OLYMON

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Please send your solution to

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It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

535. Let the triangle ABC be isosceles with $AB = AC$. Suppose that its circumcentre is O, the D is the midpoint of side AB and that E is the centroid of triangle ACD. Prove that OE is perpendicular to CD.

536. There are 21 cities, and several airlines are responsible for connections between them. Each airline serves five cities with flights both ways between all pairs of them. Two or more airlines may serve a given pair of cities. Every pair of cities is serviced by at least one direct return flight. What is the minimum number of airlines that would meet these conditions?

537. Consider all 2×2 square arrays each of whose entries is either 0 or 1. A pair (A, B) of such arrays is *compatible* if there exists a 3×3 square array in which both A and B appear as 2×2 subarrays.

For example, the two matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are compatible, as both can be found in the array

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

Determine all pairs of 2×2 arrays that are not compatible.

538. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the right bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have the same area.

539. Determine the maximum value of the expression

$$\frac{xy + 2yz + zw}{x^2 + y^2 + z^2 + w^2}$$

over all quartuple of real numbers not all zero.

540. Suppose that, if all planar cross-sections of a bounded solid figure are circles, then the solid figure must be a sphere.

541. Prove that the equation
$$x_1^{x_1} + x_2^{x_2} + \cdots + x_k^{x_k} = x_{k+1}^{x_{k+1}}$$
has no solution for which x_1, x_2, $\cdots$, x_k, x_{k+1} are all distinct nonzero integers.

Solutions

528. Let the sequence $\{x_n : n = 0, 1, 2, \cdots\}$ be defined by $x_0 = a$ and $x_1 = b$, where a and b are real numbers, and by
$$7x_n = 5x_{n-1} + 2x_{n-2}$$
for $n \geq 2$. Derive a formula for x_n as a function of a, b and n.

Solution. This can be done by the standard theory of solving linear recursions. The auxiliary equation is $7t^2 - 5t - 2 = 0$, with roots 1 and $-2/7$. Trying a solution of the form $x_n = A \cdot 1^n + B(-2/7)^n$ and plugging in the initial conditions leads to $A + B = a$ and $A - (2/7)B = b$ and the solution
$$x_n = \frac{2a + 7b}{9} + \frac{7(a - b)}{9} \cdot \left(-\frac{2}{7} \right)^n .$$

529. Let k, n be positive integers. Define $p_{n,1} = 1$ for all n and $p_{n,k} = 0$ for $k \geq n + 1$. For $2 \leq k \leq n$, we define inductively
$$p_{n,k} = k(p_{n-1,k-1} + p_{n-1,k}) .$$
Prove, by mathematical induction, that
$$p_{n,k} = \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k - r)^n .$$

Solution. Let
$$q_{n,k} = \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k - r)^n .$$
When $n = 1$, we have that $q_{1,1} = \binom{1}{0} 1 = 1$ and, for $k \geq 2$,
$$q_{1,k} = \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r k - \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r r = k[(1-1)^k - (-1)^k] + k[(1-1)^{k-1} - (-1)^{k-1}] = 0 .$$
Also $q_{n,1} = \binom{1}{0} 1^n = 1$ for $n \geq 1$. When $(n, k) = (2, 2)$, we have that $q_{2,2} = \binom{2}{0} 2^2 - \binom{2}{1} 1 = 2$ and $p_{2,2} = 2(1 + 0) = 2$. When $n = 2$ and $k \geq 3$, then
$$\begin{aligned} q_{2,k} &= \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k - r)^2 \\ &= \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r [k^2 - (2k - 1)r + r(r - 1)] \\ &= k^2 \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r + (2k - 1)k \sum_{r=0}^{k-1} \binom{k-1}{r-1} (-1)^{r-1} + k(k - 1) \sum_{r=0}^{k-1} \binom{k-2}{r-2} (-1)^{r-2} \\ &= (-1)^{k-1} k^2 + (2k - 1)k(-1)^{k-2} + k(k - 1)(-1)^{k-3} \\ &= (-1)^{k-3} [k^2 - 2k^2 + k + k^2 - k] = 0 . \end{aligned}$$

Thus, we have that $p_{n,k} = q_{n,k}$ for all n and $k = 1$ as well as for $n = 1, 2$ and all k.

The remainder of the argument can be done by induction. Suppose that $n \geq 2$ and that $k \geq 2$ and that it has been shown that $p_{n,k} = q_{n,k}$ and $p_{n,k-1} = q_{n,k-1}$. Then

$$\begin{aligned}
p_{n+1,k} &= k(p_{n,k} + p_{n,k-1}) \\
&= k\left[\sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k-r)^n + \sum_{r=0}^{k-2} \binom{k-1}{r} (-1)^r (k-1-r)^n\right] \\
&= k\left[k^n + \sum_{r=1}^{k-1} \binom{k}{r} (-1)^r (k-r)^n + \sum_{r=1}^{k-1} \binom{k-1}{r-1} (-1)^{r-1} (k-r)^n\right] \\
&= k\left[k^n + \sum_{r=1}^{k-1} \left[\binom{k}{r} - \binom{k-1}{r-1}\right] (-1)^r (k-r)^n\right] \\
&= k^{n+1} + k\sum_{r=1}^{k-1} \binom{k-1}{r} (-1)^r (k-r)^n \\
&= k^{n+1} + \sum_{r=1}^{k-1} \binom{k}{r} (-1)^r (k-r)^{n+1} \\
&= \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k-r)^{n+1} = q_{n+1,k} \ ,
\end{aligned}$$

as desired.

530. Let $\{x_1, x_2, x_3, \cdots, x_n, \cdots\}$ be a sequence is distinct positive real numbers. Prove that this sequence is a geometric progression if and only if

$$\frac{x_1}{x_2} \sum_{k=1}^{n-1} \frac{x_n^2}{x_k x_{k+1}} = \frac{x_n^2 - x_1^2}{x_2^2 - x_1^2}$$

for all $n \geq 2$.

Solution. Necessity. Suppose that $x_k = ar^{k-1}$ for some numbers a and r. Then

$$\begin{aligned}
\frac{x_1}{x_2} \sum_{k=1}^{n-1} \frac{x_n^2}{x_k x_{k+1}} &= \frac{r^{2(n-1)}}{r} \sum_{k=1}^{n-1} \frac{1}{r^{2k-1}} \\
&= (r^{2n-3})\left(\frac{1}{r} + \frac{1}{r^3} + \cdots + \frac{1}{r^{2n-3}}\right) \\
&= 1 + r^2 + \cdots + r^{2(n-2)} = \frac{r^{2(n-1)} - 1}{r^2 - 1} = \frac{x_n^2 - x_1^2}{x_2^2 - x_1^2} \ .
\end{aligned}$$

Sufficiency. Suppose that the equations of the problem holds. When $n = 2$, both sides of the equation are equal to 1 regardless of the sequence. When $n = 3$, the equation is equivalent to

$$\frac{x_1 x_3^2}{x_1 x_2^2 x_3}(x_3 + x_1) = \frac{(x_3 - x_1)(x_3 + x_1)}{x_3^2 - x_1^2} \ .$$

Since $x_3 + x_1 \neq 0$ [why?], we can divide out this factor and multiply up the denominators to get the equivalent

$$x_3(x_2^2 - x_1^2) = x_2^2(x_3 - x_1) \iff x_3 x_1^2 = x_2^2 x_1 \iff x_1 x_3 = x_2^2 \ ,$$

whence x_1, x_2, x_3 are in geometric progression.

Suppose, as an induction hypothesis, for $n \geq 4$ we know that $x_k = ar^{k-1}$ for suitable a and r and $k = 1, 2, \cdots, n-1$. Let $x_n = au_n$ for some number u_n.

Then

$$\frac{u_n^2}{r}\left(\frac{1}{r} + \frac{1}{r^3} + \cdots + \frac{1}{r^{2n-5}} + \frac{1}{r^{n-2}u_n}\right) = \frac{u_n^2 - 1}{r^2 - 1}$$

$$\iff [u_n^2(1 + r^2 + r^4 + \cdots + r^{2n-6}) + r^{n-3}u_n](r^2 - 1) = (u_n^2 - 1)(r^{2n-4})$$

$$\iff (r^{2n-4} - 1)u_n^2 + (r^{n-3}r^2 - r^{n-3})u_n = (r^{2n-4})u_n^2 - r^{2n-4}$$

$$\iff 0 = u_n^2 - (r^{n-1} - r^{n-3})u_n - r^{2n-4} = (u_n - r^{n-1})(u_n + r^{n-3}) \ .$$

The case $u_n = -r^{n-3}$ is rejected because of the condition that the sequence consists of positive terms. Hence $u_n = r^{n-1}$, as desired. The result follows.

Comment. In the absence of the positivity contiion, the second root of the quadratic can be used. For example, the finite sequences $\{1, r, r^2, -r, 1\}$ and $\{1, r, r^2, -r, -r^2\}$ both satisfies the equations for $2 \leq n \leq 5$. It would be interesting to investigate the situation further.

531. Show that the remainder of the polynomial

$$p(x) = x^{2007} + 2x^{2006} + 3x^{2005} + 4x^{2004} + \cdots + 2005x^3 + 2006x^2 + 2007x + 2008$$

is the same upon division by $x(x+1)$ as upon division by $x(x+1)^2$.

Solution 1. We have that

$$\begin{aligned} p(x) &= (x^{2007} + 2x^{2006} + x^{2005}) + 2(x^{2005} + 2x^{2004} + x^{2003}) + 3(x^{2003} + \\ &\qquad 2x^{2002} + x^{2001}) + \cdots + 1003(x^3 + 2x^2 + x) + 1004x + 2008 \\ &= x(x+1)^2(x^{2004} + 2x^{2002} + 3x^{2000} + \cdots + 1003) + (1004x + 2008) \ , \end{aligned}$$

from which the result follows with remainder $1004x + 2008$.

532. The angle bisectors BD and CE of triangle ABC meet AC and AB at D and E respectively and meet at I. If $[ABD] = [ACE]$, prove that $AI \perp ED$. Is the converse true?

Solution. Observe that

$$[ADB] : [CBD] = AD : DC = AB : BC$$

and that

$$[ACE] : [BCE] = AE : EB = AC : BC \ .$$

Now

$$\begin{aligned} [ABD] = [ACE] &\iff [DBC] = [ABC] - [ABD] = [ABC] - [ACE] = [EBC] \\ &\iff ED \| BC \iff AE : EB = AD : DC \\ &\iff AB : BC = AC : BC \iff AB = BC \\ &\iff AI \perp BC \ . \end{aligned}$$

Both the result and the converse is true. If $[ABD] = [ACE]$, the foregoing chain of implications can be read in the forward direction to deduce that $AI \perp ED$. Note that AI bisects angle A in triangle AED. Thus, if $AI \perp ED$, then it follows that triangle AED is isosceles with $AE = AD$. Then $AE : DC = AD : DC = AB : BC$ and $AE : EB = AC : BC$, whence $DC \cdot AB = AE \cdot BC = EB \cdot AC$. Therefore $DC \cdot (AE + EB) = EB \cdot (AD + CD)$, so that $DC \cdot AE = EB \cdot AD$ and $DC = EB$. Therefore $AB = AC$ and, following the foregoing implication in the backwards direction, we find that $[ABD] = [ACE]$.

533. Prove that the number

$$1 + \lfloor (5+\sqrt{17}))^{2008} \rfloor$$

is divisible by 2^{2008}.

Solution. Let $a = 5+\sqrt{17}$ and $b = 5-\sqrt{17}$, so that $a+b = 10$ and $ab = 8$. Define $x_n = a^n + b^n$. Then $x_1 = 10$, $x_2 = (a+b)^2 - 2ab = 96$ and

$$\begin{aligned} x_{n+2} &= a^{n+2} + b^{n+2} = (a+b)(a^{n+1}+b^{n+1}) - ab(a^n + b^n) \\ &= 10x_{n+1} - 8x_n \ , \end{aligned}$$

for $n \geq 0$. Note that x_1 is divisible by 2 and x_2 by 4. Suppose, as an induction hypothesis, that $x_n = 2^n u$ and $x_{n+1} = 2^{n+1}v$, for some $k \geq 0$ and integers u and v. Then

$$x_{n+2} = 5 \cdot 2^{n+2} - 2^{n+3} = 3 \cdot 2^{n+2} \ .$$

Hence, for all positive integers n, 2^n divides x_n.

Observe that $(5-\sqrt{17})^n = b^n < 1$ for each positive integer n and that $a^n + b^n$ is a positive integer. Therefore $x_n = a^n + b^n > a^n > a^n + b^n - 1 = x_n - 1$, whence $x_n = 1 + \lfloor a^n \rfloor$ and the result follows.

534. Let $\{x_n : n = 1, 2, \cdots\}$ be a sequence of distinct positive integers, with $x_1 = a$. Suppose that

$$2\sum_{k=1}^{n} \sqrt{x_i} = (n+1)\sqrt{x_n}$$

for $n \geq 2$. Determine $\sum_{k=1}^{n} x_k$.

Solution. When $n = 2$, $2(\sqrt{x_1} + \sqrt{x_2}) = 3\sqrt{x_2}$, whence $\sqrt{x_2} = 2\sqrt{x_1}$ and $x^2 = 4x_1 = 4a$. When $n = 3$,

$$2(\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}) = 4\sqrt{x_3} \Longrightarrow 2\sqrt{x_3} = 2(\sqrt{x_1} + \sqrt{x_2}) = 6\sqrt{x_1} \Longrightarrow x_3 = 9x_1 = 9a \ .$$

We conjecture that $x_k = k^2 a$ for each positive integer k.

Let $m \geq 2$ and suppose that $x_k = k^2 a$ for $1 \leq k \leq m-1$.

$$\begin{aligned} (m-1)\sqrt{x_m} &= (m+1)\sqrt{x_m} - 2\sqrt{x_m} = 2(\sqrt{x_1} + \cdots + \sqrt{x_{m-1}}) \\ &= 2\sqrt{x_1}(1 + 2 + \cdots + (m-1)) = m(m-1)\sqrt{a} \ , \end{aligned}$$

whence $\sqrt{x_m} = m\sqrt{a}$ and $x_m = m^2 a$.

Thus, $x_k = k^2 a$ for all $k \geq 1$. Therefore

$$\sum_{k=1}^{n} x_k = (1 + 4 + \cdots + n^2)a = \frac{n(n+1)(2n+1)a}{6} \ .$$