OLYMON

COMPLETE PROBLEM SET

No solutions. See yearly files.

February, 2009-xxx

PART 3

Problems 601-900

601. A convex figure lies inside a given circle. The figure is seen from every point of the circumference of the circle at right angles (that is, the two rays drawn from the point and supporting the convex figure are perpendicular). Prove that the centre of the circle is a centre of symmetry of the figure.

602. Prove that, for each pair (m, n) of integers with $1 \leq m \leq n$,

$$\sum_{i=1}^{n} i(i-1)(i-2)\cdots(i-m+1) = \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} .$$

(b) Suppose that $1 \leq r \leq n$; consider all subsets with r elements of the set $\{1, 2, 3, \cdots, n\}$. The elements of this subset are arranged in ascending order of magnitude. For $1 \leq i \leq r$, let t_i denote the ith smallest element in the subset, and let $T(n, r, i)$ denote the arithmetic mean of the elements t_i. Prove that

$$T(n, r, i) = i\left(\frac{n+1}{r+1}\right) .$$

603. For each of the following expressions severally, determine as many integer values of x as you can so that it is a perfect square. Indicate whether your list is complete or not.

(a) $1 + x$;

(b) $1 + x + x^2$;

(c) $1 + x + x^2 + x^3$;

(d) $1 + x + x^2 + x^3 + x^4$;

(e) $1 + x + x^2 + x^3 + x^4 + x^5$.

604. $ABCD$ is a square with incircle Γ. Let l be a tangent to Γ, and let A', B', C', D' be points on l such that AA', BB', CC', DD' are all prependicular to l. Prove that $AA' \cdot CC' = BB' \cdot DD'$.

605. Prove that the number $299\cdots998200\cdots029$ can be written as the sum of three perfect squares of three consecutive numbers, where there are $n-1$ nines between the first 2 and the 8, and $n-1$ zeros between the last pair of twos.

606. Let $x_1 = 1$ and let $x_{n+1} = \sqrt{x_n + n^2}$ for each positive integer n. Prove that the sequence $\{x_n : n > 1\}$ consists solely of irrational numbers and calculate $\sum_{k=1}^{n} \lfloor x_k^2 \rfloor$, where $\lfloor x \rfloor$ is the largest integer that does not exceed x.

607. Solve the equation

$$\sin x \left(1 + \tan x \tan \frac{x}{2}\right) = 4 - \cot x .$$

608. Find all positive integers n for which n, $n^2 + 1$ and $n^3 + 3$ are simultaneously prime.

609. The first term of an arithmetic progression is 1 and the sum of the first nine terms is equal to 369. The first and ninth terms of the arithmetic progression coincide respectively with the first and ninth terms of a geometric progression. Find the sum of the first twenty terms of the geometric progression.

610. Solve the system of equations

$$\log_{10}(x^3 - x^2) = \log_5 y^2$$
$$\log_{10}(y^3 - y^2) = \log_5 z^2$$
$$\log_{10}(z^3 - z^2) = \log_5 x^2$$

where $x, y, z > 1$.

611. The triangle ABC is isosceles with $AB = AC$ and I and O are the respective centres of its inscribed and circumscribed circles. If D is a point on AC for which $ID \| AB$, prove that $CI \perp OD$.

612. $ABCD$ is a rectangle for which $AB > AD$. A rotation with centre A takes B to a point B' on CD; it takes C to C' and D to D'. Let P be the point of intersection of the lines CD and $C'D'$. Prove that $CB' = DP$.

613. Let ABC be a triangle and suppose that

$$\tan\frac{A}{2} = \frac{p}{u} \qquad \tan\frac{B}{2} = \frac{q}{v} \qquad \tan\frac{C}{2} = \frac{r}{w} ,$$

where p, q, r, u, v, w are positive integers and each fraction is written in lowest terms.

(a) Verify that $pqw + pvr + uqr = uvw$.

(b) Let f be the greatest common divisor of the pair $(vw - qr, qw + vr)$, g be the greatest common divisor of the pair $(uw - pr, pw + ur)$, and h be the greatest common divisor of the pair $(uv - pq, pv + qu)$. Prove that

$$fp = vw - qr \qquad fu = qw + vr$$
$$gq = uw - pr \qquad gv = pw + ur$$
$$hr = uv - pq \qquad hw = pv + qu .$$

(c) Prove that the sides of the triangle ABC are proportional to $fpu : gqv : hrw$.

614. Determine those values of the parameter a for which there exist at least one line that is tangent to the graph of the curve $y = x^3 - ax$ at one point and normal to the graph at another.

615. The function $f(x)$ is defined for real nonzero x, takes nonzero real values and satisfies the functional equation

$$f(x) + f(y) = f(xyf(x + y)) ,$$

whenever $xy(x + y) \neq 0$. Determine all possibilities for f.

616. Let T be a triangle in the plane whose vertices are lattice points (*i.e.*, both coordinates are integers), whose edges contain no lattice points in their interiors and whose interior contains exactly one lattice point. Must this lattice point in the interior be the centroid of the T?

617. Two circles are externally tangent at A and are internally tangent to a third circle Γ at points B and C. Suppose that D is the midpoint of the chord of Γ that passses through A and is tangent there to the two smaller given circles. Suppose, further, that the centres of the three circles are not collinear. Prove that A is the incentre of triangle BCD.

618. Let a, b, c, m be positive integers for which $abcm = 1 + a^2 + b^2 + c^2$. Show that $m = 4$, and that there are actually possibilities with this value of m.

619. Suppose that $n > 1$ and that S is the set of all polynomials of the form

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 \ ,$$

whose coefficients are complex numbers. Determine the minimum value over all such polynomials of the maximum value of $|p(z)|$ when $|z| = 1$.

620. Let $a_1, a_2, \cdots, a_n$ be distinct integers. Prove that the polynomial

$$p(z) = (z - a_1)^2(z - a_2)^2 \cdots (z - a_n)^2 + 1$$

cannot be written as the product of two nonconstant polynomials with integer coefficients.

621. Determine the locus of one focus of an ellipse reflected in a variable tangent to the ellipse.

622. Let I be the centre of the inscribed circle of a triangle ABC and let u, v, w be the respective lengths of IA, IB, IC. Let P be any point in the plane and p, q, r the respective lengths of PA, PB, PC. Prove that, with the sidelengths of the triangle given conventionally as a, b, c,

$$ap^2 + bq^2 + cr^2 = au^2 + bv^2 + cw^2 + (a + b + c)z^2 \ ,$$

where z is the length of IP.

623. Given the parameters a, b, c, solve the system

$$x + y + z = a + b + c;$$

$$x^2 + y^2 + x^2 = a^2 + b^2 + c^2;$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3 \ .$$

624. Suppose that $x_i \geq 0$ and

$$\sum_{i=1}^{n} \frac{1}{1 + x_i} \leq 1 \ .$$

Prove that

$$\sum_{i=1}^{n} 2^{-x_i} \leq 1 \ .$$

625. Given an odd number of intervals, each of unit length, on the real line, let S be the set of numbers that are in an odd number of these intervals. Show that S is a finite union of disjoint intervals of total length not less than 1.

626. Let ABC be an isosceles triangle with $AB = AC$, and suppose that D is a point on the side BC with $BC > BD > DC$. Let BE and CF be diameters of the respective circumcircles of triangles ABD and ADC, and let P be the foot of the altitude from A to BC. Prove that $PD : AP = EF : BC$.

627. Let

$$f(x, y, z) = 2x^2 + 2y^2 - 2z^2 + \frac{7}{xy} + \frac{1}{z} \ .$$

There are three pairwise distinct numbers a, b, c for which

$$f(a, b, c) = f(b, c, a) = f(c, a, b) \ .$$

Determine $f(a,b,c)$. Determine three such numbers a,b,c.

628. Suppose that AP, BQ and CR are the altitudes of the acute triangle ABC, and that

$$9\overrightarrow{AP} + 4\overrightarrow{BQ} + 7\overrightarrow{CR} = \overrightarrow{O} \ .$$

Prove that one of the angles of triangle ABC is equal to $60°$.

629. Let $a > b > c > d > 0$ and $a + d = b + c$. Show that $ad < bc$.

(b) Let a, b, p, q, r, s be positive integers for which

$$\frac{p}{q} < \frac{a}{b} < \frac{r}{s}$$

and $qr - ps = 1$. Prove that $b \geq q + s$.

630. (a) Show that, if

$$\frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} = -1 \quad ,$$

then

$$\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} = 1 \quad .$$

(b) Give an example of numbers α and β that satisfy the condition in (a) and check that both equations hold.

631. The sequence of functions $\{P_n\}$ satisfies the following relations:

$$P_1(x) = x \ , \qquad P_2(x) = x^3 \ ,$$

$$P_{n+1}(x) = \frac{P_n^3(x) - P_{n-1}(x)}{1 + P_n(x)P_{n-1}(x)} \quad , \qquad n = 1, 2, 3, \cdots .$$

Prove that all functions P_n are polynomials.

632. Let a, b, c, x, y, z be positive real numbers for which $a \leq b \leq c$, $x \leq y \leq z$, $a + b + c = x + y + z$, $abc = xyz$, and $c \leq z$, Prove that $a \leq x$.

633. Let ABC be a triangle with $BC = 2 \cdot AC - 2 \cdot AB$ and D be a point on the side BC. Prove that $\angle ABD = 2\angle ADB$ if and only if $BD = 3CD$.

634. Solve the following system for real values of x and y:

$$2^{x^2+y} + 2^{x+y^2} = 8$$

$$\sqrt{x} + \sqrt{y} = 2 \ .$$

635. Two unequal spheres in contact have a common tangent cone. The three surfaces divide space into various parts, only one of which is bounded by all three surfaces; it is "ring-shaped". Being given the radii r and R of the spheres with $r < R$, find the volume of the "ring-shaped" region in terms of r and R.

636. Let ABC be a triangle. Select points D, E, F outside of ΔABC such that ΔDBC, ΔEAC, ΔFAB are all isosceles with the equal sides meeting at these outside points and with $\angle D = \angle E = \angle F$. Prove that the lines AD, BE and CF all intersect in a common point.

637. Let n be a positive integer. Determine how many real numbers x with $1 \leq x < n$ satisfy

$$x^3 - \lfloor x^3 \rfloor = (x - \lfloor x \rfloor)^3 \ .$$

638. Let x and y be real numbers. Prove that

$$\max(0, -x) + \max(1, x, y) = \max(0, x - \max(1, y)) + \max(1, y, 1 - x, y - x)$$

where $\max(a, b)$ is the larger of the two numbers a and b.

639. (a) Let $ABCDE$ be a convex pentagon such that $AB = BC$ and $\angle BCD = \angle EAB = 90°$. Let X be a point inside the pentagon such that AX is perpendicular to BE and CX is perpendicular to BD. Show that BX is perpendicular to DE.

(b) Let N be a regular nonagon, *i.e.*, a regular polygon with nine edges, having O as the centre of its circumcircle, and let PQ and QR be adjacent edges of N. The midpoint of PQ is A and the midpoint of the radius perpendicular to QR is B. Determine the angle between AO and AB.

640. Suppose that $n \geq 2$ and that, for $1 \leq i \leq n$, we have that $x_i \geq -2$ and all the x_i are nonzero with the same sign. Prove that

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) > 1 + x_1 + x_2 + \cdots + x_n \quad ,$$

641. Observe that $x^2 + 5x + 6 = (x + 2)(x + 3)$ while $x^2 + 5x - 6 = (x + 6)(x - 1)$. Determine infinitely many coprime pairs (m, n) of positive integers for which both $x^2 + mx + n$ and $x^2 + mx - n$ can be factored as a product of linear polynomials with integer coefficients.

642. In a convex polyhedron, each vertex is the endpoint of exactly three edges and each face is a concyclic polygon. Prove that the polyhedron can be inscribed in a sphere.

643. Let n^2 distinct integers be arranged in an $n \times n$ square array ($n \geq 2$). Show that it is possible to select n numbers, one from each row and column, such that if the number selected from any row is greater than another number in this row, then this latter number is less than the number selected from its column.

644. Given a point P, a line $\mathfrak{L}$ and a circle $\mathfrak{C}$, construct with straightedge and compasses an equilateral triangle PQR with one vertex at P, another vertex Q on $\mathfrak{L}$ and the third vertex R on $\mathfrak{C}$.

645. Let $n \geq 3$ be a positive integer. Are there n positive integers $a_1, a_2, \cdots, a_n$ not all the same such that for each i with $3 \leq i \leq n$ we have

$$a_i + S_i = (a_i, S_i) + [a_i, S_i] \ \ .$$

where $S_i = a_1 + a_2 + \cdots + a_i$, and where $(\cdot, \cdot)$ and $[\cdot, \cdot]$ represent the greatest common divisor and least common multiple respectively?

646. Let ABC be a triangle with incentre I. Let AI meet BC at L, and let X be the contact point of the incircle with the line BC. If D is the reflection of L in X on line BC , we construct B' and C' as the reflections of D with respect to the lines BI and CI, respectively. Show that the quadrailateral $BCC'B'$ is cyclic.

647. Find all continuous functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(x + f(y)) = f(x) + y$$

for every $x, y \in \mathbf{R}$.

648. Prove that for every positive integer n, the integer $1 + 5^n + 5^{2n} + 5^{3n} + 5^{4n}$ is composite.

649. In the triangle ABC, $\angle BAC = 20°$ and $\angle ACB = 30°$. The point M is located in the interior of triangle ABC so that $\angle MAC = \angle MCA = 10°$. Determine $\angle BMC$.

650. Suppose that the nonzero real numbers satisfy

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{xyz} .$$

Determine the minimum value of

$$\frac{x^4}{x^2 + y^2} + \frac{y^4}{y^2 + z^2} + \frac{z^4}{z^2 + x^2} .$$

651. Determine polynomials $a(t)$, $b(t)$, $c(t)$ with integer coefficients such that the equation $y^2 + 2y = x^3 - x^2 - x$ is satisfied by $(x, y) = (a(t)/c(t), b(t)/c(t))$.

652. (a) Let m be any positive integer greater than 2, such that $x^2 \equiv 1 \pmod{m}$ whenever the greatest common divisor of x and m is equal to 1. An example is $m = 12$. Suppose that n is a positive integer for which $n + 1$ is a multiple of m. Prove that the sum of all of the divisors of n is divisible by m.

(b) Does the result in (a) hold when $m = 2$?

(c) Find all possible values of m that satisfy the condition in (a).

653. Let $f(1) = 1$ and $f(2) = 3$. Suppose that, for $n \geq 3$, $f(n) = \max\{f(r) + f(n - r) : 1 \leq r \leq n - 1\}$. Determine necessary and sufficient conditions on the pair (a, b) that $f(a + b) = f(a) + f(b)$.

654. Let ABC be an arbitrary triangle with the points D, E, F on the sides BC, CA, AB respectively, so that

$$\frac{BD}{DC} \leq \frac{BF}{FA} \leq 1$$

and

$$\frac{AE}{EC} \leq \frac{AF}{FB} .$$

Prove that $[DEF] \leq \frac{1}{4}[ABC]$, with equality if and only if two at least of the three points D, E, F are midpoints of the corresponding sides.
(*Note:* $[XYZ]$ denotes the area of triangle XYZ.)

655. (a) Three ants crawl along the sides of a fixed triangle in such a way that the centroid (intersection of the medians) of the triangle they form at any moment remains constant. Show that this centroid coincides with the centroid of the fixed triangle if one of the ants travels along the entire perimeter of the triangle.

(b) Is it indeed always possible for a given fixed triangle with one ant at any point on the perimeter of the triangle to place the remaining two ants somewhere on the perimeter so that the centroid of their triangle coincides with the centroid of the fixed triangle?

656. Let ABC be a triangle and k be a real constant. Determine the locus of a point M in the plane of the triangle for which

$$|MA|^2 \sin 2A + |MB|^2 \sin 2B + |MC|^2 \sin 2C = k .$$

657. Let a, b, c be positive real numbers for which $a + b + c = abc$. Find the minimum value of

$$\sqrt{1 + \frac{1}{a^2}} + \sqrt{1 + \frac{1}{b^2}} + \sqrt{1 + \frac{1}{c^2}} .$$

658. Prove that $\tan 20^\circ + 4\sin 20^\circ = \sqrt{3}$.

659. (a) Give an example of a pair a, b of positive integers, not both prime, for which $2a-1$, $2b-1$ and $a+b$ are all primes. Determine all possibilities for which a and b are themselves prime.

(b) Suppose a and b are positive integers such that $2a-1$, $2b-1$ and $a+b$ are all primes. Prove that neither $a^b + b^a$ nor $a^a + b^b$ are multiples of $a+b$.

660. ABC is a triangle and D is a point on AB produced beyond B such that $BD = AC$, and E is a point on AC produced beyond C such that $CE = AB$. The right bisector of BC meets DE at P. Prove that $\angle BPC = \angle BAC$.

661. Let P be an arbitrary interior point of an equilateral triangle ABC. Prove that

$$|\angle PAB - \angle PAC| \geq |\angle PBC - \angle PCB| \quad .$$

662. Let n be a positive integer and $x > 0$. Prove that

$$(1+x)^{n+1} \geq \frac{(n+1)^{n+1}}{n^n} x \quad .$$

663. Find all functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that

$$x^2 y^2 (f(x+y) - f(x) - f(y)) = 3(x+y) f(x) f(y)$$

for all real numbers x and y.

664. The real numbers x, y, and z satisfy the system of equations

$$x^2 - x = yz + 1;$$

$$y^2 - y = xz + 1;$$

$$z^2 - z = xy + 1.$$

Find all solutions (x, y, z) of the system and determine all possible values of $xy + yz + zx + x + y + z$ where (x, y, z) is a solution of the system.

665. Let $f(x) = x^3 + ax^2 + bx + b$. Determine all integer pairs (a, b) for which $f(x)$ is the product of three linear factors with integer coefficients.

666. Assume that a face S of a convex polyhedron $\mathfrak{P}$ has a common edge with every other face of $\mathfrak{P}$. Show that there exists a simple (nonintersecting) closed (not necessarily planar) polygon that consists of edges of $\mathfrak{P}$ and passes through all the vertices.

667. Let A_n be the set of mappings $f : \{1, 2, 3, \cdots, n\} \longrightarrow \{1, 2, 3, \cdots, n\}$ such that, if $f(k) = i$ for some i, then f also assumes all the values $1, 2, \cdots, i-1$. Prove that the number of elements of A_n is $\sum_{k=0}^{\infty} k^n 2^{-(k+1)}$.

668. The nonisosceles right triangle ABC has $\angle CAB = 90^\circ$. The inscribed circle with centre T touches the sides AB and AC at U and V respectively. The tangent through A of the circumscribed circle meets UV produced in S. Prove that

(a) $ST \parallel BC$;

(b) $|d_1 - d_2| = r$, where r is the radius of the inscribed circle and d_1 and d_2 are the respective distances from S to AC and AB.

669. Let $n \geq 3$ be a natural number. Prove that

$$1989 | n^{n^{n^n}} - n^{n^n} \quad ,$$

i.e., the number on the right is a multiple of 1989.

670. Consider the sequence of positive integers $\{1, 12, 123, 1234, 12345, \cdots\}$ where the next term is constructed by lengthening the previous term at the right-hand end by appending the next positive integer. Note that this next integer occupies only one place, with "carrying"occurring as in addition. Thus, the ninth and tenth terms of the sequence are 123456789 and 1234567900 respectively. Determine which terms of the sequence are divisible by 7.

671. Each point in the plane is coloured with one of three distinct colours. Prove that there are two points that are unit distant apart with the same colour.

672. The Fibonacci sequence $\{F_n\}$ is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n = 0, \pm 1, \pm 2, \pm 3, \cdots$. The real number τ is the positive solution of the quadratic equation $x^2 = x + 1$.

(a) Prove that, for each positive integer n, $F_{-n} = (-1)^{n+1} F_n$.

(b) Prove that, for each integer n, $\tau^n = F_n \tau + F_{n-1}$.

(c) Let G_n be any one of the functions $F_{n+1}F_n$, $F_{n+1}F_{n-1}$ and F_n^2. In each case, prove that $G_{n+3} + G_n = 2(G_{n+2} + G_{n+1})$.

673. ABC is an isosceles triangle with $AB = AC$. Let D be the point on the side AC for which $CD = 2AD$. Let P be the point on the segment BD such that $\angle APC = 90^\circ$. Prove that $\angle ABP = \angle PCB$.

674. The sides BC, CA, AB of triangle ABC are produced to the poins R, P, Q respectively, so that $CR = AP = BQ$. Prove that triangle PQR is equilateral if and only if triangle ABC is equilateral.

675. ABC is a triangle with circumcentre O such that $\angle A$ exceeds 90° and $AB < AC$. Let M and N be the midpoints of BC and AO, and let D be the intersection of MN and AC. Suppose that $AD = \frac{1}{2}(AB + AC)$. Determine $\angle A$.

676. Determine all functions f from the set of reals to the set of reals which satisfy the functional equation

$$(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$$

for all real x and y.

677. For vectors in three-dimensional real space, establish the identity

$$[\mathbf{a} \times (\mathbf{b} - \mathbf{c})]^2 + [\mathbf{b} \times (\mathbf{c} - \mathbf{a})]^2 + [\mathbf{c} \times (\mathbf{a} - \mathbf{b})]^2 = (\mathbf{b} \times \mathbf{c})^2 + (\mathbf{c} \times \mathbf{a})^2 + (\mathbf{a} \times \mathbf{b})^2 + (\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b})^2 \quad .$$

678. For $a, b, c > 0$, prove that

$$\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} \geq \frac{3}{1 + abc} \ .$$

679. Let F_1 and F_2 be the foci of an ellipse and P be a point in the plane of the ellipse. Suppose that G_1 and G_2 are points on the ellipse for which PG_1 and PG_2 are tangents to the ellipse. Prove that $\angle F_1 P G_1 = \angle F_2 P G_2$.

680. Let $u_0 = 1$, $u_1 = 2$ and $u_{n+1} = 2u_n + u_{n-1}$ for $n \geq 1$. Prove that, for every nonnegative integer n,

$$u_n = \sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i, j, k \geq 0, i + j + 2k = n \right\} \ .$$

681. Let $\mathbf{a}$ and $\mathbf{b}$, the latter nonzero, be vectors in $\mathbb{R}^3$. Determine the value of λ for which the vector equation

$$\mathbf{a} - (\mathbf{x} \times \mathbf{b}) = \lambda \mathbf{b}$$

is solvable, and then solve it.

682. The plane is partitioned into n regions by three families of parallel lines. What is the least number of lines to ensure that $n \geq 2010$?

683. Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which

$$f(x)f(x+1) = g(h(x)) \ ,$$

684. Let x, y, z be positive reals for which $xyz = 1$. Prove that

$$\frac{x+y}{x^2+y^2} + \frac{y+z}{y^2+z^2} + \frac{z+x}{z^2+x^2} \leq \sqrt{x} + \sqrt{y} + \sqrt{z} \ .$$

685. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by

$$f(x) = x - 4\lfloor x \rfloor + \lfloor 2x \rfloor \ ,$$

where $\lfloor \cdot \rfloor$ represents the greatest integer that does not exceed the argument. Determine $f(f(x))$ and show that f is a surjective (onto) function.

686. Solve the equation

$$\sqrt{6 + 3\sqrt{2 + \sqrt{2 + x}}} + \sqrt{2 - \sqrt{2 + \sqrt{2 + x}}} = 2x \ .$$

687. Prove that

$$\frac{(1 + 2 + 3 + \cdots + n)!}{1!2! \ldots n!}$$

is a natural number for any positive integer n.

688. Solve the equation

$$2010^x + 2010^{-x} = 1 + 2x - x^2 \ .$$

689. Let BC e a diameter of the circle $\mathfrak{C}$ and let A be an interior point. Suppose that BA and CA intersect the circle $\mathfrak{C}$ at D and E respectively. If the tangents to the circle $\mathfrak{C}$ at E and D intersect at the point M, prove that $AM \perp BC$.

690. Let m_a, m_b, m_c; h_a, h_b, h_c be the lengths of the medians and the heights of triangle ABC, where the notation is used conventionally.

(a) If $a \leq b \leq c$, prove that $h_a \geq h_b \geq h_c$ and that $m_a \geq m_b \geq m_c$.

(b) If

$$\left(\frac{h_a^2}{h_b \cdot h_c}\right)^{m_a} \cdot \left(\frac{h_b)^2}{h_c \cdot h_a}\right)^{m_b} \cdot \left(\frac{h_c)^2}{h_a \cdot h_b}\right)^{m_c} = 1 \ ,$$

prove that triangle ABC is equilateral.

691. Prove that

$$\sqrt{\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z}} > \sqrt[3]{\sqrt{x} + \sqrt{y} + \sqrt{z}}$$

for positive integers x, y, z.