = 7/18$.

578. $ABEF$ is a parallelogram; C is a point on the diagonal AE and D a point on the diagonal BF for which $CD \| AB$. The segments CF and EB intersect at P; the segments ED and AF intersect at Q. Prove that $PQ \| AB$.

Solution. Consider the shear that fixes A and B and shifts E in a parallel direction to E' so that $E'B \perp AB$. This shear preserves parallelism and takes $F \rightarrow F'$, $C \rightarrow C'$, $D \rightarrow D'$, $P \rightarrow P'$, $Q \rightarrow Q'$, so that $ABE'F'$ is a rectangle. A reflection about the right bisector of AB takes $E' \leftrightarrow F'$, $C' \leftrightarrow D'$, and so $P" \leftrightarrow Q'$. Hence $PQ \| P'Q' \| AB$.

579. Solve, for real x, y, z the equation

$$\frac{y^2+z^2-x^2}{2yz} + \frac{z^2+x^2-y^2}{2zx} + \frac{x^2+y^2-z^2}{2xy} = 1 \ .$$

Solution 1. Note that none of x, y, z can vanish. We have that

$$\begin{aligned}
0 &= \frac{y^2+z^2-x^2}{2yz} = \frac{z^2+x^2-y^2}{2zx} + \frac{x^2+y^2-z^2}{2xy} - 1 \\
&= \frac{xy^2 + xz^2 - x^3 + yz^2 + x^2y - y^3 + x^2z + y^2z - z^3 - 2xyz}{2xyz} \\
&= \frac{(x+y-z)(xy+z^2) + (x^2+y^2-xy)z - (x^2+y^2-xy)(x+y)}{2xyz} \\
&= \frac{(x+y-z)(z^2-(x-y)^2)}{2xyz} = \frac{(x+y-z)(z+x-y)(z+y-x)}{2xyz} \ ,
\end{aligned}$$

whereupon (x, y, z) is a solution if and only if one of the conditions $x + y = z$, $y + z = x$ and $z + x = y$ is satisfied.

Solution 2. We must have $xyz \neq 0$ for the equation to be defined. Suppose that a, b, c are such that $y^2 + z^2 - x^2 = 2ayz$, $z^2 + x^2 - y^2 = 2bzx$, $x^2 + y^2 - z^2 = 2cxy$. Then $a + b + c = 1$. Adding pairs of the three equations yields that

$$\begin{aligned}
2z^2 &= 2z(ay + bx) \ , \\
2y^2 &= 2y(az + cx) \ , \\
2x^2 &= 2x(bz + cy) \ .
\end{aligned}$$

Hence

$$\begin{aligned}
bx + ay - z &= 0 \ , \\
cx - y + az &= 0 \ ,
\end{aligned}$$

$$-x + cy + bz = 0 \ .$$

From the first two equations, we find that

$$x : y : z = (a^2 - 1) : (-c - ab) : (-b - ac) \ .$$

Plugging this into the third equation yields that

$$1 - a^2 - c^2 - abc - b^2 - abc = 0 \Longrightarrow a^2 + b^2 + c^2 = 1 - 2abc$$

$$\Longrightarrow 1 - 2(ab + bc + ca) = (a + b + c)^2 - 2(ab + bc + bc) = 1 - 2abc$$

$$\Longrightarrow ab + bc + ca = abc \Longrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 \ ,$$

the last implication holding only if a, b, c are all nonzero.

But if any of a, b, c vanish, then two of them must vanish. Suppose that $a = b = 0$, $c = 1$. Then $z^2 = x^2 - y^2 = y^2 - x^2 = (x - y)^2$. This is impossible as $z \neq 0$.

Therefore

$$\frac{a+b}{ab} = \frac{1}{a} + \frac{1}{b} = 1 - \frac{1}{c} = \frac{c-1}{c} = \frac{-(a+b)}{c} \ .$$

Therefore, either $a + b = 0$ or $ab = -c$. Similarly, either $b + c = 0$ or $bc = -a$, and either $c + a = 0$ or $ca = -b$. It is not possible for all of $a + b = 0$, $b + c = 0$ and $c + a = 0$ to occur.

Suppose wolog, $ab = -c$. If $b + c = 0$, then $a = 1$ and $ac = -b$. The condition $a = 1$ implies that $x^2 = (y - z)^2$, whence either $x + y = z$ or $x + z = y$ (which leads to $c = 1$ or $b = 1$).

If $ab = -c$, $bc = -a$, $ca = -b$, then $(abc)^2 = -abc$, so that $a^2 = b^2 = c^2 = -abc = 1$, whence $(a, b, c) = (1, 1, -1), (1, -1, 1), (-1, 1, 1)$.

In any case, two of a, b, c equal 1 and one of them equals -1. If, say $(a, b, c) = (1, 1, -1)$, then $x^2 - (y - z)^2 = y^2 - (z - x)^2 = z^2 - (x + y)^2 = 0$, whence

$$0 = (x - y + z)(x + y - z) = (y - z + x)(y + z - x) = (z - x - y)(x + y + z) \ .$$

The solutions $x + y + z = 0$ is not possible; otherwise

$$\frac{y^2 + z^2 - x^2}{2yz} + \frac{z^2 + x^2 - y^2}{2zx} + \frac{x^2 + y^2 - z^2}{2xy} = \frac{-2yz}{2yz} + \frac{-2zx}{2zx} + \frac{-2xy}{2xy} = -3 \ .$$

Therefore $x + y - z = 0$. Similarly, if $(a, b, c) = (1, -1, 1)$, then $z + x - y = 0$, and if $(a, b, c) = (-1, 1, 1)$, then $y + z - x$. it is readily checked that these solutions work.

Comments. N. Lvov reworked the equation to the equivalent

$$(x + y + z)[x(y + z - x) + y(z + x - y) + z(x + y - z)] = 8xyz$$

which in turn is equivalent to

$$(x + y + z)[4(xy + yz + zx) - (x + y + z)^2] = 8xyz \ .$$

If we take x, y, z to the roots of the cubic equation $t^3 - at^2 + bt - c = 0$ and we find that the condition implies that $8c = a[4b - a^2]$. It remains to show that this condition on the coefficients characterizes those cubic equations for which one root is the sum of the other two. There does not seem to be a convenient way of doing this without essentially having to start from scratch and go through the workings of the given solutions. As an exercise, you might wish to show that the coefficients of any cubic equation for which one root is the sum of the other two satisfies the condition on c; the converse seems not so straightforward.

Many solvers laid out their solutions as a sucession of equations obtained from manipulating their predecessors. It is extremely important to indicate that each equation in the sequence is logically equivalent to its predecessor, for you must be assured that the solutions of the last one are exactly the solutions of the equations that have to be solved. Failure to do this gives an incomplete solution. However, this sort of layout is to be avoided where possible. It is better, as in Solution 1, to set the equation up so that you can simply manipulate one side into a form that allows you to read off the solutions. This makes for a shorter, clearer and more efficient solution. Always edit your solution to put it into the best form.

580. Two numbers m and n are two perfect squares with four decimal digits. Each digit of m is obtained by increasing the corresponding digit of n be a fixed positive integer d. What are the possible values of the pair (m, n).

Solution. Let

$$n = y^2 = p \times 10^3 + q \times 10^2 + r \times 10 + s$$

and

$$m = x^2 = (p + d) \times 10^3 + (q + d) \times 10^2 + (r + d) \times 10 + (s + d) \ ,$$

where $1 \leq p < p + d \leq 9$, $0 \leq q < q + d \leq 9$, $0 \leq r < r + d \leq 9$, $0 \leq s < s + d \leq 9$. Then

$$(x + y)(x - y) = x^2 - y^2 = d \times 1111 = d \times 11 \times 101 \ .$$

Since $10^3 \leq n < m < 10^4$, $32 \leq y < x \leq 99$, it follows that $x + y < 200$ and $x - y \leq 67$. Since the prime 101 must be a factor of either $x + y$ or $x - y$ and since each multiple of 101 exceeds 200, we must have that $x + y = 101$ and $x - y = 11d$. Since x and y must have opposite parity, d must be odd.

Since $64 \leq 2y = 101 - 11d$, $11d \leq 37$, so that $d \leq 3$. Therefore, either $d = 1$ or $d = 3$. The case $d = 1$ leads to $x + y = 101$ and $x - y = 11$, so that $(x, y) = (56, 45)$ and $(m, n) = (3136, 2025)$. The case $d = 3$ leads to $x + y = 101$ and $x - y = 33$, so that $(x, y) = (67, 34)$ and $(m, n) = (4489, 1156)$.

Thus, there are two possibilities for (m, n): $(3136, 2025)$, $(4489, 1156)$.

581. Let $n \geq 4$. The integers from 1 to n inclusive are arranged in some order around a circle. A pair (a, b) is called *acceptable* if $a < b$, a and b are not in adjacent positions around the circle and at least one of the arcs joining a and b contains only numbers that are less than both a and b. Prove that the number of acceptable pairs is equal to $n - 3$.

Solution 1. We prove the result by induction. Let $n = 4$. If 2 and 4 are not adjacent, then $(2, 4)$ is acceptable. If 2 and 4 are adjacent, then 1 must be between 3 and one of 2 and 4, in which case $(2, 3)$ or $(3, 4)$ is the only acceptable pair.

Suppose that $n \geq 5$, that the result holds for $n - 1$ numbers and that a configuration of the numbers 1 to n, inclusive is given. The number 1 must lie between two immediate neighbours u and v that are non-adjacent. Thus, the pair (u, v) is acceptable.

Now remove the number 1 and replace each remaining number r by $r' = r - 1$ to obtain a configuration of $n - 1$ numbers. We show that (r', s') is acceptable in the latter configuration if and only if (r, s) is acceptable in the given configuration.

If (r', s') is acceptable, then r' and s' are not adjacent and there is an arc of smaller numbers between them. The addition of 1 to these numbers and the insertion of 1 will not change either characteristic for (r, s). On the other hand, if $(r, s) \neq (u, v)$ is acceptable in the original configuration , then r and s are not adjacent and each arc connecting them must contain some number other than 1; one of these arcs, at least, contains only numbers less than both r and s. In the final configuration, r' and s' continue to be non-adjacent and a corresponding arc contains only numbers less than both of them.

By the induction hypothesis, there are $(n - 1) - 3 = n - 4$ acceptable pairs in the latter configuration, and so, with the inclusion of (u, v), there are $(n - 4) + 1 = n - 3$ acceptable pairs in the given configuration.

Solution 2. We formulate the more general result that, if $n \geq 3$ and any n distinct real numbers are arranged in a circle and acceptability of pairs is defined as in the problem, then there are precisely $n-3$ acceptable pairs. This is equivalent to the given problem, since there is an order-preserving one-one mapping from these numbers to $\{1, 2, \cdots, n\}$ that takes the kth largest of them to k.

We use induction. As in the previous solution, we see that it is true for $n = 3$ and $n = 4$. Let $n \geq 5$ and suppose that the largest three numbers are u, v, w. At least one of these three pairs is non-adjacent; otherwise, if w is adjacent to both u and v, then w is between u and v; since u and v are separated on both sides by at least one number, they are non-adjacent. This pair is acceptable, since a larger number can appear on at most one of the arcs connecting them.

Suppose that this acceptable pair is (u, v). Since all the numbers in at least one of the arcs connecting them are smaller, there is no acceptable pair (a, b) with a and b on different arcs joing u and v.

Consider two "circles" of numbers consisting of the $k \geq 3$ numbers of one arc A determined by (u, v) including u and v, and the $n + 2 - k$ numbers of the other arc B determined by (u, v) including u and v.

The set A contains exactly $k - 3$ acceptable pairs and the set B $n - 1 - k$ acceptable pairs, by the induction hypothesis. Each of these pairs is acceptable in the original circle of n numbers since none of the acceptable arcs includes u and v. Therefore, the original circle has $1+(k-3)+(n-1-k) = n-3$ acceptable pairs.

Solution 3. [C. Bruggeman] Suppose that $1 \leq k \leq n - 3$. Examine numbers counterclockwise from k until the first number a that exceeds k is reached; the examine numbers clockwise from k until the first number b that exceeds k is reached. Every number of the arc containing k between a and b is less than both a and b. Since there are at least three numbers exceeding k, at least one of them must be between a and b outside the arc containing k, so that a and b are not adjacent. Hence (a, b) is an acceptable pair.

We now prove that every acceptable pair is obtained exactly once in this way. Suppose that (a, b) is an acceptable pair with at least one of a and b not equal to $n - 1$ and n. Then, as one of the arcs between a and b must contain a number h bigger than at least one of them, the other arc must contain only numbers smaller than both of them. Let the largest such number be m. The m must engender the pair (a, b) by the foregoing process. Suppose that $k \leq n - 3$ is some other number other than a, b and m. Then m must lie on the arc between a and b opposite h between a and m or between m and b, or on the arc between a and b opposite m between a and h or between h and b; in each case, the pair engendered by k cannot be (a, b).

The only case remaining is $(n - 1, n)$ which may or may not be acceptable. If $(n - 1, n)$ is acceptable, then one arc connecting it must contain $n - 2$; by an argument similar to that in the last paragraph, no other element in this arc can engender $(n - 1, n)$. However, the largest element m in the other arc does not exceed $n - 3$ and it is the sole element that engenders $(n - 1, n)$.

Thus, there is a one-one correspondence between the numbers $1, 2, \cdots, n - 3$ and acceptable pairs; the desired result follows.

Comment. A. Abdi claims that the acceptable pair determine diagonals yielding a triangulation of the $n-$gon determined by the positions of the n numbers. Is this true?

582. Suppose that f is a real-valued function defined on the closed unit interval $[0, 1]$ for which $f(0) = f(1) = 0$ and $|f(x) - f(y)| < |x - y|$ when $0 \leq x < y \leq 1$. Prove that $|f(x) - f(y)| < \frac{1}{2}$ for all $x, y \in [0, 1]$. Can the number $\frac{1}{2}$ in the inequality be replaced by a smaller number and still result in a true proposition?

Solution 1. Suppose that $0 \leq x < y \leq 1$. If $y - x < \frac{1}{2}$, the result holds trivially. Suppose that $y - x \geq \frac{1}{2}$. Then

$$|f(y) - f(x)| \leq |f(1) - f(y)| + |f(x) - f(0)|$$
$$< (1 - y) + x = 1 - (y - x) \leq \frac{1}{2} ,$$

as desired.

The coefficient $\frac{1}{2}$ cannot be replaced by anything smaller. Suppose that $0 < \lambda < 1$; define

$$f_\lambda = \begin{cases} \lambda x & \text{if } 0 \le x \le \frac{1}{2} \\ \lambda(1-x) & \text{if } \frac{1}{2} < x \le 1 \ . \end{cases}$$

We show that f_λ has the desired property. However, note that $f_\lambda(\frac{1}{2}) - f_\lambda(0) = \frac{\lambda}{2}$, so that by our choice of λ, we can make the right side arbitrarily close to $\frac{1}{2}$.

If $0 \le x < y \le \frac{1}{2}$, then

$$|f(x) - f(y)| = \lambda|x - y| < |x - y| \ .$$

If $\frac{1}{2} \le x < y \le 1$, then

$$|f(x) - f(y)| = \lambda|(1-x) - (1-y)| = \lambda|x - y| < |x - y| \ .$$

Finally, suppose that $0 \le x < \frac{1}{2} < y \le 1$. Then

$$|f_\lambda(x) - f_\lambda(y)| = \lambda|x - (1-y)| = \lambda|(x+y) - 1| \ .$$

If $x + y \ge 1$, then

$$|(x+y) - 1| = (x+y) - 1 = (y-x) - (1-2x) < y - x \ ;$$

if $x + y \le 1$, then

$$|(x+y) - 1| = 1 - (x+y) = (y-x) - (2y-1) < y - x \ .$$

In either case

$$|f_\lambda(x) - f_\lambda(y)| < \lambda(y-x) < y - x = |x - y| \ .$$

Solution 2. [J. Zung] Let $0 \le x < y \le 1$. Then $|f(x)| \le x$, $|f(y) - f(x)| < y - x$ and $|f(y)| = |f(1) - f(y)| \le 1 - y$. Therefore, adding these three inequalities gives us that

$$\begin{aligned} 2|f(y) - f(x)| &= |f(y) - f(x)| + |f(y) - f(x)| \\ &\le |f(y)| + |f(x)| + |f(y) - f(x)| < (1-y) + x + (y-x) = 1 \ , \end{aligned}$$

so that $|f(y) - f(x)| < \frac{1}{2}$.

Solution 3. Since $|f(x) - f(y)| < |x - y|$, f must be continuous on $[0,1]$. [Provide an $\epsilon - \delta$ argument for this.] Therefore it assumes its maximum value M at a point $v \in [0,1]$ and its minimum value m at a point $u \in [0,1]$. We have that

$$0 \le M = f(v) = f(v) - f(0) < v \le 1$$

and

$$0 \ge m = f(u) = f(u) - f(0) > -u \ge 1 \ ,$$

since $|f(u) - f(0)| < |u - 0| = u$. Thus $|m| < u \le 1$ and $M < v \le 1$.

Suppose that $v < u$. Then

$$\begin{aligned} 2(M - m) &= M - m + (f(v) - f(u)) \\ &= f(v) + (f(1) - f(u)) + |f(u) - f(v)| \\ &< v + (1-u) + (u-v) = 1 \ . \end{aligned}$$

Suppose that $u < v$. Then

$$\begin{aligned} 2(M - m) &= M - m + (f(v) - f(u)) \\ &= |f(1) - f(v)| + |f(u)| + |f(v) - f(u)| \\ &< (1-v) + u + (v-u) = 1 \ . \end{aligned}$$

In either case, $M - m < \frac{1}{2}$. If $x, y \in [0,1]$, then $f(x)$ and $f(y)$ both lie in $[m, M]$ and so $|f(x) - f(y)| \leq M - m < \frac{1}{2}$.

Comments. The hard part of this problem was the last past, replacing the term $\frac{1}{2}$. To get a clear idea of what is being asked, reformulate the result explicitly that you want to examine: *Let f satisfy $f(0) = f(1) = 0$ and $|f(x) - f(y)| < |x - y|$ for all x, y. Prove that $|f(x) - f(y)| < \lambda$, where λ is a parameter less than $\frac{1}{2}$.* Either this result is true or it is false. If true, then you try to modify the proof given in the earlier part of the solution to see if you can get one that works. However, if it is false, you can demonstrate this by producing a *counterexample*, that is, a function f for which the hypothesis is true, but the conclusion is false. Such a counterexample would satisfy these conditions: (1) $f(0) = f(1) = 0$; (2) $|f(x) - f(y)| < |x - y|$ for $x, y \in [0,1]$; (3) **there exists** $u, v \in [0,1]$ for which $|f(u) - f(v)| > \lambda$. Note that the conclusion in the problem has the form "for all x, y" while its contradition has the form "there exists x, y".

Many solvers tried to set the solution to the first part of the problem up as a contradiction. This is completely unnecessary and complicated the solution. In many cases, the contradiction could be avoided by deleting the contradiction hypothesis and let the conclusion stand. While it is natural in solving a problem to look at it from a contradiction point of view, when you write up your solution, see if you can edit it to avoid the contradiction. The way you think about solving a problem is not always the best or most efficient way to write up the solution.

583. Suppose that $ABCD$ is a convex quadrilateral, and that the respective midpoints of AB, BC, CD, DA are K, L, M, N. Let O be the intersection point of KM and LN. Thus $ABCD$ is partitioned into four quadrilaterals. Prove that the sum of the areas of two of these that do not have a common side is equal to the sum of the areas of the other two, to wit

$$[AKON] + [CMOL] = [BLOK] + [DNOM] \ .$$

Solution 1. Using the fact that triangles with equal bases and heights have the same area, we have that $[AKO] = [BKO]$, $[BLO] = [CLO]$, $[CMO] = [DMO]$ and $[DNO] = [ANO]$. Therefore

$$\begin{aligned}[AKON] + [CMOL] &= [AKO] + [ANO] + [CLO] + [CMO] \\ &= [BKO] + [BLO] + [DNO] + [DMO] = [BLOK] + [DNOM] \ .\end{aligned}$$

Solution 2. Observe that $KN \| BD \| LM$ and $KL \| AC \| NM$, so that $KLMN$ is a parallelogram and O is the intersection of its diagonals. Therefore KM and LN bisect each other, and $[KOL] = [LOM] = [MON] = [NOK]$.

Since triangle BKL and BAC are similar with factor $\frac{1}{2}$, $[BKL] = \frac{1}{4}[BAC]$. Similarly, $[DMN] = \frac{1}{4}[DAC]$, whence $[BKL] + [DMN] = \frac{1}{4}[ABCD]$. Likewise, $[CLM] + [ANK] = \frac{1}{4}[ABCD]$.

Therefore

$$\begin{aligned}[AKON] + [CMOL] &= [ANK] + [NOK] + [CLM] + [LPM] = [ANK] + [CLM] + [NOK] + [LPM] \\ &= \frac{1}{4}[ABCD] + [KOL] + [MON] = [BKL] + [KOL] + [DMN] + [MON] \\ &= [BLOK] + [DNOM] \ .\end{aligned}$$