OLYMON

COMPLETE PROBLEM SET

No solutions. See yearly files.

March, 2004 - February, 2009

PART 2

Problems 301-600

301. Let $d = 1, 2, 3$. Suppose that M_d consists of the positive integers that *cannot* be expressed as the sum of two or more consecutive terms of an arithmetic progression consisting of positive integers with common difference d. Prove that, if $c \in M_3$, then there exist integers $a \in M_1$ and $b \in M_2$ for which $c = ab$.

302. In the following, $ABCD$ is an arbitrary convex quadrilateral. The notation $[\cdots]$ refers to the area.

(a) Prove that $ABCD$ is a trapezoid if and only if

$$[ABC] \cdot [ACD] = [ABD] \cdot [BCD] \ .$$

(b) Suppose that F is an interior point of the quadrilateral $ABCD$ such that $ABCF$ is a parallelogram. Prove that

$$[ABC] \cdot [ACD] + [AFD] \cdot [FCD] = [ABD] \cdot [BCD] \ .$$

303. Solve the equation

$$\tan^2 2x = 2 \tan 2x \tan 3x + 1 \ .$$

304. Prove that, for any complex numbers z and w,

$$(|z| + |w|) \left| \frac{z}{|z|} + \frac{w}{|w|} \right| \le 2|z + w| \ .$$

305. Suppose that u and v are positive integer divisors of the positive integer n and that $uv < n$. Is it necessarily so that the greatest common divisor of n/u and n/v exceeds 1?

306. The circumferences of three circles of radius r meet in a common point O. The meet also, pairwise, in the points P, Q and R. Determine the maximum and minimum values of the circumradius of triangle PQR.

307. Let p be a prime and m a positive integer for which $m < p$ and the greatest common divisor of m and p is equal to 1. Suppose that the decimal expansion of m/p has period $2k$ for some positive integer k, so that

$$\frac{m}{p} = .ABABABAB \ldots = (10^k A + B)(10^{-2k} + 10^{-4k} + \cdots)$$

where A and B are two distinct blocks of k digits. Prove that

$$A + B = 10^k - 1 \ .$$

(For example, $3/7 = 0.428571\ldots$ and $428 + 571 = 999$.)

308. Let a be a parameter. Define the sequence $\{f_n(x) : n = 0, 1, 2, \cdots\}$ of polynomials by

$$f_0(x) \equiv 1$$

$$f_{n+1}(x) = xf_n(x) + f_n(ax)$$

for $n \geq 0$.

(a) Prove that, for all n, x,

$$f_n(x) = x^n f_n(1/x) \ .$$

(b) Determine a formula for the coefficient of x^k $(0 \leq k \leq n)$ in $f_n(x)$.

309. Let $ABCD$ be a convex quadrilateral for which all sides and diagonals have rational length and AC and BD intersect at P. Prove that AP, BP, CP, DP all have rational length.

310. (a) Suppose that n is a positive integer. Prove that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x+k)^{k-1}(y-k)^{n-k} \ .$$

(b) Prove that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x-kz)^{k-1}(y+kz)^{n-k} \ .$$

311. Given a square with a side length 1, let P be a point in the plane such that the sum of the distances from P to the sides of the square (or their extensions) is equal to 4. Determine the set of all such points P.

312. Given ten arbitrary natural numbers. Consider the sum, the product, and the absolute value of the difference calculated for any two of these numbers. At most how many of all these calculated numbers are odd?

313. The three medians of the triangle ABC partition it into six triangles. Given that three of these triangles have equal perimeters, prove that the triangle ABC is equilateral.

314. For the real numbers a, b and c, it is known that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} = 1 \ ,$$

and

$$a + b + c = 1 \ .$$

Find the value of the expression

$$M = \frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} \ .$$

315. The natural numbers 3945, 4686 and 5598 have the same remainder when divided by a natural number x. What is the sum of the number x and this remainder?

316. Solve the equation

$$|x^2 - 3x + 2| + |x^2 + 2x - 3| = 11 \ .$$

317. Let $P(x)$ be the polynomial

$$P(x) = x^{15} - 2004x^{14} + 2204x^{13} - \cdots - 2004x^2 + 2004x \ ,$$

Calculate $P(2003)$.

318. Solve for integers x, y, z the system

$$1 = x + y + z = x^3 + y^3 + z^2 \ .$$

[Note that the exponent of z on the right is 2, not 3.]

319. Suppose that a, b, c, x are real numbers for which $abc \neq 0$ and

$$\frac{xb + (1-x)c}{a} = \frac{xc + (1-x)a}{b} = \frac{xa + (1-x)b}{c} \ .$$

Prove that $a = b = c$.

320. Let L and M be the respective intersections of the internal and external angle bisectors of the triangle ABC at C and the side AB produced. Suppose that $CL = CM$ and that R is the circumradius of triangle ABC. Prove that

$$|AC|^2 + |BC|^2 = 4R^2 \ .$$

321. Determine all positive integers k for which $k^{1/(k-7)}$ is an integer.

322. The real numbers u and v satisfy

$$u^3 - 3u^2 + 5u - 17 = 0$$

and

$$v^3 - 3v^2 + 5v + 11 = 0 \ .$$

Determine $u + v$.

323. Alfred, Bertha and Cedric are going from their home to the country fair, a distance of 62 km. They have a motorcycle with sidecar that together accommodates at most 2 people and that can travel at a maximum speed of 50 km/hr. Each can walk at a maximum speed of 5 km/hr. Is it possible for all three to cover the 62 km distance within 3 hours?

324. The base of a pyramid $ABCDV$ is a rectangle $ABCD$ with $|AB| = a$, $|BC| = b$ and $|VA| = |VB| = |VC| = |VD| = c$. Determine the area of the intersection of the pyramid and the plane parallel to the edge VA that contains the diagonal BD.

325. Solve for positive real values of x, y, t:

$$(x^2 + y^2)^2 + 2tx(x^2 + y^2) = t^2y^2 \ .$$

Are there infinitely many solutions for which the values of x, y, t are all positive integers?

Optional rider: What is the smallest value of t for a positive integer solution?

326. In the triangle ABC with semiperimeter $s = \frac{1}{2}(a + b + c)$, points U, V, W lie on the respective sides BC, CA, AB. Prove that

$$s < |AU| + |BV| + |CW| < 3s \ .$$

Give an example for which the sum in the middle is equal to $2s$.

327. Let A be a point on a circle with centre O and let B be the midpoint of OA. Let C and D be points on the circle on the same side of OA produced for which $\angle CBO = \angle DBA$. Let E be the midpoint of CD and let F be the point on EB produced for which $BF = BE$.

(a) Prove that F lies on the circle.

(b) What is the range of angle EAO?

328. Let $\mathfrak{C}$ be a circle with diameter AC and centre D. Suppose that B is a point on the circle for which $BD \perp AC$. Let E be the midpoint of DC and let Z be a point on the radius AD for which $EZ = EB$.

Prove that

(a) The length c of BZ is the length of the side of a regular pentagon inscribed in $\mathfrak{C}$.

(b) The length b of DZ is the length of the side of a regular decagon (10-gon) inscribed in $\mathfrak{C}$.

(c) $c^2 = a^2 + b^2$ where a is the length of a regular hexagon inscribed in $\mathfrak{C}$.

(d) $(a + b) : a = a : b$.

329. Let x, y, z be positive real numbers. Prove that

$$\sqrt{x^2 - xy + y^2} + \sqrt{y^2 - yz + z^2} \geq \sqrt{x^2 + xz + z^2} \ .$$

330. At an international conference, there are four official languages. Any two participants can communicate in at least one of these languages. Show that at least one of the languages is spoken by at least 60% of the participants.

331. Some checkers are placed on various squares of a $2m \times 2n$ chessboard, where m and n are odd. Any number (including zero) of checkers are placed on each square. There are an odd number of checkers in each row and in each column. Suppose that the chessboard squares are coloured alternately black and white (as usual). Prove that there are an even number of checkers on the black squares.

332. What is the minimum number of points that can be found (a) in the plane, (b) in space, such that each point in, respectively, (a) the plane, (b) space, must be at an irrational distance from at least one of them?

333. Suppose that a, b, c are the sides of triangle ABC and that a^2, b^2, c^2 are in arithmetic progression.

(a) Prove that $\cot A$, $\cot B$, $\cot C$ are also in arithmetic progression.

(b) Find an example of such a triangle where a, b, c are integers.

334. The vertices of a tetrahedron lie on the surface of a sphere of radius 2. The length of five of the edges of the tetrahedron is 3. Determine the length of the sixth edge.

335. Does the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{12}{a + b + c}$$

have infinitely many solutions in positive integers a, b, c?

336. Let $ABCD$ be a parallelogram with centre O. Points M and N are the respective midpoints of BO and CD. Prove that the triangles ABC and AMN are similar if and only if $ABCD$ is a square.

337. Let a, b, c be three real numbers for which $0 \leq c \leq b \leq a \leq 1$ and let w be a complex root of the polynomial $z^3 + az^2 + bz + c$. Must $|w| \leq 1$?

338. A *triangular triple* (a, b, c) is a set of three positive integers for which $T(a) + T(b) = T(c)$. Determine the smallest triangular number of the form $a + b + c$ where (a, b, c) is a triangular triple. (*Optional investigations:* Are there infinitely many such triangular numbers $a + b + c$? Is it possible for the three numbers of a triangular triple to each be triangular?)

339. Let a, b, c be integers with $abc \neq 0$, and u, v, w be integers, not all zero, for which

$$au^2 + bv^2 + cw^2 = 0 \ .$$

Let r be any rational number. Prove that the equation

$$ax^2 + by^2 + cz^2 = r$$

is solvable.

340. The lock on a safe consists of three wheels, each of which may be set in eight different positions. Because of a defect in the safe mechanism, the door will open if any two of the three wheels is in the correct position. What is the smallest number of combinations which must be tried by someone not knowing the correct combination to guarantee opening the safe?

341. Let s, r, R respectively specify the semiperimeter, inradius and circumradius of a triangle ABC.

(a) Determine a necessary and sufficient condition on s, r, R that the sides a, b, c of the triangle are in arithmetic progression.

(b) Determine a necessary and sufficient condition on s, r, R that the sides a, b, c of the triangle are in geometric progression.

342. Prove that there are infinitely many solutions in positive integers of the system

$$a + b + c = x + y$$
$$a^3 + b^3 + c^3 = x^3 + y^3 \ .$$

343. A sequence $\{a_n\}$ of integers is defined by

$$a_0 = 0 \ , \quad a_1 = 1 \ , \quad a_n = 2a_{n-1} + a_{n-2}$$

for $n > 1$. Prove that, for each nonnegative integer k, 2^k divides a_n if and only if 2^k divides n.

344. A function f defined on the positive integers is given by

$$f(1) = 1 \ , \quad f(3) = 3 \ , \quad f(2n) = f(n) \ ,$$

$$f(4n+1) = 2f(2n+1) - f(n)$$
$$f(4n+3) = 3f(2n+1) - 2f(n) \ ,$$

for each positive integer n. Determine, with proof, the number of positive integers no exceeding 2004 for which $f(n) = n$.

345. Let $\mathfrak{C}$ be a cube with edges of length 2. Construct a solid figure with fourteen faces by cutting off all eight corners of $\mathfrak{C}$, keeping the new faces perpendicular to the diagonals of the cuhe and keeping the newly formed faces identical. If the faces so formed all have the same area, determine the common area of the faces.

346. Let n be a positive integer. Determine the set of all integers that can be written in the form

$$\sum_{k=1}^{n} \frac{k}{a_k}$$

where $a_1, a_2, \cdots, a_n$ are all positive integers.

347. Let n be a positive integer and $\{a_1, a_2, \cdots, a_n\}$ a finite sequence of real numbers which contains at least one positive term. Let S be the set of indices k for which at least one of the numbers

$$a_k, a_k + a_{k+1}, a_k + a_{k+1} + a_{k+2}, \cdots, a_k + a_{k+1} + \cdots + a_n$$

is positive. Prove that

$$\sum\{a_k : k \in S\} > 0 \ .$$

348. (a) Suppose that $f(x)$ is a real-valued function defined for real values of x. Suppose that $f(x) - x^3$ is an increasing function. Must $f(x) - x - x^2$ also be increasing?

(b) Suppose that $f(x)$ is a real-valued function defined for real values of x. Suppose that both $f(x) - 3x$ and $f(x) - x^3$ are increasing functions. Must $f(x) - x - x^2$ also be increasing on all of the real numbers, or on at least the positive reals?

349. Let s be the semiperimeter of triangle ABC. Suppose that L and N are points on AB and CB produced (*i.e.*, B lies on segments AL and CN) with $|AL| = |CN| = s$. Let K be the point symmetric to B with respect to the centre of the circumcircle of triangle ABC. Prove that the perpendicular from K to the line NL passes through the incentre of triangle ABC.

350. Let $ABCDE$ be a pentagon inscribed in a circle with centre O. Suppose that its angles are given by $\angle B = \angle C = 120°$, $\angle D = 130°$, $\angle E = 100°$. Prove that BD, CE and AO are concurrent.

351. Let $\{a_n\}$ be a sequence of real numbers for which $a_1 = 1/2$ and, for $n \geq 1$,

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1} \ .$$

Prove that, for all n, $a_1 + a_2 + \cdots + a_n < 1$.

352. Let $ABCD$ be a unit square with points M and N in its interior. Suppose, further, that MN produced does not pass through any vertex of the square. Find the smallest value of k for which, given any position of M and N, at least one of the twenty triangles with vertices chosen from the set $\{A, B, C, D, M, N\}$ has area not exceeding k.

353. The two shortest sides of a right-angled triangle, a and b, satisfy the inequality:

$$\sqrt{a^2 - 6a\sqrt{2} + 19} + \sqrt{b^2 - 4b\sqrt{3} + 16} \leq 3 \ .$$

Find the perimeter of this triangle.

354. Let ABC be an isosceles triangle with $AC = BC$ for which $|AB| = 4\sqrt{2}$ and the length of the median to one of the other two sides is 5. Calculate the area of this triangle.

355. (a) Find all natural numbers k for which $3^k - 1$ is a multiple of 13.

(b) Prove that for any natural number k, $3^k + 1$ is not a multiple of 13.

356. Let a and b be real parameters. One of the roots of the equation $x^{12} - abx + a^2 = 0$ is greater than 2. Prove that $|b| > 64$.

357. Consider the circumference of a circle as a set of points. Let each of these points be coloured red or blue. Prove that, regardless of the choice of colouring, it is always possible to inscribe in this circle an isosceles triangle whose three vertices are of the same colour.

358. Find all integers x which satisfy the equation

$$\cos\left(\frac{\pi}{8}(3x - \sqrt{9x^2 + 160x + 800})\right) = 1 \ .$$

359. Let ABC be an acute triangle with angle bisectors AA_1 and BB_1, with A_1 and B_1 on BC and AC, respectively. Let J be the intersection of AA_1 and BB_1 (the incentre), H be the orthocentre and O the

circumcentre of the triangle ABC. The line OH intersects AC at P and BC at Q. Given that C, A_1, J and B_1 are vertices of a concyclic quadrilateral, prove that $PQ = AP + BQ$.

360. Eliminate θ from the two equations

$$x = \cot\theta + \tan\theta$$

$$y = \sec\theta - \cos\theta\ ,$$

to get a polynomial equation satisfied by x and y.

361. Let $ABCD$ be a square, M a point on the side BC, and N a point on the side CD for which $BM = CN$. Suppose that AM and AN intersect BD and P and Q respectively. Prove that a triangle can be constructed with sides of length $|BP|$, $|PQ|$, $|QD|$, one of whose angles is equal to $60°$.

362. The triangle ABC is inscribed in a circle. The interior bisectors of the angles A, B, C meet the circle again at U, V, W, respectively. Prove that the area of triangle UVW is not less than the area of triangle ABC.

363. Suppose that x and y are positive real numbers. Find all real solutions of the equation

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} = \sqrt{xy} + \frac{x+y}{2}\ .$$

364. Determine necessary and sufficient conditions on the positive integers a and b such that the vulgar fraction a/b has the following property: *Suppose that one successively tosses a coin and finds at one time, the fraction of heads is less than a/b and that at a later time, the fraction of heads is greater than a/b; then at some intermediate time, the fraction of heads must be exactly a/b.*

365. Let $p(z)$ be a polynomial of degree greater than 4 with complex coefficients. Prove that $p(z)$ must have a pair u, v of roots, not necessarily distinct, for which the real parts of both u/v and v/u are positive. Show that this does not necessarily hold for polynomials of degree 4.

366. What is the largest real number r for which

$$\frac{x^2+y^2+z^2+xy+yz+zx}{\sqrt{x}+\sqrt{y}+\sqrt{z}} \geq r$$

holds for all positive real values of x, y, z for which $xyz = 1$.

367. Let a and c be fixed real numbers satisfying $a \leq 1 \leq c$. Determine the largest value of b that is consistent with the condition

$$a + bc \leq b + ac \leq c + ab\ .$$

368. Let A, B, C be three distinct points of the plane for which $AB = AC$. Describe the locus of the point P for which $\angle APB = \angle APC$.

369. $ABCD$ is a rectangle and APQ is an inscribed equilateral triangle for which P lies on BC and Q lies on CD.

(a) For which rectangles is the configuration possible?

(b) Prove that, when the configuration is possible, then the area of triangle CPQ is equal to the sum of the areas of the triangles ABP and ADQ.

370. A deck of cards has nk cards, n cards of each of the colours C_1, C_2, $\cdots$, C_k. The deck is thoroughly shuffled and dealt into k piles of n cards each, P_1, P_2, $\cdots$, P_k. A game of solitaire proceeds as follows: The top card is drawn from pile P_1. If it has colour C_i, it is discarded and the top card is drawn from pile P_i. If it has colour C_j, it is discarded and the top card is drawn from pile P_j. The game continues

in this way, and will terminate when the nth card of colour C_1 is drawn and discarded, as at this point, there are no further cards left in pile P_1. What is the probability that every card is discarded when the game terminates?

371. Let X be a point on the side BC of triangle ABC and Y the point where the line AX meets the circumcircle of triangle ABC. Prove or disprove: if the length of XY is maximum, then AX lies between the median from A and the bisector of angle BAC.

372. Let b_n be the number of integers whose digits are all $1, 3, 4$ and whose digits sum to n. Prove that b_n is a perfect square when n is even.

373. For each positive integer n, define

$$a_n = 1 + 2^2 + 3^3 + \cdots + n^n \ .$$

Prove that there are infinitely many values of n for which a_n is an odd composite number.

374. What is the maximum number of numbers that can be selected from $\{1, 2, 3, \cdots, 2005\}$ such that the difference between any pair of them is not equal to 5?

375. Prove or disprove: there is a set of concentric circles in the plane for which both of the following hold:

(i) each point with integer coordinates lies on one of the circles;

(ii) no two points with integer coefficients lie on the same circle.

376. A soldier has to find whether there are mines buried within or on the boundary of a region in the shape of an equilateral triangle. The effective range of his detector is one half of the height of the triangle. If he starts at a vertex, explain how he can select the shortest path for checking that the region is clear of mines.

377. Each side of an equilateral triangle is divided into 7 equal parts. Lines through the division points parallel to the sides divide the triangle into 49 smaller equilateral triangles whose vertices consist of a set of 36 points. These 36 points are assigned numbers satisfying both the following conditions:

(a) the number at the vertices of the original triangle are 9, 36 and 121;

(b) for each rhombus composed of two small adjacent triangles, the sum of the numbers placed on one pair of opposite vertices is equal to the sum of the numbers placed on the other pair of opposite vertices.

Determine the sum of all the numbers. Is such a choice of numbers in fact possible?

378. Let $f(x)$ be a nonconstant polynomial that takes only integer values when x is an integer, and let P be the set of all primes that divide $f(m)$ for at least one integer m. Prove that P is an infinite set.

379. Let n be a positive integer exceeding 1. Prove that, if a graph with $2n + 1$ vertices has at least $3n + 1$ edges, then the graph contains a circuit (*i.e.*, a closed non-self-intersecting chain of edges whose terminal point is its initial point) with an even number of edges. Prove that this statement does not hold if the number of edges is only $3n$.

380. Factor each of the following polynomials as a product of polynomials of lower degree with integer coefficients:

(a) $(x + y + z)^4 - (y + z)^4 - (z + x)^4 - (x + y)^4 + x^4 + y^4 + z^4$;

(b) $x^2(y^3 - z^3) + y^2(z^3 - x^3) + z^2(x^3 - y^3)$;

(c) $x^4 + y^4 - z^4 - 2x^2y^2 + 4xyz^2$;

(d) $(yz + zx + xy)^3 - y^3z^3 - z^3x^3 - x^3y^3$;

(e) $x^3y^3 + y^3z^3 + z^3x^3 - x^4yz - xy^4z - xyz^4$;

(f) $2(x^4 + y^4 + z^4 + w^4) - (x^2 + y^2 + z^2 + w^2)^2 + 8xyzw$;

(g) $6(x^5 + y^5 + z^5) - 5(x^2 + y^2 + z^2)(x^3 + y^3 + z^3)$.

381. Determine all polynomials $f(x)$ such that, for some positive integer k,

$$f(x^k) - x^3 f(x) = 2(x^3 - 1)$$

for all values of x.

382. Given an odd number of intervals, each of unit length, on the real line, let S be the set of numbers that are in an odd number of these intervals. Show that S is a finite union of disjoint intervals of total length not less than 1.

383. Place the numbers $1, 2, \cdots, 9$ in a 3×3 unit square so that

(a) the sums of numbers in each of the first two rows are equal;

(b) the sum of the numbers in the third row is as large as possible;

(c) the column sums are equal;

(d) the numbers in the last row are in descending order.

Prove that the solution is unique.

384. Prove that, for each positive integer n,

$$(3 - 2\sqrt{2})(17 + 12\sqrt{2})^n + (3 + 2\sqrt{2})(17 - 12\sqrt{2})^n - 2$$

is the square of an integer.

385. Determine the minimum value of the product $(a + 1)(b + 1)(c + 1)(d + 1)$, given that $a, b, c, d \geq 0$ and

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 1 \ .$$

386. In a round-robin tournament with at least three players, each player plays one game against each other player. The tournament is said to be *competitive* if it is impossible to partition the players into two sets, such that each player in one set beat each player in the second set. Prove that, if a tournament is not competitive, it can be made so by reversing the result of a single game.

387. Suppose that a, b, u, v are real numbers for which $av - bu = 1$. Prove that

$$a^2 + u^2 + b^2 + v^2 + au + bv \geq \sqrt{3} \ .$$

Give an example to show that equality is possible. (Part marks will be awarded for a result that is proven with a smaller bound on the right side.)

388. A class with at least 35 students goes on a cruise. Seven small boats are hired, each capable of carrying 300 kilograms. The combined weight of the class is 1800 kilograms. It is determined that any group of 35 students can fit into the boats without exceeding the capacity of any one of them. Prove that it is unnecessary to leave any student off the cruise.

389. Let each of m distinct points on the positive part of the $x-$axis be joined by line segments to n distinct points on the positive part of the $y-$axis. Obtain a formula for the number of intersections of these segments (exclusive of endpoints), assuming that no three of the segments are concurrent.

390. Suppose that $n \geq 2$ and that $x_1, x_2, \cdots, x_n$ are positive integers for which $x_1 + x_2 + \cdots + x_n = 2(n+1)$. Show that there exists an index r with $0 \leq r \leq n - 1$ for which the following $n - 1$ inequalities hold:

$$x_{r+1} \leq 3$$

$$x_{r+1} + x_{r+2} \leq 5$$

$$\cdots$$

$$x_{r+1} + x_{r+2} + \cdots + r_{r+i} \leq 2i + 1$$

$$\cdots$$

$$x_{r+1} + x_{r+2} + \cdots + x_n \leq 2(n-r) + 1$$

$$\cdots$$

$$x_{r+1} + \cdots + x_n + x_1 + \cdots + x_j \leq 2(n+j-r) + 1$$

$$\cdots$$

$$x_{r+1} + x_{r+2} + \cdots + x_n + x_1 + \cdots + x_{r-1} \leq 2n - 1$$

where $1 \leq i \leq n-r$ and $1 \leq j \leq r-1$. Prove that, if all the inequalities are strict, then r is unique, and that, otherwise, there are exactly two such r.

391. Show that there are infinitely many nonsimilar ways that a square with integer side lengths can be partitioned into three nonoverlapping polygons with integer side lengths which are similar, but no two of which are congruent.

392. Determine necessary and sufficient conditions on the real parameter a, b, c that

$$\frac{b}{cx+a} + \frac{c}{ax+b} + \frac{a}{bx+c} = 0$$

has exactly one real solution.

393. Determine three positive rational numbers x, y, z whose sum s is rational and for which $x - s^3$, $y - s^3$, $z - s^3$ are all cubes of rational numbers.

394. The average age of the students in Ms. Ruler's class is 17.3 years, while the average age of the boys is 17.5 years. Give a cogent argument to prove that the average age of the girls cannot also exceed 17.3 years.

395. None of the nine participants at a meeting speaks more than three languages. Two of any three speakers speak a common language. Show that there is a language spoken by at least three participants.

396. Place 32 white and 32 black checkers on a 8×8 square chessboard. Two checkers of different colours form a *related pair* if they are placed in either the same row or the same column. Determine the maximum and the minimum number of related pairs over all possible arrangements of the 64 checkers.

397. The altitude from A of triangle ABC intersects BC in D. A circle touches BC at D, intersectes AB at M and N, and intersects AC at P and Q. Prove that

$$(AM + AN) : AC = (AP + AQ) : AB \ .$$

398. Given three disjoint circles in the plane, construct a point in the plane so that all three circles subtend the same angle at that point.

399. Let n and k be positive integers for which $k < n$. Determine the number of ways of choosing k numbers from $\{1, 2, \cdots, n\}$ so that no three consecutive numbers appear in any choice.

400. Let a_r and b_r $(1 \leq r \leq n)$ be real numbers for which $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and

$$b_1 \geq a_1 \ , \quad b_1 b_2 \geq a_1 a_2 \ , \quad b_1 b_2 b_3 \geq a_1 a_2 a_3 \ , \quad \cdots \quad , b_1 b_2 \cdots b_n \geq a_1 a_2 \cdots a_n \ .$$

Show that

$$b_1 + b_2 + \cdots + b_n \geq a_1 + a_2 + \cdots + a_n \ .$$

401. Five integers are arranged in a circle. The sum of the five integers is positive, but at least one of them is negative. The configuration is changed by the following moves: at any stage, a negative integer is selected and its sign is changed; this negative integer is added to each of its neighbours (*i.e.*, its absolute value is subtracted from each of its neighbours).

Prove that, regardless of the negative number selected for each move, the process will eventually terminate with all integers nonnegative in exactly the same number of moves with exactly the same configuration.

402. Let the sequences $\{x_n\}$ and $\{y_n\}$ be defined, for $n \geq 1$, by $x_1 = x_2 = 10$, $x_{n+2} = x_{n+1}(x_n + 1) + 1$ $(n \geq 1)$ and $y_1 = y_2 = -10$, $y_{n+2} = y_{n+1}(y_n + 1) + 1$ $(n \geq 1)$. Prove that there is no number that is a term of both sequences.

403. Let $f(x) = |1 - 2x| - 3|x + 1|$ for real values of x.

(a) Determine all values of the real parameter a for which the equation $f(x) = a$ has two different roots u and v that satisfy $2 \leq |u - v| \leq 10$.

(b) Solve the equation $f(x) = \lfloor x/2 \rfloor$.

404. Several points in the plane are said to be *in general position* if no three are collinear.

(a) Prove that, given 5 points in general position, there are always four of them that are vertices of a convex quadrilateral.

(b) Prove that, given 400 points in general position, there are at least 80 nonintersecting convex quadrilaterals, whose vertices are chosen from the given points. (Two quadrilaterals are nonintersecting if they do not have a common point, either in the interior or on the perimeter.)

(c) Prove that, given 20 points in general position, there are at least 969 convex quadrilaterals whose vertices are chosen from these points. (**Bonus:** Derive a formula for the number of these quadrilaterals given n points in general position.)

405. Suppose that a permutation of the numbers from 1 to 100, inclusive, is given. Consider the sums of all triples of consecutive numbers in the permutation. At most how many of these sums can be odd?

406. Let a, b. c be natural numbers such that the expression

$$\frac{a+1}{b} + \frac{b+1}{c} + \frac{c+1}{a}$$

is also equal to a natural number. Prove that the greatest common divisor of a, b and c, $\gcd(a, b, c)$, does not exceed $\sqrt[3]{ab + bc + ca}$, *i.e.*,

$$\gcd(a, b, c) \leq \sqrt[3]{ab + bc + ca} \ .$$

407. Is there a pair of natural numbers, x and y, for which

(a) $x^3 + y^4 = 2^{2003}$?

(b) $x^3 + y^4 = 2^{2005}$?

Provide reasoning for your answers to (a) and (b).

408. Prove that a number of the form $a000 \cdots 0009$ (with $n + 2$ digits for which the first digit a is followed by n zeros and the units digit is 9) cannot be the square of another integer.

409. Find the number of ways of dealing n cards to two persons ($n \geq 2$), where the persons may receive unequal (positive) numbers of cards. Disregard the order in which the cards are received.

410. Prove that $\log n \geq k \log 2$, where n is a natural number and k the number of distinct primes that divide n.

411. Let b be a positive integer. How many integers are there, each of which, when expressed to base b, is equal to the sum of the squares of its digits?

412. Let A and B be the midpoints of the sides, EF and ED, of an equilateral triangle DEF. Extend AB to meet the circumcircle of triangle DEF at C. Show that B divides AC according to the golden section. (That is, show that $BC : AB = AB : AC$.)

413. Let I be the incentre of triangle ABC. Let A', B' and C' denote the intersections of AI, BI and CI, respectively, with the incircle of triangle ABC. Continue the process by defining I' (the incentre of triangle $A'B'C'$), then $A''B''C''$, *etc.*. Prove that the angles of triangle $A^{(n)}B^{(n)}C^{(n)}$ get closer and closer to $\pi/3$ as n increases.

414. Let $f(n)$ be the greatest common divisor of the set of numbers of the form $k^n - k$, where $2 \leq k$, for $n \geq 2$. Evaluate $f(n)$. In particular, show that $f(2n) = 2$ for each integer n.

415. Prove that

$$\cos\frac{\pi}{7} = \frac{1}{6} + \frac{\sqrt{7}}{6}\left(\cos\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right) + \sqrt{3}\sin\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right)\right).$$

416. Let P be a point in the plane.

(a) Prove that there are three points A, B, C for which $AB = BC$, $\angle ABC = 90°$, $|PA| = 1$, $|PB| = 2$ and $|PC| = 3$.

(b) Determine $|AB|$ for the configuration in (a).

(c) A rotation of $90°$ about B takes C to A and P to Q. Determine $\angle APQ$.

417. Show that for each positive integer n, at least one of the five numbers 17^n, 17^{n+1}, 17^{n+2}, 17^{n+3}, 17^{n+4} begins with 1 (at the left) when written to base 10.

418. (a) Show that, for each pair m,n of positive integers, the minimum of $m^{1/n}$ and $n^{1/m}$ does not exceed $3^{1/2}$.

(b) Show that, for each positive integer n,

$$\left(1 + \frac{1}{\sqrt{n}}\right)^2 \geq n^{1/n} \geq 1 .$$

(c) Determine an integer N for which

$$n^{1/n} \leq 1.00002005$$

whenever $n \geq N$. Justify your answer.

419. Solve the system of equations

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = t$$

for x, y, z not all equal. Determine xyz.

420. Two circle intersect at A and B. Let P be a point on one of the circles. Suppose that PA meets the second circle again at C and PB meets the second circle again at D. For what position of P is the length of the segment CD maximum?

421. Let $ABCD$ be a tetrahedron. Prove that

$$|AB| \cdot |CD| + |AC| \cdot |BD| \geq |AD| \cdot |BC| \ .$$

422. Determine the smallest two positive integers n for which the numbers in the set $\{1, 2, \cdots, 3n-1, 3n\}$ can be partitioned into n disjoint triples $\{x, y, z\}$ for which $x + y = 3z$.

423. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

424. Simplify

$$\frac{x^3 - 3x + (x^2-1)\sqrt{x^2-4} - 2}{x^3 - 3x + (x^2-1)\sqrt{x^2-4} + 2}$$

to a fraction whose numerator and denominator are of the form $u\sqrt{v}$ with u and v each linear polynomials. For which values of x is the equation valid?

425. Let $\{x_1, x_2, \cdots, x_n, \cdots\}$ be a sequence of nonzero real numbers. Show that the sequence is an arithmetic progression if and only if, for each integer $n \geq 2$,

$$\frac{1}{x_1x_2} + \frac{1}{x_2x_3} + \cdots + \frac{1}{x_{n-1}x_n} = \frac{n-1}{x_1x_n} \ .$$

426. (a) The following paper-folding method is proposed for trisecting an acute angle.

(1) transfer the angle to a rectangular sheet so that its vertex is at one corner P of the sheet with one ray along the edge PY; let the angle be XPY;

(2) fold up PY over QZ to fall on RW, so that $PQ = QR$ and $PY \| QZ \| RW$, with QZ between PY and RW;

(3) fold across a line AC with A on the sheet and C on the edge PY so that P falls on a point P' on QZ and R on a point R' on PX;

(4) suppose that the fold AC intersects the fold QZ at B and carries Q to Q'; make a fold along BQ'.

It is claimed that the fold BQ' passes through P and trisects angle XPY.

Explain why the fold described in (3) is possible. Does the method work? Why?

(b) What happens with a right angle?

(c) Can the method be adapted for an obtuse angle?

427. The radius of the inscribed circle and the radii of the three escribed circles of a triangle are consecutive terms of a geometric progression. Determine the largest angle of the triangle.

428. **a**, **b** and **c** are three lines in space. Neither **a** nor **b** is perpendicular to **c**. Points P and Q vary on **a** and **b**, respectively, so that PQ is perpendicular to **c**. The plane through P perpendicular to **b** meets **c** at R, and the plane through Q perpendicular to **a** meets **c** at S. Prove that RS is of constant length.

429. Prove that

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \binom{kn}{n} = (-1)^{n+1} n^n \ .$$

430. Let triangle ABC be such that its excircle tangent to the segment AB is also tangent to the circle whose diameter is the segment BC. If the lengths of the sides BC, CA and AB of the triangle form, in this order, an arithmetic sequence, find the measure of the angle ACB.

431. Prove the following trigonometric identity, for any natural number n:

$$\sin\frac{\pi}{4n+2}\cdot\sin\frac{3\pi}{4n+2}\cdot\sin\frac{5\pi}{4n+2}\cdots\sin\frac{(2n-1)\pi}{4n+2}=\frac{1}{2^n}\ .$$

432. Find the exact value of:

(a)

$$\sqrt{\frac{1}{6}+\frac{\sqrt{5}}{18}}-\sqrt{\frac{1}{6}-\frac{\sqrt{5}}{18}}\ ;$$

(b)

$$\sqrt{1+\frac{2}{5}}\cdot\sqrt{1+\frac{2}{6}}\cdot\sqrt{1+\frac{2}{7}}\cdot\sqrt{1+\frac{2}{8}}\cdots\sqrt{1+\frac{2}{57}}\cdot\sqrt{1+\frac{2}{58}}\ .$$

433. Prove that the equation

$$x^2+2y^2+98z^2=77777\ldots777$$

does not have a solution in integers, where the right side has 2006 digits, all equal to 7.

434. Find all natural numbers n for which 2^n+n^{2004} is equal to a prime number.

435. A circle with centre I is the incircle of the convex quadrilateral $ABCD$. The diagonals AC and BD intersect at the point E. Prove that, if the midpoints of the segments AD, BC and IE are collinear, then $AB=CD$.

436. In the Euro-African volleyball tournament, there were nine more teams participating from Europe than from Africa. In total, the European won nine times as many points as were won by all of the African teams. In this tournamet, each team played exactly once against each other team; there were no ties; the winner of a game gets 1 point, the loser 0. What is the greatest possible score of the best African team?

437. Let a, b, c be the side lengths and m_a, m_b, m_c the lengths of their respective medians, of an arbitrary triangle ABC. Show that

$$\frac{3}{4}<\frac{m_a+m_b+m_c}{a+b+c}<1\ .$$

Furthermore, show that one cannot find a smaller interval to bound the ratio.

438. Determine all sets (x,y,z) of real numbers for which

$$x+y=2 \qquad\text{and}\qquad xy-z^2=1\ .$$

439. A natural number n, less than or equal to 500, has the property that when one chooses a number m randomly among $\{1,2,3,\cdots,500\}$, the probability that m divides n (*i.e.*, n/m is an integer) is 1/100. Find the largest such n.

440. You are to choose 10 distinct numbers from $\{1,2,3,\cdots,2006\}$. Show that you can choose such numbers with a sum greater than 10039 in more ways than you can choose such numbers with a sum less than 10030.

441. Prove that, no matter how 15 points are placed inside a circle of radius 2 (including the boundary), there exists a circle of radius 1 (including the boundary) containing at least 3 of the 15 points.

442. Prove that the regular tetrahedron has minimum diameter among all tetrahedra that circumscribe a given sphere. (The diameter of a tetrahedron is the length of its longest edge.)

443. For $n \geq 3$, show that $n-1$ straight lines are sufficient to go through the interior of every square of an $n \times n$ chessboard. Are $n-1$ lines necessary?

444. (a) Suppose that a 6×6 square grid of unit squares (chessboard) is tiled by 1×2 rectangles (dominoes). Prove that it can be decomposed into two rectangles, tiled by disjoint subsets of the dominoes.

(b) Is the same thing true for an 8×8 array?

(c) Is the same thing true for a 6×8 array?

445. Two parabolas have parallel axes and intersect in two points. Prove that their common chord bisects the segments whose endpoints are the points of contact of their common tangent.

446. Suppose that you have a 3×3 grid of squares. A *line* is a set of three squares in the same row, the same column or the same diagonal; thus, there are eight lines.

Two players A and B play a game. They take alternate turns, A putting a 0 in any unoccupied square of the grid and B putting a 1. The first player is A, and the game cannot go on for more than nine moves. (The play is similar to noughts-and-crosses, or tictactoe.) A move is *legitimate* if it does not result in two lines of squares being filled in with different sums. The winner is the last player to make a legitimate move.

(For example, if there are three 0s down the diagonal, then B can place a 1 in any vacant square provided it completes no other line, for then the sum would differ from the diagonal sum. If there are two zeros at the top of the main diagonal and two ones at the left of the bottom line, then the lower right square cannot be filled by either player, as it would result in two lines with different sums.)

(a) What is the maximum number of legitimate moves possible in a game?

(b) What is the minimum number of legitimate moves possible in a game that would not leave a legitimate move available for the next player?

(c) Which player has a winning strategy? Explain.

447. A high school student asked to solve the surd equation

$$\sqrt{3x-2} - \sqrt{2x-3} = 1$$

gave the following answer: *Squaring both sides leads to*

$$3x - 2 - 2x - 3 = 1$$

so $x = 6$. The answer is, in fact, correct.

Show that there are infinitely many real quadruples (a, b, c, d) for which this method leads to a correct solution of the surd equation

$$\sqrt{ax-b} - \sqrt{cx-d} = 1 \ .$$

448. A criminal, having escaped from prison, travelled for 10 hours before his escape was detected. He was then pursued and gained upon at 3 miles per hour. When his pursuers had been 8 hours on the way, they met an express (train) going in the opposite direction at the same rate as themselves, which had met the criminal 2 hours and 24 minutes earlier. In what time from the beginning of the pursuit will the criminal be overtaken? [from *The high school algebra* by Robertson and Birchard, approved for Ontario schools in 1886]

449. Let $S = \{x : x > -1\}$. Determine all functions from S to S which both

(a) satisfies the equation $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all $x, y \in S$, and

(b) $f(x)/x$ is strictly increasing or strictly decreasing on each of the two intervals $\{x : -1 < x < 0\}$ and $\{x : x > 0\}$.

450. The 4-sectors of an angle are the three lines through its vertex that partition the angle into four equal parts; adjacent 4-sectors of two angles that share a side consist of the 4-sector through each vertex that is closest to the other vertex.

Prove that adjacent 4-sectors of the angles of a parallelogram meet in the vertices of a square if and only if the parallelogram has four equal sides.

451. Let a and b be positive integers and let $u = a + b$ and $v = lcm\ (a, b)$. Prove that

$$gcd\ (u, v) = gcd\ (a, b)\ .$$

452. (a) Let m be a positive integer. Show that there exists a positive integer k for which the set

$$\{k + 1, k + 2, \ldots, 2k\}$$

contains exactly m numbers whose binary representation has exactly three digits equal to 1.

(b) Determine all intgers m for which there is exactly one such integer k.

453. Let A, B be two points on a circle, and let AP and BQ be two rays of equal length that are tangent to the circle that are directed counterclockwise from their tangency points. Prove that the line AB intersects the segment PQ at its midpoint.

454. Let ABC be a non-isosceles triangle with circumcentre O, incentre I and orthocentre H. Prove that the angle OIH exceeds $90°$.

455. Let $ABCDE$ be a pentagon for which the position of the base AB and the lengths of the five sides are fixed. Find the locus of the point D for all such pentagons for which the angles at C and E are equal.

456. Let $n + 1$ cups, labelled in order with the numbers $0, 1, 2, \cdots, n$, be given. Suppose that $n + 1$ tokens, one bearing each of the numbers $0, 1, 2, \cdots, n$ are distributed randomly into the cups, so that each cup contains exactly one token.

We perform a sequence of moves. At each move, determine the smallest number k for which the cup with label k has a token with label m not equal to k. Necessarily, $k < m$. Remove this token; move all the tokens in cups labelled $k + 1, k + 2, \cdots, m$ to the respective cups labelled $k, k + 1, m - 1$; drop the token with label m into the cup with label m. Repeat.

Prove that the process terminates with each token in its own cup (token k in cup k for each k) in not more that $2^n - 1$ moves. Determine when it takes exactly $2^n - 1$ moves.

457. Suppose that $u_1 > u_2 > u_3 > \cdots$ and that there are infinitely many indices n for which $u_n \geq 1/n$. Prove that there exists a positive integer N for which

$$u_1 + u_2 + u_3 + \cdots + u_N > 2006\ .$$

458. Let ABC be a triangle. Let A_1 be the reflected image of A with axis BC, B_1 the reflected image of B with axis CA and C_1 the reflected image of C with axis AB. Determine the possible sets of angles of triangle ABC for which $A_1B_1C_1$ is equilateral.

459. At an International Conference, there were exactly 2006 participants. The organizers observed that: (1) among any three participants, there were two who spoke the same language; and (2) every participant spoke at most 5 languages. Prove that there is a group of at least 202 participants who speak the same language.

460. Given two natural numbers x and y for which

$$3x^2 + x = 4y^2 + y\ ,$$

prove that their positive difference is a perfect square. Determine a nontrivial solution of this equation.

461. Suppose that x and y are integers for which $x^2 + y^2 \neq 0$. Determine the minimum value of the function

$$f(x,y) \equiv |5x^2 + 11xy - 5y^2| \ .$$

462. For any positive real numbers a, b, c, d, establish the inequality

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+d}} + \sqrt{\frac{c}{d+a}} + \sqrt{\frac{d}{a+b}} > 2 \ .$$

463. In *Squareland*, a newly-created country in the shape of a square with side length of 1000 km, there are 51 cities. The country can afford to build at most 11000 km of roads. Is it always possible, within this limit, to design a road map that provides a connection between any two cities in the country?

464. A square is partitioned into non-overlapping rectangles. Consider the circumcircles of all the rectangles. Prove that, if the sum of the areas of all these circles is equal to the area of the circumcircle of the square, then all the rectangles must be squares, too.

465. For what positive real numbers a is

$$\sqrt[3]{2+\sqrt{a}} + \sqrt[3]{2-\sqrt{a}}$$

an integer?

466. For a positive integer m, let $\overline{m}$ denote the sum of the digits of m. Find all pairs of positive integers $(m.n)$ with $m < n$ for which $(\overline{m})^2 = n$ and $(\overline{n})^2 = m$.

467. For which positive integers n does there exist a set of n distinct positive integers such that

(a) each member of the set divides the sum of all members of the set, *and*

(b) none of its proper subsets with two or more elements satisfies the condition in (a)?

468. Let a and b be positive real numbers satisfying $a + b \geq (a-b)^2$. Prove that

$$x^a(1-x)^b + x^b(1-x)^a \leq \frac{1}{2^{a+b-1}}$$

for $0 \leq x \leq 1$, with equality if and only if $x = \frac{1}{2}$.

469. Solve for t in terms of a, b in the equation

$$\sqrt{\frac{t^3+a^3}{t+a}} + \sqrt{\frac{t^3+b^3}{t+b}} = \sqrt{\frac{a^3-b^3}{a-b}}$$

where $0 < a < b$.

470. Let ABC, ACP and BCQ be nonoverlapping triangles in the plane with angles CAP and CBQ right. Let M be the foot of the perpendicular from C to AB. Prove that lines AQ, BP and CM are concurrent if and only if $\angle BCQ = \angle ACP$.

471. Let I and O denote the incentre and the circumcentre, respectively, of triangle ABC. Assume that triangle ABC is not equilateral. Prove that $\angle AIO \leq 90°$ if and only if $2BC \leq AB + CA$, with equality holding only simultaneously.

472. Find all integers x for which

$$(4-x)^{4-x} + (5-x)^{5-x} + 10 = 4^x + 5^x \ .$$

473. Let $ABCD$ be a quadrilateral; let M and N be the respective midpoint of AB and BC; let P be the point of interesection of AN and BD, and Q be the point of intersection of DM amd AC. Suppose the $3BP = BD$ and $3AQ = AC$. Prove that $ABCD$ is a parallelogram.

474. Solve the equation for positive real x:

$$(2^{\log_5 x} + 3)^{\log_5 2} = x - 3 \ .$$

475. Let z_1, z_2, z_3, z_4 be distinct complex numbers for which $|z_1| = |z_2| = |z_3| = |z_4|$. Suppose that there is a real number $t \neq 1$ for which

$$|tz_1 + z_2 + z_3 + z_4| = |z_1 + tz_2 + z_3 + z_4| = |z_1 + z_2 + tz_3 + z_4| \ .$$

Show that, in the complex plane, z_1, z_2, z_3, z_4 lie at the vertices of a rectangle.

476. Let p be a positive real number and let $|x_0| \leq 2p$. For $n \geq 1$, define

$$x_n = 3x_{n-1} - \frac{1}{p^2}x_{n-1}^3 \ .$$

Determine x_n as a function of n and x_0.

477. Let S consist of all real numbers of the form $a + b\sqrt{2}$, where a and b are integers. Find all functions that map S into the set $\mathbf{R}$ of reals such that (1) f is increasing, and (2) $f(x+y) = f(x) + f(y)$ for all x, y in S.

478. Solve the equation

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + x}}} + \sqrt{3}\sqrt{2 - \sqrt{2 + \sqrt{2 + x}}} = 2x$$

for $x \geq 0$

479. Let x, y, z be positive integer for which

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$$

and the greatest common divisor of x and z is 1. Prove that $x + y$, $x - z$ and $y - z$ are all perfect squares. Give two examples of triples (x, y, z) that satisfy these conditions.

480. Let a and b be positive real numbers for which $60^a = 3$ and $60^b = 5$. Without the use of a calculator or of logarithms, determine the value of

$$12^{\frac{1-a-b}{2(1-b)}} \ .$$

481. In a certain town of population $2n + 1$, one knows those to whom one is known. For any set A of n citizens, there is some person among the other $n + 1$ who knows everyone on A. Show that some citizen of the town knows all the others.

482. A trapezoid whose parallel sides have the lengths a and b is partitioned into two trapezoids of equal area by a line segment of length c parallel to these sides. Determine c as a function of a and b.

483. Let A and B be two points on the circumference of a circle, and E be the midpoint of arc AB (either arc will do). Let P be any point on the minor arc EB and N the foot of the perpendicular from E to AP. Prove that $AN = NP + PB$.

484. ABC is a triangle with $\angle A = 40°$ and $\angle B = 60°$. Let D and E be respective points of AB and AC for which $\angle DCB = 70°$ and $\angle EBC = 40°$. Furthermore, let F be the point of intersection of DC and EB. Prove that $AF \perp BC$.

485. From the foot of each altitude of the triangle, perpendiculars are dropped to the other two sides. Prove that the six feet of these perpendiculars lie on a circle.

486. Determine all quintuplets (a, b, c, d, u) of nonzero integers for which

$$\frac{a}{b} = \frac{c}{d} = \frac{ab+u}{cd+u} \ .$$

487. ABC is an isosceles triangle with $\angle A = 100°$ and $AB = AC$. The bisector of angle B meets AC in D. Show that $BD + AD = BC$.

488. A host is expecting a number of children, which is either 7 or 11. She has 77 marbles as gifts, and distributes them into n bags in such a way that whether 7 or 11 children come, each will receive a number of bags so that all 77 marbles will be shared equally among the children. What is the minimum value of n?

489. Suppose n is a positive integer not less than 2 and that $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq 0$,

$$\sum_{i=1}^{n} x_i \leq 400 \qquad \text{and} \qquad \sum_{i=1}^{n} x_i^2 \geq 10^4 \ .$$

Prove that $\sqrt{x_1} + \sqrt{x_2} \geq 10$. is it possible to have equality throughout? [*Bonus:* Formulate and prove a generalization.]

490. (a) Let a, b, c be real numbers. Prove that

$$\min \ [(a-b)^2, (b-c)^2, (c-a)^2] \leq \frac{1}{2}[a^2 + b^2 + c^2] \ .$$

(b) Does there exist a number k for which

$$\min \ [(a-b)^2, (a-c)^2, (a-d)^2, (b-c)^2, (b-d)^2, (c-d)^2] \leq k[a^2 + b^2 + c^2 + d^2]$$

for any real numbers a, b, c, d? If so, determine the smallest such k.
[*Bonus:* Determine if there is a generalization.]

491. Given that x and y are positive real numbers for which $x + y = 1$ and that m and n are positive integers exceeding 1, prove that

$$(1 - x^m)^n + (1 - y^n)^m > 1 \ .$$

492. The faces of a tetrahedron are formed by four congruent triangles. if α is the angle between a pair of opposite edges of the tetrahedron, show that

$$\cos \alpha = \frac{\sin(B - C)}{\sin(B + C)}$$

where B and C are the angles adjacent to one of these edges in a face of the tetrahedron.

493. Prove that there is a natural number with the following characteristics: (a) it is a multiple of 2007; (b) the first four digits in its decimal representation are 2009; (c) the last four digits in its decimal representation are 2009.

494. (a) Find all real numbers x that satisfy the equation

$$(8x - 56)\sqrt{3 - x} = 30x - x^2 - 97 \ .$$

(b) Find all real numbers x that satisfy the equation

$$\sqrt{x} + \sqrt[3]{x+7} = \sqrt[4]{x+80}\ .$$

495. Let $n \geq 3$. A regular $n-$gon has area S. Squares are constructed externally on its sides, and the vertices of adjacent squares that are not vertices of the polygon are connected to form a $2n-$sided polygon, whose area is T. Prove that $T \leq 4(\sqrt{3}+1)S$. For what values of n does equality hold?

496. Is the hundreds digit of $N = 2^{2006} + 2^{2007} + 2^{2008}$ even or odd? Justify your answer.

497. Given $n \geq 4$ points in the plane with no three collinear, construct all segments connecting two of these points. It is known that the length of each of these segments is a positive integer. Prove that the lengths of at least $1/6$ of the segments are multiples of 3.

498. Let a be a real parameter. Consider the simultaneous sytem of two equations:

$$\frac{1}{x+y} + x = a - 1\ ; \tag{1}$$

$$\frac{x}{x+y} = a - 2\ . \tag{2}$$

(a) For what value of the parameter a does the system have exactly one solution?

(b) Let $2 < a < 3$. Suppose that (x, y) satisfies the sytem. For which value of a in the stated range does $(x/y) + (y/x)$ reach its maximum value?

499. The triangle ABC has all acute angles. The bisector of angle ACB intersects AB at L. Segments LM and LN with $M \in AC$ and $N \in BC$ are constructed, perpendicular to the sides AC and BC respectively. Suppose that AN and BM intersect at P. Prove that CP is perpendicular to AB.

500. Find all sets of distinct integers $1 < a < b < c < d$ for which $abcd - 1$ is divisible by $(a-1)(b-1)(c-1)(d-1)$.

501. Given a list of $3n$ not necessarily distinct elements of a set S, determine necessary and sufficient conditions under which these $3n$ elements can be divided into n triples, none of which consist of three distinct elements.

502. A set consisting of n men and n women is partitioned at random into n disjoint pairs of people. What are the expected value and variance of the number of male-female couples that result? (The *expected value* E is the average of the number N of male-female couples over all possibilities, *i.e.* the sum of the numbers of male-female couples for the possibilities divided by the number of possibilities. The *variance* is the average of the difference $(E-N)^2$ over all possibilities, *i.e.* the sum of the values of $(E-N)^2$ for the possibilities divided by the number of possibilities.)

503. A natural number is *perfect* if it is the sum of its proper positive divisors. Prove that no two consecutive numbers can both be perfect.

504. Find all functions f taking the real numbers into the real numbers for which the following conditions hold simultaneously:

(a) $f(x + f(y) + yf(x)) = y + f(x) + xf(y)$ for every real pair (x, y);

(b) $\{f(x)/x : x \neq 0\}$ is a finite set.

505. What is the largest cubical present that can be completely wrapped (without cutting) by a unit square of wrapping paper?

506. A two-person game is played as follows. A position consists of a pair (a, b) of positive integers. Playes move alternately. A move consists of decreasing the larger number in the current position by any positive multiple of the smaller number, as long as the result remains positive. The first player unable to make a move loses. (This happens, for example, when $a = b$.) Determine those positions (a, b) from which the first player can guarantee a win with optimal play.

507. Prove that, if a, b, c are positive reals, then

$$\log^2 \frac{ab}{c} + \log^2 \frac{bc}{a} + \log^2 \frac{ca}{b} + \frac{3}{4} \geq \log(abc) \ .$$

508. Let a, b, c be integers exceeding 1 for which both $\log_a b + \log_b a$ and $\log_a^2 b + \log_b^2 a$ are rational. Prove that, for every positive integer n, $\log_a^n b + \log_b^n a$ is rational.

509. Let $ABCDA'B'C'D'$ be a cube where the point O is the centre of the face $ABCD$ and $|AB| = 2a$. Calculate the distance from the point B to the line of intersection of the planes $A'B'O$ and $ADD'A'$ and the distance between AB' and BD. (AA', BB', CC', DD' are edges of the cube.)

510. Solve the equation

$$\sqrt[3]{x^2+2} + \sqrt[3]{4x^2+3x-2} = \sqrt[3]{3x^2+x+5} + \sqrt[3]{2x^2+2x-5} \ .$$

511. Find the sum of the last 100 digits of the number

$$A = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2005 \cdot 2006 + 2007 \ .$$

512. Prove that

$$\binom{3n}{n} = \sum_{k=0}^{n} \binom{2n}{k} \binom{n}{k}$$

when $n \geq 1$.

513. Solve the equation

$$2^{1-2\sin^2 x} = 2 + \log_2(1 - \sin^2 x) \ .$$

514. Prove that there do not exist polynomials $f(x)$ and $g(x)$ with complex coefficients for which

$$\log_b x = \frac{f(x)}{g(x)}$$

where b is any base exceeding 1.

515. Let n be a fixed positive integer exceeding 1. To any choice of n real numbers x_i satisfying $0 \leq x_i \leq 1$, we can associate the sum

$$\sum \{|x_i - x_j| : 1 \leq i < j \leq n\} \ .$$

What is the maximum possible value of this sum and for which values of the x_i is it assumed?

516. Let $n \geq 1$. Is it true that, for any $2n+1$ positive real numbers $x_1, x_2, \cdots, x_{2n+1}$, we have that

$$\frac{x_1x_2}{x_3} + \frac{x_2x_3}{x_4} + \cdots + \frac{x_{2n+1}x_1}{x_2} \geq x_1 + x_2 + \cdots + x_{2n+1} \ ,$$

with equality if and only if all the x_i are equal?

517. A man bought four items in a *Seven-Eleven* store. The clerk entered the four prices into a pocket calculator and *multiplied* to get a result of 7.11 dollars. When the customer objected to this procedure, the clerk realized that he should have added and redid the calculation. To his surprise, he again got the answer 7.11. What did the four items cost?

518. Let I be the incentre of triangle ABC, and let AI, BI, CI, produced, intersect the circumcircle of triangle ABC at the respective points D, E, F. Prove that $EF \perp AD$.

519. Let AB be a diameter of a circle and X any point other than A and B on the circumference of the circle. Let t_A, t_B and t_X be the tangents to the circle at the respective points A, B and X. Suppose that AX meets t_B at Z and BX meets t_A at Y. Show that the three lines YZ, t_X and AB are either concurrent (i.e. passing through a common point) or parallel.

520. The *diameter* of a plane figure is the largest distance between any pair of points in the figure. Given an equilateral triangle of side 1, show how, by a stright cut, one can get two pieces that can be rearranged to form a figure with maximum diameter

(a) if the resulting figure is convex (*i.e.* the line segment joining any two of its points must lie inside the figure);

(b) if the resulting figure is not necessaarily convex, but it is connected (*i.e.* any two points in the figure can be connected by a curve lying inside the figure).

521. On a 8×8 chessboard, either $+1$ or -1 is written in each square cell. Let A_k be the product of all the numbers in the kth row, and B_k the product of all the numbers in the kth column of the board $(k = 1, 2, \cdots, 8)$. Prove that the number

$$A_1 + A_2 + \cdots + A_8 + B_1 + B_2 + \cdots + B_8$$

is a multiple of 4.

522. (a) Prove that, in each scalene triangle, the angle bisector from one of its vertices is always "between" the median and the altitude from the same vertex.

(b) Find the measures of the angles of a triangle if the lengths of the median, the angle bisector and the altitude from one of its vertices are in the ratio $\sqrt{5} : \sqrt{2} : 1$.

523. Let ABC be an isosceles triangle with $AB = AC$. The segments BC and AC are used as hypotenuses to construct three right triangles BCM, BCN and ACP. Prove that, if $\angle ACP + \angle BCM + \angle BCN = 90°$, then the triangle MPN is isosceles.

524. Solve the irrational equation

$$\frac{7}{\sqrt{x^2-10x+26}+\sqrt{x^2-10x+29}+\sqrt{x^2-10x+41}} = x^4 - 9x^3 + 16x^2 + 15x + 26 \ .$$

525. The circle inscribed in the triangle ABC divides the median from A into three segments of the same length. If the area of ABC is $6\sqrt{14}$, calculate the lengths of its sides.

526. For the non-negative numbers a, b, c, prove the inequality

$$4(a + b + c) \geq 3(a + \sqrt{ab} + \sqrt[3]{abc}) \ .$$

When does equality hold?

527. Consider the set A of the $2n-$digit natural numbers, with 1 and 2 each occurring n times as a digit, and the set B of the $n-$digit numbers all of whose digits are 1, 2, 3, 4 with the digits 1 and 2 occurring

with equal frequency. Show that A and B contain the same number of elements (*i.e.*, have the same cardinality).

528. Let the sequence $\{x_n : n = 0, 1, 2, \cdots\}$ be defined by $x_0 = a$ and $x_1 = b$, where a and b are real numbers, and by

$$7x_n = 5x_{n-1} + 2x_{n-2}$$

for $n \geq 2$. Derive a formula for x_n as a function of a, b and n.

529. Let k, n be positive integers. Define $p_{n,1} = 1$ for all n and $p_{n,k} = 0$ for $k \geq n + 1$. For $2 \leq k \leq n$, we define inductively

$$p_{n,k} = k(p_{n-1,k-1} + p_{n-1,k}) \ .$$

Prove, by mathematical induction, that

$$p_{n,k} = \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k-r)^n \ .$$

530. Let $\{x_1, x_2, x_3, \cdots, x_n, \cdots\}$ be a sequence is distinct positive real numbers. Prove that this sequence is a geometric progression if and only if

$$\frac{x_1}{x_2} \sum_{k=1}^{n-1} \frac{x_n^2}{x_k x_{k+1}} = \frac{x_n^2 - x_1^2}{x_2^2 - x_1^2}$$

for all $n \geq 2$.

531. Show that the remainder of the polynomial

$$p(x) = x^{2007} + 2x^{2006} + 3x^{2005} + 4x^{2004} + \cdots + 2005x^3 + 2006x^2 + 2007x + 2008$$

is the same upon division by $x(x + 1)$ as upon division by $x(x + 1)^2$.

532. The angle bisectors BD and CE of triangle ABC meet AC and AB at D and E respectively and meet at I. If $[ABD] = [ACE]$, prove that $AI \perp ED$. is the converse true?

533. Prove that the number

$$1 + \lfloor (5 + \sqrt{17}))^{2008} \rfloor$$

is divisible by 2^{2008}.

534. Let $\{x_n : n = 1, 2, \cdots\}$ be a sequence of distinct positive integers, with $x_1 = a$. Suppose that

$$2 \sum_{k=1}^{n} \sqrt{x_i} = (n + 1)\sqrt{x_n}$$

for $n \geq 2$. Determine $\sum_{k=1}^{n} x_k$.

535. Let the triangle ABC be isosceles with $AB = AC$. Suppose that its circumcentre is O, the D is the midpoint of side AB and that E is the centroid of triangle ACD. Prove that OE is perpendicular to CD.

536. There are 21 cities, and several airlines are responsible for connections between them. Each airline serves five cities with flights both ways between all pairs of them. Two or more airlines may serve a given pair of cities. Every pair of cities is serviced by at least one direct return flight. What is the minimum number of airlines that would meet these conditions?

537. Consider all 2×2 square arrays each of whose entries is either 0 or 1. A pair (A, B) of such arrays is *compatible* if there exists a 3×3 square array in which both A and B appear as 2×2 subarrays.

For example, the two matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are compatible, as both can be found in the array

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

Determine all pairs of 2×2 arrays that are not compatible.

538. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the right bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have the same area.

539. Determine the maximum value of the expression

$$\frac{xy + 2yz + zw}{x^2 + y^2 + z^2 + w^2}$$

over all quartuple of real numbers not all zero.

540. Suppose that, if all planar cross-sections of a bounded solid figure are circles, then the solid figure must be a sphere.

541. Prove that the equation

$$x_1^{x_1} + x_2^{x_2} + \cdots + x_k^{x_k} = x_{k+1}^{x_{k+1}}$$

has no solution for which $x_1, x_2, \cdots, x_k, x_{k+1}$ are all distinct nonzero integers.

542. Solve the system of equations

$$\lfloor x \rfloor + 3\{y\} = 3.9 ,$$
$$\{x\} + 3\lfloor y \rfloor = 3.4 .$$

543. Let $a > 0$ and b be real parameters, and suppose that f is a function taking the set of reals to itself for which

$$f(a^3x^3 + 3a^2bx^2 + 3ab^2x) \le x \le a^3f(x)^3 + 3a^2bf(x)^2 + 3ab^2f(x) ,$$

for all real x. Prove that f is a one-one function that takes the set of real numbers onto itself (*i.e.*, f is a *bijection*).

544. Define the real sequences $\{a_n : n \ge 1\}$ and $\{b_n : n \ge 1\}$ by $a_1 = 1$, $a_{n+1} = 5a_n + 4$ and $5b_n = a_n + 1$ for $n \ge 1$.

(a) Determine $\{a_n\}$ as a function of n.

(b) Prove that $\{b_n : n \ge 1\}$ is a geometric progression and evaluate the sum

$$S \equiv \frac{\sqrt{b_1}}{\sqrt{b_2} - \sqrt{b_1}} + \frac{\sqrt{b_2}}{\sqrt{b_3} - \sqrt{b_2}} + \cdots + \frac{\sqrt{b_n}}{\sqrt{b_{n+1}} - \sqrt{b_n}} .$$

545. Suppose that x and y are real numbers for which $x^3 + 3x^2 + 4x + 5 = 0$ and $y^3 - 3y^2 + 4y - 5 = 0$. Determine $(x + y)^{2008}$.

546. Let $a, a_1, a_2, \cdots, a_n$ be a set of positive real numbers for which

$$a_1 + a_2 + \cdots + a_n = a$$

and

$$\sum_{k=1}^{n} \frac{1}{a - a_k} = \frac{n+1}{a} \ .$$

Prove that

$$\sum_{k=1}^{n} \frac{a_k}{a - a_k} = 1 \ .$$

547. Let A, B, C, D be four points on a circle, and let E be the fourth point of the parallelogram with vertices A, B, C. Let AD and BC intersect at M, AB and DC intersect at N, and EC and MN intersect at F. Prove that the quadrilateral $DENF$ is concyclic.

548. In a sphere of radius R is inscribed a regular hexagonal truncated pyramid whose big base is inscribed in a great circle of the sphere (i.e., a whose centre is the centre of the sphere). The length of the side of the big base is three times the length of the side of a small base. Find the volume of the truncated pyramid as a function of R.

549. The set E consists of 37 two-digit natural numbers, none of them a multiple of 10. Prove that, among the elements of E, we can find at least five numbers, such that any two of them have different tens digits and different units digits.

550. The functions $f(x)$ and $g(x)$ are defined by the equations: $f(x) = 2x^2 + 2x - 4$ and $g(x) = x^2 - x + 2$.

(a) Find all real numbers x for which $f(x)/g(x)$ is a natural number.

(b) Find the solutions of the inequality

$$\sqrt{f(x)} + \sqrt{g(x)} \geq 2 \ .$$

551. The numbers 1, 2, 3 and 4 are written on the circumference of a circle, in this order. Alice and Bob play the following game: On each turn, Alice adds 1 to two adjacent numbers, while Bob switches the places of two adjacent numbers. Alice wins the game, if after her turn, all numbers on the circle are equal. Does Bob have a strategy to prevent Alice from winning the game? Justify your answer.

552. Two real nonnegative numbers a and b satisfy the inequality $ab \geq a^3 + b^3$. Prove that $a + b \leq 1$.

553. The convex quadrilateral $ABCD$ is concyclic with side lengths $|AB| = 4$, $|BC| = 3$, $|CD| = 2$ and $|DA| = 1$. What is the length of the radius of the circumcircle of $ABCD$? Provide an exact value of the answer.

554. Determine all real pairs (x, y) that satisfy the system of equations:

$$3\sqrt[3]{x^2y^5} = 4(y^2 - x^2)$$

$$5\sqrt[3]{x^4y} = y^2 + x^2 \ .$$

555. Let ABC be a triangle, all of whose angles do not exceed $90°$. The points K on side AB, M on side AC and N on side BC are such that $KM \perp AC$ and $KN \perp BC$. Prove that the area $[ABC]$ of triangle ABC is at least 4 times as great as the area $[KMN]$ of triangle KMN, *i.e.*, $[ABC] \geq 4[KMN]$. When does equality hold?

556. Let x, y, z be positive real numbers for which $x + y + z = 4$. Prove the inequality

$$\frac{1}{2xy + xz + yz} + \frac{1}{xy + 2xz + yz} + \frac{1}{xy + xz + 2yz} \le \frac{1}{xyz} .$$

557. Suppose that the polynomial $f(x) = (1+x+x^2)^{1004}$ has the expansion $a_0 + a_1x + a_2x^2 + \cdots + a_{2008}x^{2008}$. Prove that $a_0 + a_2 + \cdots + a_{2008}$ is an odd integer.

558. Determine the sum

$$\sum_{m=0}^{n-1} \sum_{k=0}^{m} \binom{n}{k} .$$

559. Let ϵ be one of the roots of the equation $x^n = 1$, where n is a positive integer. Prove that, for any polynomial $f(x) = a_0 + a_x + \cdots + a_n x^n$ with real coefficients, the sum $\sum_{k=1}^{n} f(1/\epsilon^k)$ is real.

560. Suppose that the numbers $x_1, x_2, \cdots, x_n$ all satisfy $-1 \le x_i \le 1$ $(1 \le i \le n)$ and $x_1^3 + x_2^3 + \cdots + x_n^3 = 0$. Prove that

$$x_1 + x_2 + \cdots + x_n \le \frac{n}{3} .$$

561. Solve the equation

$$\left(\frac{1}{10}\right)^{\log_{(1/4)}(\sqrt[4]{x}-1)} - 4^{\log_{10}(\sqrt[4]{x}+5)} = 6 ,$$

for $x \ge 1$.

562. The circles $\mathfrak{C}$ and $\mathfrak{D}$ intersect at the two points A and B. A secant through A intersects $\mathfrak{C}$ at C and $\mathfrak{D}$ at D. On the segments CD, BC, BD, consider the respective points M, N, K for which $MN \| BD$ and $MK \| BC$. On the arc BC of the circle $\mathfrak{C}$ that does not contain A, choose E so that $EN \perp BC$, and on the arc BD of the circle $\mathfrak{D}$ that does not contain A, choose F so that $FK \perp BD$. Prove that angle EMF is right.

563. (a) Determine infinitely many triples (a, b, c) of integers for which a, b, c are not in arithmetic progression and $ab + 1$, $bc + 1$, $ca + 1$ are all squares.

(b) Determine infinitely many triples (a, b, c) of integers for which a, b, c are in arithemetic progression and $ab + 1$, $bc + 1$, $ca + 1$ are all squares.

(c) Determine infinitely many triples (u, v, w) of integers for which $uv - 1$, $vw - 1$, $wu - 1$ are all squares. (Can it be arranged that u, v, w are in arithmetic progression?)

564. Let $x_1 = 2$ and

$$x_{n+1} = \frac{2x_n}{3} + \frac{1}{3x_n}$$

for $n \ge 1$. Prove that, for all $n > 1$, $1 < x_n < 2$.

565. Let ABC be an acute-angled triangle. Points A_1 and A_2 are located on side BC so that the four points are ordered B, A_1, A_2, C; similarly B_1 and B_2 are on CA in the order C, B_1, B_2, A and C_1 and C_2 on side AB in order A, C_1, C_2, B. All the angles AA_1A_2, AA_2A_1, BB_1B_2, BB_2B_1, CC_1C_2, CC_2C_1 are equal to θ. Let $\mathfrak{T}_1$ be the triangle bounded by the lines AA_1, BB_1, CC_1 and $\mathfrak{T}_2$ the triangle bounded by the lines AA_2, BB_2, CC_2. Prove that all six vertices of the triangles are concyclic.

566. A deck of cards numbered 1 to n (one card for each number) is arranged in random order and placed on the table. If the card numbered k is on top, remove the kth card counted from the top and place it on top of the pile, not otherwise disturbing the order of the cards. Repeat the process. Prove that the card numbered 1 will eventually come to the top, and determine the maximum number of moves that is required to achieve this.

567. (a) Let A, B, C, D be four distinct points in a straight line. For any points X, Y on the line, let XY denote the *directed* distance between them. In other words, a positive direction is selected on the line and $XY = \pm|XY|$ according as the direction X to Y is positive or negative. Define

$$(AC, BD) = \frac{AB/BC}{AD/DC} = \frac{AB \times CD}{BC \times DA} .$$

Prove that $(AB, CD) + (AC, BD) = 1$.

(b) In the situation of (a), suppose in addition that $(AC, BD) = -1$. Prove that

$$\frac{1}{AC} = \frac{1}{2}\left(\frac{1}{AB} + \frac{1}{AD}\right) ,$$

and that

$$OC^2 = OB \times OD ,$$

where O is the midpoint of AC. Deduce from the latter that, if Q is the midpoint of BD and if the circles on diameters AC and BD intersect at P, $\angle OPQ = 90°$.

(c) Suppose that A, B, C, D are four distinct on one line and that P, Q, R, S are four distinct points on a second line. Suppose that AP, BQ, CR and DS all intersect in a common point V. Prove that $(AC, BD) = (PR, QS)$.

(d) Suppose that $ABQP$ is a quadrilateral in the plane with no two sides parallel. Let AQ and BP intersect in U, and let AP and BQ intersect in V. Suppose that VU and PQ produced meet AB at C and D respectively, and that VU meets PQ at W. Prove that

$$(AB, CD) = (PQ, WD) = -1 .$$

568. Let ABC be a triangle and the point D on BC be the foot of the altitude AD from A. Suppose that H lies on the segment AD and that BH and CH intersect AC and AB at E and F respectively.

Prove that $\angle FDH = \angle HDE$.

569. Let A, W, B, U, C, V be six points in this order on a circle such that AU, BV and CW all intersect in the common point P at angles of $60°$. Prove that

$$|PA| + |PB| + |PC| = |PU| + |PV| + |PW| .$$

570. Let a be an integer. Consider the diophantine equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a}{xyz}$$

where x, y, z are integers for which the greatest common divisor of xyz and a is 1.

(a) Determine all integers a for which there are infinitely many solutions to the equation that satisfy the condition.

(b) Determine an infinite set of integers a for which there are solutions to the equation for which the condition is satisfied and x, y, z are all positive. [Optional: Given N ¿ 0, are there infinitely many a for which there are at least N positive solutions satisfying the condition?]

571. Let ABC be a triangle and U, V, W points, not vertices, on the respective sides BC, CA, AB, for which the segments AU, BV, CW intersect in a common point O. Prove that

$$\frac{|OU|}{|AU|} + \frac{|OV|}{|BV|} + \frac{|OW|}{|CW|} = 1 ,$$

and

$$\frac{|AO|}{|OU|} \cdot \frac{|BO|}{|OV|} \cdot \frac{|CO|}{|OW|} = \frac{|AO|}{|OU|} + \frac{|BO|}{|OV|} + \frac{|CO|}{|OW|} + 2 \ .$$

572. Let $ABCD$ be a convex quadrilateral that is not a parallelogram. On the sides AB, BC, CD, DA, construct isosceles triangles KAB, MBC, LCD, NDA exterior to the quadrilateral $ABCD$ such that the angles K, M, L, N are right. Suppose that O is the midpoint of BD. Prove that one of the triangles MON and LOK is a $90°$ rotation of the other around O.

What happens when $ABCD$ is a parallelogram?

573. A point O inside the hexagon $ABCDEF$ satisfies the conditions $\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle EOF = 60°$, $OA > OC > OE$ and $OB > OD > OF$. Prove that $|AB| + |CD| + |EF| < |BC| + |DE| + |FA|$.

574. A fair coin is tossed at most n times. The tossing stops before n tosses if there is a run of an odd number of heads followed by a tail. Determine the expected number of tosses.

575. A partition of the positive integer n is a set of positive integers (repetitions allowed) whose sum is n. For example, the partitions of 4 are (4), (3,1), (2,2), (2,1,1), (1,1,1,1); of 5 are (5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1); and of 6 are (6), (5,1), (4,2), (3,3), (4,1,1), (3,2,1), (2,2,2), (3,1,1,1), (2,2,1,1), (2,1,1,1), (1,1,1,1,1,1).

Let $f(n)$ be the number of 2's that occur in all partitions of n and $g(n)$ the number of times a number occurs exactly once in a partition. For example, $f(4) = 3$, $f(5) = 4$, $f(6) = 8$, $g(4) = 4$, $g(5) = 8$ and $g(6) = 11$. Prove that, for $n \geq 2$, $f(n) = g(n-1)$.

576. (a) Let $a \geq b > c$ be the radii of three circles each of which is tangent to a common line and is tangent externally to the other two circles. Determine c in terms of a and b.

(b) Let a, b, c, d be the radii of four circles each of which is tangent to the other three. Determine a relationship among a, b, c, d

577. $ABCDEF$ is a regular hexagon of area 1. Determine the area of the region inside the hexagon thst belongs to none of the triangles ABC, BCD, CDE, DEF, EFA and FAB.

578. $ABEF$ is a parallelogram; C is a point on the side AE and D a point on the aide BF for which $CD \| AB$. The sements CF and EB intersect at P; the segments ED and AF intersect at Q. Prove that $PQ \| AB$.

579. Solve, for real x, y, z the equation

$$\frac{y^2 + z^2 - x^2}{2yz} + \frac{z^2 + x^2 - y^2}{2zx} + \frac{x^2 + y^2 - z^2}{2xy} = 1 \ .$$

580. Two numbers m and n are two perfect squares with four decimal digits. Each digit of m is obtained by increasing the corresponding digit of n be a fixed positive integer d. What are the possible values of the pair (m, n).

581. Let $n \geq 4$. The integers from 1 to n inclusive are arranged in some order around a circle. A pair (a, b) is called *acceptable* if $a < b$, a and b are not in adjacent positions around the circle and at least one of the arcs joining a and b contains only numbers that are less than both a and b. Prove that the number of acceptable pairs is equal to $n - 3$.

582. Suppose that f is a real-valued function defined on the closed unit interval $[0, 1]$ for which $f(0) = f(1) = 0$ and $|f(x) - f(y)| < |x - y|$ when $0 \leq x < y \leq 1$. Prove that $|f(x) - f(y)| < \frac{1}{2}$ for all $x, y \in [0, 1]$. Can the number $\frac{1}{2}$ in the inequality be replaced by a smaller number and still result in a true proposition?

583. Suppose that $ABCD$ is a convex quadrilateral, and that the respective midpoints of AB, BC, CD, DA are K, L, M, N. Let O be the intersection point of KM and KN. Thus $ABCD$ is partitioned into four quadrilaterals. Prove that the sum of the areas of two of these that do not have a common side is equal to the sum of the areas of the other two, to wit

$$[AKON] + [CMOL] = [BLOK] + [DNOM] \ .$$

584. Let n be an integer exceeding 2 and suppose that $x_1, x_2, \cdots, x_n$ are real numbers for which $\sum_{i=1}^n x_i = 0$ and $\sum_{i=1}^n x_i^2 = n$. Prove that there are two numbers among the x_i whose product does not exceed -1.

585. Calculate the number

$$a = \lfloor \sqrt{n-1} + \sqrt{n} + \sqrt{n+1} \rfloor^2 \ ,$$

where $\lfloor x \rfloor$ denotes the largest integer than does not exceed x and n is a positive integer exceeding 1.

586. The function defined on the set $\mathbf{C^*}$ of all nonzero complex numbers satisfies the equation

$$f(z)f(iz) = z^2 \ ,$$

for all $z \in \mathbf{C}^*$. Prove that the function $f(z)$ is odd, *i,e.*, $f(-z) = -f(z)$ for all $z \in \mathbf{C}^*$. Give an example of a function that satisfies this condition.

587. Solve the equation

$$\tan 2x \tan \left(2x + \frac{\pi}{3}\right) \tan \left(2x + \frac{2\pi}{3}\right) = \sqrt{3} \ .$$

588. Let the function $f(x)$ be defined for $0 \le x \le \pi/3$ by

$$f(x) = \sec \left(\frac{\pi}{6} - x\right) + \sec \left(\frac{\pi}{6} + x\right) \ .$$

Determine the set of values (its image or range) assumed by the function.

589. In a circle, A is a variable point and B and C are fixed points. The internal bisector of the angle BAC intersects the circle at D and the line BC at G; the external bisector of the angle BAC intersects the circle at E and the line BC at F. Find the locus of the intersection of the lines DF and EG.

590. Let $SABC$ be a regular tetrahedron. The points M, N, P belong to the edges SA, SB and SC respectively such that $MN = NP = PM$. Prove that the planes MNP and ABC are parallel.

591. The point O is arbitrarily selected from the interior of the angle KAM. A line g is constructed through the point O, intersecting the ray AK at the point B and the ray AM at the point C. Prove that the value of the expression

$$\frac{1}{[AOB]} + \frac{1}{[AOC]}$$

does not depend on the choice of the line g. [*Note:* $[MNP]$ denotes the area of triangle MNP.]

592. The incircle of the triangle ABC is tangent to the sides BC, CA and AB at the respective points D, E and F. Points K from the line DF and L from the line EF are such that $AK \| BL \| DE$. Prove that:

(a) the points A, E, F and K are concyclic, and the points B, D, F and L are concyclic;

(b) the points C, K and L are collinear.

593. Consider all natural numbers M with the following properties:

(i) the four rightmost digits of M are 2008;

(ii) for some natural numbers $p > 1$ and $n > 1$, $M = p^n$.

Determine all numbers n for which such numbers M exist.

594. For each natural number N, denote by $S(N)$ the sum of the digits of N. Are there natural numbers N which satisfy the condition severally:

(a) $S(N) + S(N^2) = 2008$;

(b) $S(N) + S(N^2) = 2009$?

595. What are the dimensions of the greatest $n \times n$ square chessboard for which it is possible to arrange 111 coins on its cells so that the numbers of coins on any two adjacent cells (*i.e.* that share a side) differ by 1?

596. A 12×12 square array is composed of unit squares. Three squares are removed from one of its major diagonals. Is it possible to cover completely the remaining part of the array by 47 rectangular tiles of size 1×3 without overlapping any of them?

597. Find all pairs of natural numbers (x, y) that satisfy the equation

$$2x(xy - 2y - 3) = (x + y)(3x + y) \ .$$

598. Let $a_1, a_2, \cdots, a_n$ be a finite sequence of positive integers. If possible, select two indices j, k with $1 \leq j < k \leq n$ for which a_j does not divide a_k; replace a_j by the greatest common divisor of a_j and a_k, and replace a_k by the least common multiple of a_j and a_k. Prove that, if the process is repeated, it must eventually stop, and the final sequence does not depend on the choices made.

599. Determine the number of distinct solutions x with $0 \leq x \leq \pi$ for each of the following equations. Where feasible, give an explicit representation of the solution.

(a) $8 \cos x \cos 2x \cos 4x = 1$;

(b) $8 \cos x \cos 4x \cos 5x = 1$.

600. Let $0 < a < b$. Prove that, for any positive integer n,

$$\frac{b+a}{2} \leq \sqrt[n]{\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}} \leq \sqrt[n]{\frac{a^n + b^n}{2}} \ .$$