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Problems 528-590

Notes. $\lfloor x \rfloor$, the *floor of* x, is the largest integer n that does not exceed x, *i.e.*, that integer n for which $n \leq x < n+1$. $\{x\}$, the *fractional part of* x, is equal to $x - \lfloor x \rfloor$. The notation $[PQR]$ denotes the area of the triangle PQR. A *geometric progression* is a sequence for which the ratio of two successive terms is always the same; its nth term has the general form ar^{n-1}. A *truncated pyramid* is a pyramid with a similar pyramid on a base parallel to the base of the first pyramid removed. A polyhedron is inscribed in a sphere if each of its vertices lies on the surface of the sphere.

528. Let the sequence $\{x_n : n = 0, 1, 2, \cdots\}$ be defined by $x_0 = a$ and $x_1 = b$, where a and b are real numbers, and by

$$7x_n = 5x_{n-1} + 2x_{n-2}$$

for $n \geq 2$. Derive a formula for x_n as a function of a, b and n.

529. Let k, n be positive integers. Define $p_{n,1} = 1$ for all n and $p_{n,k} = 0$ for $k \geq n+1$. For $2 \leq k \leq n$, we define inductively

$$p_{n,k} = k(p_{n-1,k-1} + p_{n-1,k}) \ .$$

Prove, by mathematical induction, that

$$p_{n,k} = \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k-r)^n \ .$$

530. Let $\{x_1, x_2, x_3, \cdots, x_n, \cdots\}$ be a sequence is distinct positive real numbers. Prove that this sequence is a geometric progression if and only if

$$\frac{x_1}{x_2} \sum_{k=1}^{n-1} \frac{x_n^2}{x_k x_{k+1}} = \frac{x_n^2 - x_1^2}{x_2^2 - x_1^2}$$

for all $n \geq 2$.

531. Show that the remainder of the polynomial

$$p(x) = x^{2007} + 2x^{2006} + 3x^{2005} + 4x^{2004} + \cdots + 2005x^3 + 2006x^2 + 2007x + 2008$$

is the same upon division by $x(x+1)$ as upon division by $x(x+1)^2$.

532. The angle bisectors BD and CE of triangle ABC meet AC and AB at D and E respectively and meet at I. If $[ABD] = [ACE]$, prove that $AI \perp ED$. is the converse true?

533. Prove that the number

$$1 + \lfloor (5 + \sqrt{17}))^{2008} \rfloor$$

is divisible by 2^{2008}.

534. Let $\{x_n : n = 1, 2, \cdots\}$ be a sequence of distinct positive integers, with $x_1 = a$. Suppose that

$$2\sum_{k=1}^{n} \sqrt{x_i} = (n+1)\sqrt{x_n}$$

for $n \geq 2$. Determine $\sum_{k=1}^{n} x_k$.

535. Let the triangle ABC be isosceles with $AB = AC$. Suppose that its circumcentre is O, the D is the midpoint of side AB and that E is the centroid of triangle ACD. Prove that OE is perpendicular to CD.

536. There are 21 cities, and several airlines are responsible for connections between them. Each airline serves five cities with flights both ways between all pairs of them. Two or more airlines may serve a given pair of cities. Every pair of cities is serviced by at least one direct return flight. What is the minimum number of airlines that would meet these conditions?

537. Consider all 2×2 square arrays each of whose entries is either 0 or 1. A pair (A, B) of such arrays is *compatible* if there exists a 3×3 square array in which both A and B appear as 2×2 subarrays.

For example, the two matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are compatible, as both can be found in the array

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

Determine all pairs of 2×2 arrays that are not compatible.

538. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the right bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have the same area.

539. Determine the maximum value of the expression

$$\frac{xy + 2yz + zw}{x^2 + y^2 + z^2 + w^2}$$

over all quartuple of real numbers not all zero.

540. Suppose that, if all planar cross-sections of a bounded solid figure are circles, then the solid figure must be a sphere.

541. Prove that the equation

$$x_1^{x_1} + x_2^{x_2} + \cdots + x_k^{x_k} = x_{k+1}^{x_{k+1}}$$

has no solution for which $x_1, x_2, \cdots, x_k, x_{k+1}$ are all distinct nonzero integers.

542. Solve the system of equations

$$\lfloor x \rfloor + 3\{y\} = 3.9 ,$$

$$\{x\} + 3\lfloor y \rfloor = 3.4 .$$

543. Let $a > 0$ and b be real parameters, and suppose that f is a function taking the set of reals to itself for which

$$f(a^3x^3 + 3a^2bx^2 + 3ab^2x) \leq x \leq a^3f(x)^3 + 3a^2bf(x)^2 + 3ab^2f(x) ,$$

for all real x. Prove that f is a one-one function that takes the set of real numbers onto itself (*i.e.*, f is a *bijection*).

544. Define the real sequences $\{a_n : n \geq 1\}$ and $\{b_n : n \geq 1\}$ by $a_1 = 1$, $a_{n+1} = 5a_n + 4$ and $5b_n = a_n + 1$ for $n \geq 1$.

(a) Determine $\{a_n\}$ as a function of n.

(b) Prove that $\{b_n : n \geq 1\}$ is a geometric progression and evaluate the sum

$$S \equiv \frac{\sqrt{b_1}}{\sqrt{b_2} - \sqrt{b_1}} + \frac{\sqrt{b_2}}{\sqrt{b_3} - \sqrt{b_2}} + \cdots + \frac{\sqrt{b_n}}{\sqrt{b_{n+1}} - \sqrt{b_n}} .$$

545. Suppose that x and y are real numbers for which $x^3 + 3x^2 + 4x + 5 = 0$ and $y^3 - 3y^2 + 4y - 5 = 0$. Determine $(x+y)^{2008}$.

546. Let $a, a_1, a_2, \cdots, a_n$ be a set of positive real numbers for which

$$a_1 + a_2 + \cdots + a_n = a$$

and

$$\sum_{k=1}^{n} \frac{1}{a - a_k} = \frac{n+1}{a} .$$

Prove that

$$\sum_{k=1}^{n} \frac{a_k}{a - a_k} = 1 .$$

547. Let A, B, C, D be four points on a circle, and let E be the fourth point of the parallelogram with vertices A, B, C. Let AD and BC intersect at M, AB and DC intersect at N, and EC and MN intersect at F. Prove that the quadrilateral $DENF$ is concyclic.

548. In a sphere of radius R is inscribed a regular hexagonal truncated pyramid whose big base is inscribed in a great circle of the sphere (i.e., a whose centre is the centre of the sphere). The length of the side of the big base is three times the length of the side of a small base. Find the volume of the truncated pyramid as a function of R.

549. The set E consists of 37 two-digit natural numbers, none of them a multiple of 10. Prove that, among the elements of E, we can find at least five numbers, such that any two of them have different tens digits and different units digits.

550. The functions $f(x)$ and $g(x)$ are defined by the equations: $f(x) = 2x^2 + 2x - 4$ and $g(x) = x^2 - x + 2$.

(a) Find all real numbers x for which $f(x)/g(x)$ is a natural number.

(b) Find the solutions of the inequality

$$\sqrt{f(x)} + \sqrt{g(x)} \geq 2 .$$

551. The numbers 1, 2, 3 and 4 are written on the circumference of a circle, in this order. Alice and Bob play the following game: On each turn, Alice adds 1 to two adjacent numbers, while Bob switches the places of two adjacent numbers. Alice wins the game, if after her turn, all numbers on the circle are equal. Does Bob have a strategy to prevent Alice from winning the game? Justify your answer.

552. Two real nonnegative numbers a and b satisfy the inequality $ab \geq a^3 + b^3$. Prove that $a + b \leq 1$.

553. The convex quadrilateral $ABCD$ is concyclic with side lengths $|AB| = 4$, $|BC| = 3$, $|CD| = 2$ and $|DA| = 1$. What is the length of the radius of the circumcircle of $ABCD$? Provide an exact value of the answer.

554. Determine all real pairs (x, y) that satisfy the system of equations:

$$3\sqrt[3]{x^2y^5} = 4(y^2 - x^2)$$

$$5\sqrt[3]{x^4y} = y^2 + x^2 \ .$$

555. Let ABC be a triangle, all of whose angles do not exceed $90°$. The points K on side AB, M on side AC and N on side BC are such that $KM \perp AC$ and $KN \perp BC$. Prove that the area $[ABC]$ of triangle ABC is at least 4 times as great as the area $[KMN]$ of triangle KMN, *i.e.*, $[ABC] \geq 4[KMN]$. When does equality hold?

556. Let x, y, z be positive real numbers for which $x + y + z = 4$. Prove the inequality

$$\frac{1}{2xy + xz + yz} + \frac{1}{xy + 2xz + yz} + \frac{1}{xy + xz + 2yz} \leq \frac{1}{xyz} \ .$$

557. Suppose that the polynomial $f(x) = (1+x+x^2)^{1004}$ has the expansion $a_0 + a_1x + a_2x^2 + \cdots + a_{2008}x^{2008}$. Prove that $a_0 + a_2 + \cdots + a_{2008}$ is an odd integer.

558. Determine the sum

$$\sum_{m=0}^{n-1} \sum_{k=0}^{m} \binom{n}{k} \ .$$

559. Let ϵ be one of the roots of the equation $x^n = 1$, where n is a positive integer. Prove that, for any polynomial $f(x) = a_0 + a_x + \cdots + a_nx^n$ with real coefficients, the sum $\sum_{k=1}^{n} f(1/\epsilon^k)$ is real.

560. Suppose that the numbers $x_1, x_2, \cdots, x_n$ all satisfy $-1 \leq x_i \leq 1$ $(1 \leq i \leq n)$ and $x_1^3 + x_2^3 + \cdots + x_n^3 = 0$. Prove that

$$x_1 + x_2 + \cdots + x_n \leq \frac{n}{3} \ .$$

561. Solve the equation

$$\left(\frac{1}{10}\right)^{\log_{(1/4)}(\sqrt[4]{x}-1)} - 4^{\log_{10}(\sqrt[4]{x}+5)} = 6 \ ,$$

for $x \geq 1$.

562. The circles $\mathfrak{C}$ and $\mathfrak{D}$ intersect at the two points A and B. A secant through A intersects $\mathfrak{C}$ at C and $\mathfrak{D}$ at D. On the segments CD, BC, BD, consider the respective points M, N, K for which $MN \| BD$ and $MK \| BC$. On the arc BC of the circle $\mathfrak{C}$ that does not contain A, choose E so that $EN \perp BC$, and on the arc BD of the circle $\mathfrak{D}$ that does not contain A, choose F so that $FK \perp BD$. Prove that angle EMF is right.

563. (a) Determine infinitely many triples (a, b, c) of integers for which a, b, c are not in arithmetic progression and $ab + 1$, $bc + 1$, $ca + 1$ are all squares.

(b) Determine infinitely many triples (a, b, c) of integers for which a, b, c are in arithemetic progression and $ab + 1$, $bc + 1$, $ca + 1$ are all squares.

(c) Determine infinitely many triples (u, v, w) of integers for which $uv - 1$, $vw - 1$, $wu - 1$ are all squares. (Can it be arranged that u, v, w are in arithmetic progression?)

564. Let $x_1 = 2$ and

$$x_{n+1} = \frac{2x_n}{3} + \frac{1}{3x_n}$$

for $n \geq 1$. Prove that, for all $n > 1$, $1 < x_n < 2$.

565. Let ABC be an acute-angled triangle. Points A_1 and A_2 are located on side BC so that the four points are ordered B, A_1, A_2, C; similarly B_1 and B_2 are on CA in the order C, B_1, B_2, A and C_1 and C_2 on side AB in order A, C_1, C_2, B. All the angles AA_1A_2, AA_2A_1, BB_1B_2, BB_2B_1, CC_1C_2, CC_2C_1 are equal to θ. Let $\mathfrak{T}_1$ be the triangle bounded by the lines AA_1, BB_1, CC_1 and $\mathfrak{T}_2$ the triangle bounded by the lines AA_2, BB_2, CC_2. Prove that all six vertices of the triangles are concyclic.

566. A deck of cards numbered 1 to n (one card for each number) is arranged in random order and placed on the table. If the card numbered k is on top, remove the kth card counted from the top and place it on top of the pile, not otherwise disturbing the order of the cards. Repeat the process. Prove that the card numbered 1 will eventually come to the top, and determine the maximum number of moves that is required to achieve this.

567. (a) Let A, B, C, D be four distinct points in a straight line. For any points X, Y on the line, let XY denote the *directed* distance between them. In other words, a positive direction is selected on the line and $XY = \pm|XY|$ according as the direction X to Y is positive or negative. Define

$$(AC, BD) = \frac{AB/BC}{AD/DC} = \frac{AB \times CD}{BC \times DA} .$$

Prove that $(AB, CD) + (AC, BD) = 1$.

(b) In the situation of (a), suppose in addition that $(AC, BD) = -1$. Prove that

$$\frac{1}{AC} = \frac{1}{2}\left(\frac{1}{AB} + \frac{1}{AD}\right) ,$$

and that

$$OC^2 = OB \times OD ,$$

where O is the midpoint of AC. Deduce from the latter that, if Q is the midpoint of BD and if the circles on diameters AC and BD intersect at P, $\angle OPQ = 90°$.

(c) Suppose that A, B, C, D are four distinct on one line and that P, Q, R, S are four distinct points on a second line. Suppose that AP, BQ, CR and DS all intersect in a common point V. Prove that $(AC, BD) = (PR, QS)$.

(d) Suppose that $ABQP$ is a quadrilateral in the plane with no two sides parallel. Let AQ and BP intersect in U, and let AP and BQ intersect in V. Suppose that VU and PQ produced meet AB at C and D respectively, and that VU meets PQ at W. Prove that

$$(AB, CD) = (PQ, WD) = -1 .$$

568. Let ABC be a triangle and the point D on BC be the foot of the altitude AD from A. Suppose that H lies on the segment AD and that BH and CH intersect AC and AB at E and F respectively.

Prove that $\angle FDH = \angle HDE$.

569. Let A, W, B, U, C, V be six points in this order on a circle such that AU, BV and CW all intersect in the common point P at angles of $60°$. Prove that

$$|PA| + |PB| + |PC| = |PU| + |PV| + |PW| .$$

570. Let a be an integer. Consider the diophantine equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a}{xyz}$$

where x, y, z are integers for which the greatest common divisor of xyz and a is 1.

(a) Determine all integers a for which there are infinitely many solutions to the equation that satisfy the condition.

(b) Determine an infinite set of integers a for which there are solutions to the equation for which the condition is satisfied and x, y, z are all positive. [Optional: Given N ¿ 0, are there infinitely many a for which there are at least N positive solutions satisfying the condition?]

571. Let ABC be a triangle and U, V, W points, not vertices, on the respective sides BC, CA, AB, for which the segments AU, BV, CW intersect in a common point O. Prove that

$$\frac{|OU|}{|AU|} + \frac{|OV|}{|BV|} + \frac{|OW|}{|CW|} = 1 ,$$

and

$$\frac{|AO|}{|OU|} \cdot \frac{|BO|}{|OV|} \cdot \frac{|CO|}{|OW|} = \frac{|AO|}{|OU|} + \frac{|BO|}{|OV|} + \frac{|CO|}{|OW|} + 2 .$$

572. Let $ABCD$ be a convex quadrilateral that is not a parallelogram. On the sides AB, BC, CD, DA, construct isosceles triangles KAB, MBC, LCD, NDA exterior to the quadrilateral $ABCD$ such that the angles K, M, L, N are right. Suppose that O is the midpoint of BD. Prove that one of the triangles MON and LOK is a $90°$ rotation of the other around O.

What happens when $ABCD$ is a parallelogram?

573. A point O inside the hexagon $ABCDEF$ satisfies the conditions $\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle EOF = 60°$, $OA > OC > OE$ and $OB > OD > OF$. Prove that $|AB| + |CD| + |EF| < |BC| + |DE| + |FA|$.

574. A fair coin is tossed at most n times. The tossing stops before n tosses if there is a run of an odd number of heads followed by a tail. Determine the expected number of tosses.

575. A partition of the positive integer n is a set of positive integers (repetitions allowed) whose sum is n. For example, the partitions of 4 are (4), (3,1), (2,2), (2,1,1), (1,1,1,1); of 5 are (5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1); and of 6 are (6), (5,1), (4,2), (3,3), (4,1,1), (3,2,1), (2,2,2), (3,1,1,1), (2,2,1,1), (2,1,1,1), (1,1,1,1,1,1).

Let $f(n)$ be the number of 2's that occur in all partitions of n and $g(n)$ the number of times a number occurs exactly once in a partition. For example, $f(4) = 3$, $f(5) = 4$, $f(6) = 8$, $g(4) = 4$, $g(5) = 8$ and $g(6) = 11$. Prove that, for $n \geq 2$, $f(n) = g(n-1)$.

576. (a) Let $a \geq b > c$ be the radii of three circles each of which is tangent to a common line and is tangent externally to the other two circles. Determine c in terms of a and b.

(b) Let a, b, c, d be the radii of four circles each of which is tangent to the other three. Determine a relationship among a, b, c, d

577. $ABCDEF$ is a regular hexagon of area 1. Determine the area of the region inside the hexagon thst belongs to none of the triangles ABC, BCD, CDE, DEF, EFA and FAB.

578. $ABEF$ is a parallelogram; C is a point on the diagonal AE and D a point on the diagonal BF for which $CD \| AB$. The segments CF and EB intersect at P; the segments ED and AF intersect at Q. Prove that $PQ \| AB$.

579. Solve, for real x, y, z the equation

$$\frac{y^2 + z^2 - x^2}{2yz} + \frac{z^2 + x^2 - y^2}{2zx} + \frac{x^2 + y^2 - z^2}{2xy} = 1 .$$

580. Two numbers m and n are two perfect squares with four decimal digits. Each digit of m is obtained by increasing the corresponding digit of n be a fixed positive integer d. What are the possible values of the pair (m, n).

581. Let $n \geq 4$. The integers from 1 to n inclusive are arranged in some order around a circle. A pair (a, b) is called *acceptable* if $a < b$, a and b are not in adjacent positions around the circle and at least one of the arcs joining a and b contains only numbers that are less than both a and b. Prove that the number of acceptable pairs is equal to $n - 3$.

582. Suppose that f is a real-valued function defined on the closed unit interval $[0, 1]$ for which $f(0) = f(1) = 0$ and $|f(x) - f(y)| < |x - y|$ when $0 \leq x < y \leq 1$. Prove that $|f(x) - f(y)| < \frac{1}{2}$ for all $x, y \in [0, 1]$. Can the number $\frac{1}{2}$ in the inequality be replaced by a smaller number and still result in a true proposition?

583. Suppose that $ABCD$ is a convex quadrilateral, and that the respective midpoints of AB, BC, CD, DA are K, L, M, N. Let O be the intersection point of KM and LN. Thus $ABCD$ is partitioned into four quadrilaterals. Prove that the sum of the areas of two of these that do not have a common side is equal to the sum of the areas of the other two, to wit

$$[AKON] + [CMOL] = [BLOK] + [DNOM] \ .$$

584. Let n be an integer exceeding 2 and suppose that $x_1, x_2, \cdots, x_n$ are real numbers for which $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} x_i^2 = n$. Prove that there are two numbers among the x_i whose product does not exceed -1.

585. Calculate the number

$$a = \lfloor \sqrt{n-1} + \sqrt{n} + \sqrt{n+1} \rfloor^2 \ ,$$

where $\lfloor x \rfloor$ denotes the largest integer than does not exceed x and n is a positive integer exceeding 1.

586. The function defined on the set $\mathbf{C^*}$ of all nonzero complex numbers satisfies the equation

$$f(z)f(iz) = z^2 \ ,$$

for all $z \in \mathbf{C}^*$. Prove that the function $f(z)$ is odd, *i,e.*, $f(-z) = -f(z)$ for all $z \in \mathbf{C}^*$. Give an example of a function that satisfies this condition.

587. Solve the equation

$$\tan 2x \tan\left(2x + \frac{\pi}{3}\right) \tan\left(2x + \frac{2\pi}{3}\right) = \sqrt{3} \ .$$

588. Let the function $f(x)$ be defined for $0 \leq x \leq \pi/3$ by

$$f(x) = \sec\left(\frac{\pi}{6} - x\right) + \sec\left(\frac{\pi}{6} + x\right) \ .$$

Determine the set of values (its image or range) assumed by the function.

589. In a circle, A is a variable point and B and C are fixed points. The internal bisector of the angle BAC intersects the circle at D and the line BC at G; the external bisector of the angle BAC intersects the circle at E and the line BC at F. Find the locus of the intersection of the lines DF and EG.

590. Let $SABC$ be a regular tetrahedron. The points M, N, P belong to the edges SA, SB and SC respectively such that $MN = NP = PM$. Prove that the planes MNP and ABC are parallel.

Solutions

528. Let the sequence $\{x_n : n = 0, 1, 2, \cdots\}$ be defined by $x_0 = a$ and $x_1 = b$, where a and b are real numbers, and by

$$7x_n = 5x_{n-1} + 2x_{n-2}$$

for $n \geq 2$. Derive a formula for x_n as a function of a, b and n.

Solution. This can be done by the standard theory of solving linear recursions. The auxiliary equation is $7t^2 - 5t - 2 = 0$, with roots 1 and $-2/7$. Trying a solution of the form $x_n = A \cdot 1^n + B(-2/7)^n$ and plugging in the initial conditions leads to $A + B = a$ and $A - (2/7)B = b$ and the solution

$$x_n = \frac{2a + 7b}{9} + \frac{7(a - b)}{9} \cdot \left(-\frac{2}{7} \right)^n .$$

529. Let k, n be positive integers. Define $p_{n,1} = 1$ for all n and $p_{n,k} = 0$ for $k \geq n + 1$. For $2 \leq k \leq n$, we define inductively

$$p_{n,k} = k(p_{n-1,k-1} + p_{n-1,k}) .$$

Prove, by mathematical induction, that

$$p_{n,k} = \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k - r)^n .$$

Solution. Let

$$q_{n,k} = \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k - r)^n .$$

When $n = 1$, we have that $q_{1,1} = \binom{1}{0} 1 = 1$ and, for $k \geq 2$,

$$q_{1,k} = \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r k - \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r r = k[(1 - 1)^k - (-1)^k] + k[(1 - 1)^{k-1} - (-1)^{k-1}] = 0 .$$

Also $q_{n,1} = \binom{1}{0} 1^n = 1$ for $n \geq 1$. When $(n, k) = (2, 2)$, we have that $q_{2,2} = \binom{2}{0} 2^2 - \binom{2}{1} 1 = 2$ and $p_{2,2} = 2(1 + 0) = 2$. When $n = 2$ and $k \geq 3$, then

$$\begin{aligned}
q_{2,k} &= \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k - r)^2 \\
&= \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r [k^2 - (2k - 1)r + r(r - 1)] \\
&= k^2 \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r + (2k - 1)k \sum_{r=0}^{k-1} \binom{k-1}{r-1} (-1)^{r-1} + k(k - 1) \sum_{r=0}^{k-1} \binom{k-2}{r-2} (-1)^{r-2} \\
&= (-1)^{k-1} k^2 + (2k - 1)k(-1)^{k-2} + k(k - 1)(-1)^{k-3} \\
&= (-1)^{k-3} [k^2 - 2k^2 + k + k^2 - k] = 0 .
\end{aligned}$$

Thus, we have that $p_{n,k} = q_{n,k}$ for all n and $k = 1$ as well as for $n = 1, 2$ and all k.

The remainder of the argument can be done by induction. Suppose that $n \geq 2$ and that $k \geq 2$ and that it has been shown that $p_{n,k} = q_{n,k}$ and $p_{n,k-1} = q_{n,k-1}$. Then

$$\begin{aligned}
p_{n+1,k} &= k(p_{n,k} + p_{n,k-1}) \\
&= k\left[\sum_{r=0}^{k-1}\binom{k}{r}(-1)^r(k-r)^n + \sum_{r=0}^{k-2}\binom{k-1}{r}(-1)^r(k-1-r)^n\right] \\
&= k\left[k^n + \sum_{r=1}^{k-1}\binom{k}{r}(-1)^r(k-r)^n + \sum_{r=1}^{k-1}\binom{k-1}{r-1}(-1)^{r-1}(k-r)^n\right] \\
&= k\left[k^n + \sum_{r=1}^{k-1}\left[\binom{k}{r} - \binom{k-1}{r-1}\right](-1)^r(k-r)^n\right] \\
&= k^{n+1} + k\sum_{r=1}^{k-1}\binom{k-1}{r}(-1)^r(k-r)^n \\
&= k^{n+1} + \sum_{r=1}^{k-1}\binom{k}{r}(-1)^r(k-r)^{n+1} \\
&= \sum_{r=0}^{k-1}\binom{k}{r}(-1)^r(k-r)^{n+1} = q_{n+1,k} \ ,
\end{aligned}$$

as desired.

530. Let $\{x_1, x_2, x_3, \cdots, x_n, \cdots\}$ be a sequence is distinct positive real numbers. Prove that this sequence is a geometric progression if and only if

$$\frac{x_1}{x_2}\sum_{k=1}^{n-1}\frac{x_n^2}{x_k x_{k+1}} = \frac{x_n^2 - x_1^2}{x_2^2 - x_1^2}$$

for all $n \geq 2$.

Solution. Necessity. Suppose that $x_k = ar^{k-1}$ for some numbers a and r. Then

$$\begin{aligned}
\frac{x_1}{x_2}\sum_{k=1}^{n-1}\frac{x_n^2}{x_k x_{k+1}} &= \frac{r^{2(n-1)}}{r}\sum_{k=1}^{n-1}\frac{1}{r^{2k-1}} \\
&= (r^{2n-3})\left(\frac{1}{r} + \frac{1}{r^3} + \cdots + \frac{1}{r^{2n-3}}\right) \\
&= 1 + r^2 + \cdots + r^{2(n-2)} = \frac{r^{2(n-1)} - 1}{r^2 - 1} = \frac{x_n^2 - x_1^2}{x_2^2 - x_1^2} \ .
\end{aligned}$$

Sufficiency. Suppose that the equations of the problem holds. When $n = 2$, both sides of the equation are equal to 1 regardless of the sequence. When $n = 3$, the equation is equivalent to

$$\frac{x_1 x_3^2}{x_1 x_2^2 x_3}(x_3 + x_1) = \frac{(x_3 - x_1)(x_3 + x_1)}{x_3^2 - x_1^2} \ .$$

Since $x_3 + x_1 \neq 0$ [why?], we can divide out this factor and multiply up the denominators to get the equivalent

$$x_3(x_2^2 - x_1^2) = x_2^2(x_3 - x_1) \Longleftrightarrow x_3 x_1^2 = x_2^2 x_1 \Longleftrightarrow x_1 x_3 = x_2^2 \ ,$$

whence x_1, x_2, x_3 are in geometric progression.

Suppose, as an induction hypothesis, for $n \geq 4$ we know that $x_k = ar^{k-1}$ for suitable a and r and $k = 1, 2, \cdots, n-1$. Let $x_n = au_n$ for some number u_n.

Then

$$\frac{u_n^2}{r}\left(\frac{1}{r} + \frac{1}{r^3} + \cdots + \frac{1}{r^{2n-5}} + \frac{1}{r^{n-2}u_n}\right) = \frac{u_n^2 - 1}{r^2 - 1}$$
$$\Longleftrightarrow [u_n^2(1 + r^2 + r^4 + \cdots + r^{2n-6}) + r^{n-3}u_n](r^2 - 1) = (u_n^2 - 1)(r^{2n-4})$$
$$\Longleftrightarrow (r^{2n-4} - 1)u_n^2 + (r^{n-3}r^2 - r^{n-3})u_n = (r^{2n-4})u_n^2 - r^{2n-4}$$
$$\Longleftrightarrow 0 = u_n^2 - (r^{n-1} - r^{n-3})u_n - r^{2n-4} = (u_n - r^{n-1})(u_n + r^{n-3}) \ .$$

The case $u_n = -r^{n-3}$ is rejected because of the condition that the sequence consists of positive terms. Hence $u_n = r^{n-1}$, as desired. The result follows.

Comment. In the absence of the positivity contiion, the second root of the quadratic can be used. For example, the finite sequences $\{1, r, r^2, -r, 1\}$ and $\{1, r, r^2, -r, -r^2\}$ both satisfies the equations for $2 \leq n \leq 5$. It would be interesting to investigate the situation further.

531. Show that the remainder of the polynomial

$$p(x) = x^{2007} + 2x^{2006} + 3x^{2005} + 4x^{2004} + \cdots + 2005x^3 + 2006x^2 + 2007x + 2008$$

is the same upon division by $x(x+1)$ as upon division by $x(x+1)^2$.

Solution 1. We have that

$$\begin{aligned} p(x) &= (x^{2007} + 2x^{2006} + x^{2005}) + 2(x^{2005} + 2x^{2004} + x^{2003}) + 3(x^{2003} + \\ &\qquad 2x^{2002} + x^{2001}) + \cdots + 1003(x^3 + 2x^2 + x) + 1004x + 2008 \\ &= x(x+1)^2(x^{2004} + 2x^{2002} + 3x^{2000} + \cdots + 1003) + (1004x + 2008) \ , \end{aligned}$$

from which the result follows with remainder $1004x + 2008$.

532. The angle bisectors BD and CE of triangle ABC meet AC and AB at D and E respectively and meet at I. If $[ABD] = [ACE]$, prove that $AI \perp ED$. Is the converse true?

Solution. Observe that

$$[ADB] : [CBD] = AD : DC = AB : BC$$

and that

$$[ACE] : [BCE] = AE : EB = AC : BC \ .$$

Now

$$\begin{aligned} [ABD] = [ACE] &\Longleftrightarrow [DBC] = [ABC] - [ABD] = [ABC] - [ACE] = [EBC] \\ &\Longleftrightarrow ED \| BC \Longleftrightarrow AE : EB = AD : DC \\ &\Longleftrightarrow AB : BC = AC : BC \Longleftrightarrow AB = BC \\ &\Longleftrightarrow AI \perp BC \ . \end{aligned}$$

Both the result and the converse is true. If $[ABD] = [ACE]$, the foregoing chain of implications can be read in the forward direction to deduce that $AI \perp ED$. Note that AI bisects angle A in triangle AED. Thus, if $AI \perp ED$, then it follows that triangle AED is isosceles with $AE = AD$. Then $AE : DC = AD : DC = AB : BC$ and $AE : EB = AC : BC$, whence $DC \cdot AB = AE \cdot BC = EB \cdot AC$. Therefore $DC \cdot (AE + EB) = EB \cdot (AD + CD)$, so that $DC \cdot AE = EB \cdot AD$ and $DC = EB$. Therefore $AB = AC$ and, following the foregoing implication in the backwards direction, we find that $[ABD] = [ACE]$.

533. Prove that the number

$$1 + \lfloor (5 + \sqrt{17}))^{2008} \rfloor$$

is divisible by 2^{2008}.

Solution. Let $a = 5+\sqrt{17}$ and $b = 5-\sqrt{17}$, so that $a+b = 10$ and $ab = 8$. Define $x_n = a^n + b^n$. Then $x_1 = 10$, $x_2 = (a+b)^2 - 2ab = 96$ and

$$\begin{aligned} x_{n+2} &= a^{n+2} + b^{n+2} = (a+b)(a^{n+1}+b^{n+1}) - ab(a^n + b^n) \\ &= 10x_{n+1} - 8x_n \ , \end{aligned}$$

for $n \geq 0$. Note that x_1 is divisible by 2 and x_2 by 4. Suppose, as an induction hypothesis, that $x_n = 2^n u$ and $x_{n+1} = 2^{n+1}v$, for some $k \geq 0$ and integers u and v. Then

$$x_{n+2} = 5 \cdot 2^{n+2} - 2^{n+3} = 3 \cdot 2^{n+2} \ .$$

Hence, for all positive integers n, 2^n divides x_n.

Observe that $(5-\sqrt{17})^n = b^n < 1$ for each positive integer n and that $a^n + b^n$ is a positive integer. Therefore $x_n = a^n + b^n > a^n > a^n + b^n - 1 = x_n - 1$, whence $x_n = 1 + \lfloor a^n \rfloor$ and the result follows.

534. Let $\{x_n : n = 1, 2, \cdots\}$ be a sequence of distinct positive integers, with $x_1 = a$. Suppose that

$$2\sum_{k=1}^{n} \sqrt{x_i} = (n+1)\sqrt{x_n}$$

for $n \geq 2$. Determine $\sum_{k=1}^{n} x_k$.

Solution. When $n = 2$, $2(\sqrt{x_1} + \sqrt{x_2}) = 3\sqrt{x_2}$, whence $\sqrt{x_2} = 2\sqrt{x_1}$ and $x^2 = 4x_1 = 4a$. When $n = 3$,

$$2(\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}) = 4\sqrt{x_3} \Longrightarrow 2\sqrt{x_3} = 2(\sqrt{x_1} + \sqrt{x_2}) = 6\sqrt{x_1} \Longrightarrow x_3 = 9x_1 = 9a \ .$$

We conjecture that $x_k = k^2 a$ for each positive integer k.

Let $m \geq 2$ and suppose that $x_k = k^2 a$ for $1 \leq k \leq m-1$.

$$\begin{aligned} (m-1)\sqrt{x_m} &= (m+1)\sqrt{x_m} - 2\sqrt{x_m} = 2(\sqrt{x_1} + \cdots + \sqrt{x_{m-1}}) \\ &= 2\sqrt{x_1}(1 + 2 + \cdots + (m-1)) = m(m-1)\sqrt{a} \ , \end{aligned}$$

whence $\sqrt{x_m} = m\sqrt{a}$ and $x_m = m^2 a$.

Thus, $x_k = k^2 a$ for all $k \geq 1$. Therefore

$$\sum_{k=1}^{n} x_k = (1 + 4 + \cdots + n^2)a = \frac{n(n+1)(2n+1)a}{6} \ .$$

535. Let the triangle ABC be isosceles with $AB = AC$. Suppose that its circumcentre is O, the D is the midpoint of side AB and that E is the centroid of triangle ACD. Prove that OE is perpendicular to CD.

Solution 1. Let F be the midpoint of AC, so that DF is a median of triangle ADC and so contains the point E. The centroid, G, of triangle ABC lies on the median CD as well as on the right bisector of BC. Since $DE \| BC$ and the circumcentre O of triangle ABC lies on the right bisector of BC, we have that $DE \perp AO$.

Let H be the midpoint of CD. Since FH is a midline of triangle ACD, $FH \| AD$. Since $DG = \frac{1}{3}CD = \frac{2}{3}DH$ and $DE = \frac{2}{3}DF$, $EG \| FH \| AB$. Since O lies on the right bisector of AB, $DO \perp EG$. Therefore,

O is the intersection of two altitudes from D and G and so is the orthocentre of triangle DEG. Therefore $OE \perp CD$.

Solution 2. [N. Gurram] Place the configuration in a complex plane with O at 0 and A, B, C, respectively, at $2ai$, $-2b-2ci$, $2b-2ci$. Since $|OA| = |OB|$, $a^2 = b^2 + c^2$.

The point D is located at $-b + (a-c)i$ and E at

$$\frac{1}{3}[2ai + (-b + (a-c)i) + (2b - 2ci)] = \frac{1}{3}[b + 3(a-c)i] \ .$$

Note that $OE \perp CD$ if and only if $\frac{1}{3}[b+3(a-c)i]$ is i times a real multiple of $(2b-2ci)-(-b+(a-c)i) = 3b-(a+c)i$. Since

$$\begin{aligned} \frac{b+3(a-c)i}{3b-(a-c)i} &= \frac{[b+3(a-c)i][3b+(a+c)i]}{9b^2+(a-c)^2} \\ &= \frac{3[b^2-(a^2-c^2)]+[9b(a-c)+b(a+c)]i}{9b^2+(a-c)^2} \\ &= \frac{2b(5a-4c)i}{9b^2+(a-c)^2} \ , \end{aligned}$$

is pure imaginary, the result follows.

Solution 3. Assign coordinates to the points: $A \sim (2a, 2b)$, $B \sim (4a, 0)$, $C \sim (0,0)$, and $O \sim (2a, k)$ where $4a^2 + k^2 = (2b-k)^2$ or $k = b - (a^2/b)$. Then $D \sim (3a, b)$ and $E \sim (5a/3, b)$. The slope of OE is $(-a^2/b)/(a/3) = -3a/b$ and the slope of CD is $b/3a$. Therefore $OE \perp CD$.

536. There are 21 cities, and several airlines are responsible for connections between them. Each airline serves five cities with flights both ways between all pairs of them. Two or more airlines may serve a given pair of cities. Every pair of cities is serviced by at least one direct return flight. What is the minimum number of airlines that would meet these conditions?

Solution 1. Since there are 210 pairs of cities and each airline serves 10 pairs, at least 21 airlines are required. In fact, we can get by with exactly 21 airlines. Label the cities from 0 to 20 inclusive, and let the kth airline service the set of five cities

$$\{k, k+2, k+7, k+8, k+11\}$$

where the numbers are taken modulo 21. Observe that the differences between two numbers of such sets for any airline cover all the numbers from 1 to 10. Given any two cities labelled i and j, the difference between the two labels (possibly adjusted modulo 21) is equal to some number between 1 and 10, and we can select a value of k for which the two labels appear in the cities services by the kth airline.

Solution 2. Suppose that there are m airlines, and that each airline maintains an office in each city that it serves. Then there are $5m$ offices. Consider any particular city: it is connected to four other cities by each airline that serves it, so that there must be at least $20/4 = 5$ offices in the city. Therefore, there are at least 5×21 offices in all the cities. Thus, $5k \geq 5 \times 21$ and so $k \geq 21$.

An example can be given as in the first solution.

537. Consider all 2×2 square arrays each of whose entries is either 0 or 1. A pair (A, B) of such arrays is *compatible* if there exists a 3×3 square array in which both A and B appear as 2×2 subarrays.

For example, the two arrays

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are compatible, as both can be found in the array

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

Determine all pairs of 2×2 arrays that are not compatible.

Solution. Let a_{ij} and b_{ij} be the respective entries in the ith row and jth column of the arrays A and B, where $1 \leq i, j \leq 2$. If any of the following hold: $a_{11} = b_{22}$, $a_{21} = b_{12}$, $a_{12} = b_{21}$, $a_{22} = b_{11}$, then the arrays are compatible as they can be inserted into a 3×3 array overlapping at a corner. Therefore, if two arrays are not compatible, we must have that $b_{ij} = 1 - a_{ji}$ for each i and j.

Suppose that two matrices A and B related in this way have two unequal entries. Wolog, we may assume that $a_{11} = 0$ and $a_{12} = 1$. Then $b_{22} = 1$ and $b_{21} = 0$. Then the two matrices can be fitted into a 3×3 array with the bottom row of B coinciding with the top row of A. Hence, if A and B are not compatible, then each must have all of its entries the same. Therefore, the only noncompatible pairs (A, B) have one matrix containing only 1s and the other only 0s.

538. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the right bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have the same area.

Solution 1. [N. Gurram] Let AC and BD intersect at E. Let R and S be the respective feet of the perpendiculars from P to AC and BD. Observe that $PRES$ is a rectangle, so that $|RE| = |PS|$ and $|SE| = |PR|$. T here are essentially two cases to consider, according as P lies in triangle AEB or triangle AED.

First, suppose that P lies in triangle AEB. Then

$$\begin{aligned} [ABP] &= [ABE] - [AEP] - [BEP] \\ &= \frac{1}{2}[|AE||BE| - |AE||ES| - |BE||ER|] \\ &= \frac{1}{2}[(|AE| - |ER|)(|BE| - |ES|) - |ER||ES|] \\ &= \frac{1}{2}[|AR||BS| - |ER||ES|] \ . \end{aligned}$$

Likewise,

$$\begin{aligned} [CDP] &= [CDE] + [CEP] + [DEP] \\ &= \frac{1}{2}[|CE||DE| + |CE||ES| + |DE||ER|] \\ &= \frac{1}{2}[(|CE| + |ER|)(|DE| + |ES|) - |ER||ES|] \\ &= \frac{1}{2}[|CR||DS| - |ER||ES|] \ . \end{aligned}$$

Secondly, suppose that P lies in triangle AED. Then

$$\begin{aligned} [ABP] &= [ABE] + [AEP] - [BEP] \\ &= \frac{1}{2}[|AE||BE| + |AE||ES| - |BE||ER|] \\ &= \frac{1}{2}[(|AE| - |ER|)(|BE| + |ES|) + |ER||ES|] \\ &= \frac{1}{2}[|AR||BS| + |ER||ES|] \ . \end{aligned}$$

Likewise,

$$\begin{aligned}[CDP] &= [CDE] - [CEP] + [DEP] \\ &= \frac{1}{2}[|CE||DE| - |CE||ES| + |DE||ER|] \\ &= \frac{1}{2}[(|CE| + |ER|)(|DE| - |ES|) + |ER||ES|] \\ &= \frac{1}{2}[|CR||DS| + |ER||ES|] \ .\end{aligned}$$

In either case, we find that

$$[ABP] - [CDP] = \frac{1}{2}(|AR||BS| - |CR||DS| \ .$$

Suppose that $ABCD$ is concyclic. Then P is the centre of the circumcircle of $ABCD$, and hence of each of the triangle ABC and ABD. Therefore, R is the midpoint of AC and S the midpoint of $BD/$ Therefore, $|AR| = |CR|$ and $|BS| = |DS|$, so that $[ABP] = [CDP]$.

On the other hand, suppose that $ABCD$ is not concyclic. Then, wolog, we may suppose that $|AP| = |BP| > |CP| = |DP|$. By looking at right triangles, we see that $|AR| > |CR|$ and $|BS| > |DS|$, so that $[ABP] > [CDP]$. The result follows.

Solution 2. [J. Zung] We first establish Brahmagupta's theorem: *Suppose that $ABCD$ is a concyclic quadrilateral and that AC and BD intersect at right angles at E. Let Q be the point on CD for which $EQ \perp CD$, and let QE produced meet AB at M. Then M is the midpoint of AB.*

To prove this, note that

$$\begin{aligned}\angle MEB = \angle DEQ = 90^\circ - \angle EDQ = 90^\circ - \angle EDC \\ = \angle DCA = \angle DBA = \angle EBM \ ,\end{aligned}$$

whence $MB = ME$. Similarly, $MA = ME$. ♠

In the problem, let $ABCD$ be concyclic with circumcentre P, and M and N be the respective midpoints of AB and CD. Since P is the circumcentre of $ABCD$, we have that $PN \perp CD$, so that $PN \| ME$ by Brahmagupta's theorem. Similarly, $PM \| NE$ so that $PMNE$ is a paralleloram.

Therefore,

$$[CDP] = |ND||PN| = |NE||PN| = |PM||ME| = |PM||AM| = [ABP] \ .$$

[Z.Q. Liu] Suppose that the respective midpoints of AB and CD are M and N, that AC and BD intersect at E, that the right bisectors of AB and CD meet at P, and that AB and DC produced meet at K. Observe that, because of the right triangle ABE and CDE, $MA = MB = ME$ and $NC = ND = NE$.

Suppose that $[ABP] = [CDP]$. Then

$$|AM||MP| = |DN||NP| \Longrightarrow |ME||MP| = |NE|NP| \Longrightarrow ME : NE = NP : MP \ .$$

Also

$$\begin{aligned}\angle MEN = \angle MEA + \angle NED + 90^\circ = \angle MAE + \angle NDE + 90^\circ \\ = \angle KAD + \angle KDA = 180^\circ - \angle MKN = \angle MPN \ ,\end{aligned}$$

since $\angle KMP = \angle KNP = 90^\circ$. Therefore triangles MEN and MPN are similar. But as their common side MN corresponds in the similarity, the two triangles are congruent and so $MENP$ is a parallelogram.

Suppose that ME produced meets CD at R. Then $MR \perp CD$ and

$$\angle BDC = \angle EDR = \angle NER = \angle AEM = \angle MAE = \angle BAC \ ,$$

from which we conclude that $ABCD$ is concyclic.

Solution 3. (part) [T. Tang] As before, let the diagonals intersect at E, the right bisectors of AB and CD intersect at P and M and N be the respective midpoints of AB and CD. Suppose that $ABCD$ is concyclic. Then P is the circumcentre,

$$\begin{aligned}\angle NCP &= 90^\circ - \angle NPC = 90^\circ - \frac{1}{2}\angle DPC \\ &= 90^\circ - \angle DBC = 90^\circ - \angle EBC \\ &= \angle BCE = \angle BCA = \frac{1}{2}\angle APB = \angle MPB \ ,\end{aligned}$$

and

$$\angle PCD = 90^\circ - \angle NCP = 90^\circ - \angle MPB = \angle MBP \ .$$

As $PB = PC$, triangle PMB and CNP are congruent, so that $[APB] = 2[PMB] = 2[CNP] = [CPD]$.

539. Determine the maximum value of the expression

$$\frac{xy + 2yz + zw}{x^2 + y^2 + z^2 + w^2}$$

over all quartuple of real numbers not all zero.

Solution 1. Observe that

$$0 \leq [x - (\sqrt{2} - 1)y]^2 = x^2 + (3 - 2\sqrt{2})y^2 - 2(\sqrt{2} - 1)xy \ ,$$

$$0 \leq [w - (\sqrt{2} - 1)z]^2 = w^2 + (3 - 2\sqrt{2})w^2 - 2(\sqrt{2} - 1)zw \ ,$$

and

$$0 \leq 2(\sqrt{2} - 1)(y - z)^2 = 2(\sqrt{2} - 1)y^2 + 2(\sqrt{2} - 1)z^2 - 4(\sqrt{2} - 1)yz \ ,$$

with equality if and only if $x = (\sqrt{2} - 1)y = (\sqrt{2} - 1)z = w$. Adding the inequalities yields

$$2(\sqrt{2} - 1)(xy + 2yz + zw) \leq x^2 + y^2 + z^2 + w^2 \ .$$

Therefore, the maximum value of the expression is $[2(\sqrt{2}-1)]^{-1} = \frac{1}{2}(\sqrt{2}+1)$, and this maximum is assumed, for example, when $(x, y, z, w) = (\sqrt{2} - 1, 1, 1, \sqrt{2} - 1)$.

Solution 2. Since the expression is homogeneous of degree 0, we may wolog assume that $x^2+y^2+w^2+z^2 = 1$. Select θ so that $0 \leq \theta \leq \pi/2$ and $y^2 + z^2 = \sin^2\theta$ and $x^2 + w^2 = \cos^2\theta$. Then $2yz \leq \sin^2\theta$ and, by the Cauchy-Schwarz Inequality, $xy + zw \leq \sin\theta\cos\theta$. Therefore

$$\begin{aligned}xy + 2yz + zw &\leq \sin^2\theta + \sin\theta\cos\theta \\ &= \frac{1}{2}[1 - \cos 2\theta + \sin 2\theta] \\ &= \frac{1}{2}\left[1 + \sqrt{2}\sin\left(2\theta - \frac{\pi}{4}\right)\right] \\ &\leq \frac{1}{2}[1 + \sqrt{2}] \ ,\end{aligned}$$

with equality if and only if $\theta = \frac{3\pi}{8}$.

Solution 3. [H. Spink] Let u and v be nonnegative real numbers for which $2u^2 = x^2+w^2$ and $2v^2 = y^2+z^2$. Then $2yz \leq 2v^2$, $xy + zw \leq 2uv$ (by the Cauchy-Schwarz Inequality) and $x^2 + y^2 + z^2 + w^2 = 2(u^2 + v^2)$.

The given expression is not greater than $(v^2 + uv)/(u^2 + v^2)$. Equality occurs when $x = w$ and $y = z$. This vanishes when $v = 0$, When $v \neq 0$, we can write it as

$$f(w) \equiv \frac{1+w}{1+w^2}$$

where $w = u/v$. Thus, it suffices to determine the maximum of this last expression over positive values of w.

$f(w)$ assumes the positive real value λ if and only if the equation $f(w) = \lambda$ is solvable. This equation can be rewritten as

$$\begin{aligned} 0 &= \lambda w^2 - w + (\lambda - 1) \\ &= \frac{1}{4\lambda}[4\lambda^2 w^2 - 4\lambda w + 4\lambda(\lambda - 1)] \\ &= \frac{1}{4\lambda}[(2\lambda w - 1)^2 + (2\lambda - 1)^2 - 2] \ . \end{aligned}$$

The equation is solvable if and only if

$$(2\lambda - 1)^2 \leq 2 \Longleftrightarrow \lambda \leq \frac{\sqrt{2}+)}{2} \ .$$

The value of w that yields this value of λ is

$$\frac{1}{2\lambda} = \frac{1}{\sqrt{2}+1} = \sqrt{2} - 1 \ .$$

The expression takes its maximum value of $\frac{1}{2}(\sqrt{2}+1)$ when $(x, y, z, w) = (\sqrt{2}-1, 1, 1, \sqrt{2}-1)$.

Solution 4. [J. Zung] Let the expression to be maximized by u and set $x = a + b$, $y = a - b$, $z = c + d$, $w = c - d$. Then

$$u = \frac{ac + c^2 + bd - d^2}{a^2 + b^2 + c^2 + d^2} \ .$$

When q and s are positive, then $(p+r)/(q+s)$ lies between p/q and r/s, with equality if and only if $p/q = r/s$. Applying this to u, we see that it lies between

$$\frac{bd - d^2}{b^2 + d^2} \qquad \text{and} \qquad \frac{ac + c^2}{a^2 + c^2} \ .$$

The term on the left, being no greater than, $(bd + d^2)/(b^2 + d^2)$ is less than the maximum value over all (a, c) of the term on the right. So we maximize the function of a and c. Since it vanishes when $c = 0$ and clearly takes positive values, we may assume $c \neq 0$ and that $w = a/c$. Thus, we maximize $(1 + w)/(1 + w^2)$. This can be done as in Solution 3 to obtain the maximum value $\frac{1}{2}(\sqrt{2}+1)$.

However, we are not quite done. To ensure that u can assume this value, it seems that we need to find (b, d) so that $(bd - d^2)/(b^2 + d^2)$ equals this maximum value of $(ac + c^2)/(a^2 + c^2)$. But there is a way out: u is equal to the maximum when $b = d = 0$, and this occurs when $x = w$ and $y = z$, leading to the solution given previously.

Solution 5. [P. Wen] We are looking for the smallest value of u for which

$$\frac{xy + 2yz + zw}{x^2 + y^2 + z^2 + w^2} \leq u$$

for all reals x, y, z, w, not all vanishing. Since $|xy + 2yz + zw| \leq |x||y| + 2|y||z| + |z||w|$, it is enough to consider only nonnegative values of the variables. Since the left side takes the value 1 when $x = y = z = w$, and eligible u satisfies $u \geq 1$.

The inequality can be rewritten

$$\begin{aligned} 0 &\le u(x^2+y^2+z^2+w^2)-(xy+2yz+zw) \\ &= (y-z)^2+(u-1)(y^2+z^2)+u(x^2+w^2)-xy-zw \\ &= (y-z)^2+[\sqrt{u-1}y-\sqrt{u}x]^2+[\sqrt{u-1}z-\sqrt{w}w]^2+(\sqrt{(u-1)u}-1)(xy+zw)\ . \end{aligned}$$

This is to hold for all $x,y,z,w \ge 0$. When $y=z$ and $\sqrt{u}x=\sqrt{u-1}y$, the first three terms on the right vanish; for the fourth to be nonnegative, we require that

$$2\sqrt{(u-1)u}-1 \ge 0 \quad \Longleftrightarrow 1 \le 4(u-1)u \quad \Longleftrightarrow \quad 2 \le (2u-1)^2 \quad 2u-1 \ge \sqrt{2}\ .$$

Thus $u \ge \frac{1}{2}(\sqrt{2}+1)$.

When $u=\frac{1}{2}(\sqrt{2}+1)$, $\sqrt{(u-1)/u}=\sqrt{2}-1$, and we find that the expression in the problem assumes the value $\frac{1}{2}(\sqrt{2}+1)$ when $(x,y,z,w)=(\sqrt{2}-1,1,1,\sqrt{2}-1)$. Thus, the maximum value is $\frac{1}{2}(\sqrt{2}+1)$.

540. Suppose that, if all planar cross-sections of a bounded solid figure are circles, then the solid figure must be a sphere.

Solution. Since the solid figure is bounded, there exists two point A and B whose distance, r, apart is maximum. Let σ be any plane that passes through the segment AB. It intersects the solid figure in a circle, and no two points on this circle can be further than r apart. Therefore, AB is a diameter of this circle, and the solid figure is the solid of revolution of this circle about the segment AB.

541. Prove that the equation

$$x_1^{x_1}+x_2^{x_2}+\cdots+x_k^{x_k}=x_{k+1}^{x_{k+1}}$$

has no solution for which x_1, x_2, $\cdots$, x_k, x_{k+1} are all distinct nonzero integers.

Solution. Consider a sum of the following type:

$$\sum_{r=2}^{\infty}\epsilon_r r^{-r}=\epsilon_2\frac{1}{2^2}+\epsilon_3\frac{1}{3^3}+\epsilon_4\frac{1}{4^4}+\epsilon_5\frac{1}{5^5}+\cdots\ ,$$

where each ϵ_r is one of the numbers -1, 0, 1 and at most finitely many ϵ_r are nonzero. Since

$$\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\cdots<\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots\le\frac{1}{2}\ ,$$

it follows that each sum must exceed $-1/2$ and be less than $1/2$. Furthermore, since, for each index $s \ge 2$,

$$\frac{1}{(s+1)^{(s+1)}}+\frac{1}{(s+2)^{(s+2)}}+\frac{1}{(s+3)^{(s+3)}}+\frac{1}{(s+4)^{(s+4)}}+\cdots$$

$$<\frac{1}{(s+1)^{(s+1)}}+\frac{1}{(s+1)^{(s+2)}}+\frac{1}{(s+1)^{(s+3)}}+\frac{1}{(s+1)^{(s+4)}}+\cdots\le\frac{1}{s(s+1)^s}<\frac{1}{s^s}\ ,$$

it follows that for each sum, the absolute value of the first nonzero term exceeds the absolute value of the sum of the remaining terms, so that no sum can vanish. Therefore, all sums of the prescribed type are nonintegral rationals between $-1/2$ and $1/2$.

Suppose that there is a solution in integers to the equation of the problem; wolog, we may take $x_1 < x_2 < x_3 < \cdots < x_k$. If $x_1 \le -2$, then by the result of the previous paragraph, the sum of the terms of the left side is not an integer. Therefore x_{k+1}^{k+1} is not an integer, so that $x_{k+1} \le -2$. Shifting this term to the left side, we get a sum equal to zero consisting of two parts, a sum of the type $\sum_{r=2}^{\infty}\epsilon_r r^{-r}$ which is a noninteger and a sum of integer terms $x_i^{x_i}$ corresponding to any terms $x_i \geq -1$. This is impossible. Therefore, for all i, $x_i \geq -1$.

There is no solution in the case that $k = 1$. When $k \geq 2$, we must have that $x_1 \geq -1$, $x_k \geq 2$ and $x_k \geq k-1$. Therefore $x_{k+1}^{k+1} \geq x_k^{x_k} - 1$, whence $x_{k+1} > x_k$. Also

$$x_1^{x_1} + x_2^{x_2} + \cdots + x_k^{x_k} < kx_k^{x_k} \leq (x_k+1)x_k^{x_k} < (x_k+1)^{(x_k+1)} \leq x_{k+1}^{x_{k+1}} .$$

It follows that the equation is not solvable for distinct integers values of the x_i.

542. Solve the system of equations

$$\lfloor x \rfloor + 3\{y\} = 3.9 ,$$
$$\{x\} + 3\lfloor y \rfloor = 3.4 .$$

Solution. Let $x = a + u$ and $y = b + v$, where a and b are integers and $0 \leq u, v < 1$. Then the equations become $a + 3v = 3.9$ and $u + 3b = 3.4$. Adding the equations yields that $x + 3y = 7.3$.

Since $3b \leq 3.4 = u + 3b < 3b + 1$, it follows that $0.8 < b < 1.2$, whence $b = 1$ and $u = 0.4$. Since $a \leq 3.9 = a + 3v < a + 3$, it follows that $0.9 < a < 3.9$, whence $a = 1, 2, 3$ and $3v = 3.9 - a$. Hence $(a, v) = (1, 29/30), (2, 19/30), (3, 0.3)$.

The solutions (x, y) of the system are $(7/5, 59/30)$, $(12/5, 49/30)$ and $(17/5, 39/30) = (3.4, 1.3)$

543. Let $a > 0$ and b be real parameters, and suppose that f is a function taking the set of reals to itself for which

$$f(a^3x^3 + 3a^2bx^2 + 3ab^2x) \leq x \leq a^3f(x)^3 + 3a^2bf(x)^2 + 3ab^2f(x) ,$$

for all real x. Prove that f is a one-one function that takes the set of real numbers onto itself (*i.e.*, f is a *bijection*).

Solution. Let

$$g(x) = a^3x^3 + 3a^2bx^2 + 3ab^2x = (ax+b)^3 - b^3$$

for all real x. Then g is a one-one increasing function from the reals onto the reals. Let h be the composition inverse of g; then $h(g(x)) = g(h(x)) = x$ for all real x. The given condition is that

$$f(g(x)) \leq x \leq g(f(x))$$

for all real x. For each x, we have that

$$f(x) = f(g(h(x)) \leq h(x)$$

from the left inequality. Since $x \leq g(f(x))$ and h is increasing, we have that

$$h(x) \leq h(g(f(x)) = f(x) .$$

Therefore $f(x) = h(x)$. Applying this to the problem at hand, we find that

$$f(x) = \frac{\sqrt[3]{x+b^3} - b}{a}$$

and the result follows.

Comment. Note that it is possible to have the condition satisfied when $g(x)$ is decreasing, for example when $g(x) = -x^3$ and $f(x) = -x^{1/3}$. However, it does not seem clear that $f(x)$ is necessarily equal to $h(x)$ in this case.

544. Define the real sequences $\{a_n : n \geq 1\}$ and $\{b_n : n \geq 1\}$ by $a_1 = 1$, $a_{n+1} = 5a_n + 4$ and $5b_n = a_n + 1$ for $n \geq 1$.

(a) Determine $\{a_n\}$ as a function of n.

(b) Prove that $\{b_n : n \geq 1\}$ is a geometric progression and evaluate the sum

$$S \equiv \frac{\sqrt{b_1}}{\sqrt{b_2} - \sqrt{b_1}} + \frac{\sqrt{b_2}}{\sqrt{b_3} - \sqrt{b_2}} + \cdots + \frac{\sqrt{b_n}}{\sqrt{b_{n+1}} - \sqrt{b_n}} \ .$$

Solution. (a) We have that $a_{n+1} + 1 = 5(a_n + 1) = \cdots = 5^n(a_1 + 1) = 2 \cdot 5^n$, whence $a_n = 2 \cdot 5^{n-1} - 1$ for each positive integer n.

(b) Note that $b_n = 2 \cdot 5^{n-2}$, so that $\{b_n\}$ is a geometric progression. Since $b_{k+1} = 5b_k$, we have that $\sqrt{b_{k+1}} - \sqrt{b_k} = (\sqrt{5} - 1)\sqrt{b_k}$ for each positive integer k. it follows that $S = n/(\sqrt{5} - 1)$.

545. Suppose that x and y are real numbers for which $x^3 + 3x^2 + 4x + 5 = 0$ and $y^3 - 3y^2 + 4y - 5 = 0$. Determine $(x + y)^{2008}$.

Solution 1. The equations can be rewritten as

$$(x + 1)^3 + (x + 1) + 3 = 0$$

and

$$(y - 1)^3 + (y - 1) - 3 = 0 \ .$$

The left sides are both increasing with respect to their variables, so that x and y are uniquely determined by these equations. Adding these equations yields

$$\begin{aligned} 0 &= (x + 1)^3 + (y - 1)^3 + (x + 1) + (y - 1) \\ &= (x + y)[(x + 1)^2 - (x + 1)(y - 1) + (y - 1)^2] + (x + y) \\ &= (x + y)[x^2 - xy + y^2 + 3(x - y) + 4] \ . \end{aligned}$$

The second factor is one fourth of

$$4x^2 + 4(3 - y)x + 4(y^2 - 3y + 12) = (2x - y + 3)^2 + 3(y - 1)^2 + 4 \ ,$$

which is always positive. Hence $x + y = 0$ and the value of $(x + y)^{2008}$ is also 0.

Solution 2. As in Solution 1, we establish that each equation has exactly one real root. We note that if $x = u$ is a real solution of the first equation, then $y = -u$ is a real solution of the second. Therefore $x + y = 0$ and so $(x + y)^{2008} = 0$.

546. Let $a, a_1, a_2, \cdots, a_n$ be a set of positive real numbers for which

$$a_1 + a_2 + \cdots + a_n = a$$

and

$$\sum_{k=1}^{n} \frac{1}{a - a_k} = \frac{n + 1}{a} \ .$$

Prove that

$$\sum_{k=1}^{n} \frac{a_k}{a - a_k} = 1 \ .$$

Solution. Observe that

$$\sum_{k=1}^{n} \left(\frac{a_k}{a - a_k} - 1 \right) - n = \sum_{k=1}^{m} \frac{a}{a - a_k} - n = 1 \ .$$

from which the desired result follows.

547. Let A, B, C, D be four points on a circle, and let E be the fourth point of the parallelogram with vertices A, B, C. Let AD and BC intersect at M, AB and DC intersect at N, and EC and MN intersect at F. Prove that the quadrilateral $DENF$ is concyclic.

Solution. Since $\angle DCM = \angle BAD$, triangles DCM and ABM are similar and $AB : DC = BM : DM$. In triangle BCN, we have that

$$CN : BN = \sin \angle NBC : \sin \angle DCM \ .$$

In triangle DCM we have that

$$CM : DM = \sin \angle CDM : \sin \angle DCM \ .$$

Since

$$\angle NBC = 180° - \angle ABC = \angle ADC = 180° - \angle CDM \ ,$$

it follows that $CN : BN = CM : DM$.

Since $CF \| BN$, we have that $CF : BN = CM : BM$. Since $AB = EC$, we have that

$$BC \cdot CF = AB \cdot \frac{BN \cdot CM}{BM} \ .$$

Also

$$DC \cdot CN = DC \cdot \frac{CM \cdot BN}{DM} = \frac{DC}{DM} \cdot BN \cdot CM = \frac{AB}{BM} \cdot BN \cdot CM = EC \cdot CF \ .$$

From $DC \cdot CN = DC \cdot CN$, we deduce that the quadrilateral $DENF$ is concyclic.

548. In a sphere of radius R is inscribed a regular hexagonal truncated pyramid whose big base is inscribed in a great circle of the sphere (i.e., a whose centre is the centre of the sphere). The length of the side of the big base is three times the length of the side of a small base. Find the volume of the truncated pyramid as a function of R.

Solution. The big base, being the inscribed regular hexagon in a circle of radius R, has area $(3\sqrt{3}/2)R^2$. The small base is the inscribed hexagon in a circle of radius $R/3$, and this circle is distant $\sqrt{R^2 - (R/3)^2} = 2\sqrt{2}R/3$ from the big base. We can conceive of the frustum as consisting of the full pyramid on the big base with the full pyramid on the small base taken away. The height of the former pyramid is $\sqrt{2}R$ and of the latter $\sqrt{2}R/3$. The volume of the pyramid on the big base is $(1/3)(3\sqrt{3}/2)R^2(\sqrt{2})R = (\sqrt{6}/2)R^3$. The volume of the pyramid on the small base that is removed is $1/27$ of this, so that the volume of the frustum is $(26/27)(\sqrt{6}/2)R^3 = (13\sqrt{6}/27)R^3$.

Comment. If B is the area of the big base and S the area of the small base of the frustum, then the volume of the frustum is given by $(h/3)(B + S + \sqrt{BS})$, where h is the height of the frustum. In this case $B = 9S$, so that the volume of the frustum is $(2\sqrt{2}/9)R((1/9) + 1 + (1/3))(3\sqrt{3}/2)R = (13\sqrt{6}/27)R^3$.

549. The set E consists of 37 two-digit natural numbers, none of them a multiple of 10. Prove that, among the elements of E, we can find at least five numbers, such that any two of them have different tens digits and different units digits.

Solution. Call a set of nine numbers with the same tens digit a *decade*. By the Pigeon-Hole Principle, there is at least one decade with at least five numbers of E in it; wolog, let it be the tens decade. There are at least 28 numbers in E that are not in the tens decade; one of the remaining decades, say the twenties, must have at least four members of E. There are at least 19 members of E that are not in the tens or twenties decade; at least one of the remaining decades, say the thirties, has at least three members of E. Similarly, a fourth decade, say the forties, has at least two members and a fifth decade, say the fifties, has at least one member.

We can select the five element subset of E working back from the fifth decade. Select any number from the fifties, a number from the forties with a different tens digit, a number from the thirties with a tens digit differing from the two already determined, a number from the twenties with a tens digit differing from the three already determined and finally a number from the tens with a fifth tens digit. This will serve the purpose.

550. The functions $f(x)$ and $g(x)$ are defined by the equations: $f(x) = 2x^2 + 2x - 4$ and $g(x) = x^2 - x + 2$.

(a) Find all real numbers x for which $f(x)/g(x)$ is a natural number.

(b) Find the solutions of the inequality

$$\sqrt{f(x)} + \sqrt{g(x)} \geq \sqrt{2}\ .$$

Solution. (a) We have that

$$\frac{f(x)}{g(x)} = \frac{2x^2+2x-4}{x^2-x+2} = \frac{2(x+2)(x-1)}{x^2-x+2} = 2 + \frac{4(x-2)}{x^2-x+2}\ .$$

Observe that $x^2 - x + 2 = (x - \frac{1}{2})^2 + \frac{7}{4} > 0$.

Since $x^2 - 5x + 10 = (x - \frac{5}{2})^2 + \frac{15}{4} > 0$, $4x - 8 < x^2 - x + 2$, so that $4(x-2)/(x^2 - x + 2) < 1$. Hence $f(x)/g(x)$ cannot take integer values exceeding 2.

$$f(x)/g(x) = 2 \Longleftrightarrow x = 2\ ;$$

$$f(x)/g(x) = 1 \Longleftrightarrow 4x - 8 = -(x^2 - x + 2) \Longleftrightarrow x^2 + 3x - 6 = 0\ .$$

Therefore, $f(x)/g(x)$ is a natural number if and only if $x = 2$ or $x = \frac{1}{2}(-3 \pm \sqrt{3})$.

Comment. It is not too hard to find all values of x for which $f(x)/g(x)$ assumes integer values. Note that, if $x \geq 2$ or $x \leq -5$, then $x^2+3x-6 > 0$, so that $4x-8 > -(x^2-x+2)$ and $4(x-2)/(x^2-x+2) > -1$. Thus, if $f(x)/g(x)$ assumes integer values, then $|x| \leq 5$ and

$$\left|\frac{4(x-2)}{x^2-x+2}\right| \leq \frac{12}{3/2} = 8\ .$$

Thus, $f(x)/g(x)$ can takes only integer values between -6 and 2, and we can check each case.

(b) We require that $f(x) \geq 0$, so that $x \leq -2$ or $x \geq 1$. The functions $f(x) = 2(x+2)(x-1)$ and $g(x) = x(x-1)+2$ are both decreasing on $(-\infty, -2]$ and increasing on $[1, +\infty)$. Since $\sqrt{f(-2)} + \sqrt{g(-2)} = 0 + 2\sqrt{2} > 2$, $\sqrt{f(1)} + \sqrt{g(1)} = \sqrt{2}$, the inequality is satisfied on the set $(-\infty, -2] \cup [1, +\infty)$.

Note. There was an error in the statement of (b), where $\sqrt{2}$ was given as 2. In this case, the inequality is satisfied on the set $(\infty, -2] \cup [\alpha, \infty)$, where α lies between 1 and 2, and satisfies the equation $\sqrt{f(\alpha)} + \sqrt{g(\alpha)} = 2$.

551. The numbers 1, 2, 3 and 4 are written on the circumference of a circle, in this order. Alice and Bob play the following game: On each turn, Alice adds 1 to two adjacent numbers, while Bob switches the

places of two adjacent numbers. Alice wins the game, if after her turn, all numbers on the circle are equal. Does Bob have a strategy to prevent Alice from winning the game? Justify your answer.

Solution. Bob can prevent Alice from winning the game whenever Alice has the first move. The configuration of numbers begins with even and odd numbers alternating; Bob's strategy is to always present Alice with this situation. Then, whatever Alice does, she must leave two odd and two even numbers and therefore at least two distinct numbers.

To do this, Bob must switch whatever pair of numbers Alice selects to add 1 to. Alice's move changes the parity of the numbers in the two positions and Bob's move switches the parity back to what it was before.

552. Two real nonnegative numbers a and b satisfy the inequality $ab \geq a^3 + b^3$. Prove that $a + b \leq 1$.

Solution 1. Note that $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and that $a^2 - ab + b^2$ is always positive. Then

$$\begin{aligned} 1 - (a + b) &= 1 - \frac{a^3 + b^3}{a^2 - ab + b^2} \geq 1 - \frac{ab}{a^2 - ab + b^2} \\ &= \frac{a^2 - 2ab + b^2}{a^2 - ab + b^2} = \frac{(a - b)^2}{a^2 - ab + b^2} \geq 0 \ , \end{aligned}$$

from which the desired result follows.

Solution 2. We note two inequalities: (1) $(a + b)^2 \geq 4ab$ and (2) $4(a^3 + b^3) \geq (a + b)^3$. The first is a consequence of the arithmetic-geometric means inequality, while the second can be obtained either from $(a + b)(a - b)^2 \geq 0$ or from the power-mean inequality $[\frac{1}{2}(a^3 + b^3)]^{1/3} \geq \frac{1}{2}(a + b)$. It follows that

$$(a + b)^2 \geq 4ab \geq 4(a^3 + b^3) \geq (a + b)^3 \ ,$$

from which the result is obtained.

553. The convex quadrilateral $ABCD$ is concyclic with side lengths $|AB| = 4$, $|BC| = 3$, $|CD| = 2$ and $|DA| = 1$. What is the length of the radius of the circumcircle of $ABCD$? Provide an exact value of the answer.

Solution. Let $\alpha = \angle DAB$ and $\beta = \angle ABC$, so that $\angle BCD = 180° - \alpha$ and $\angle CDA = 180° - \beta$. Then, by the Law of Cosines,

$$1 + 16 - 8\cos\alpha = |BD|^2 = 9 + 4 + 12\cos\alpha \ ,$$

whence $\cos\alpha = 1/5$ and $|BD| = \sqrt{77/5}$. The circumradius R of $ABCD$ satisfies $2R\sin\alpha = |BD|$, whence

$$R = \frac{\sqrt{77/5}}{2\sqrt{24/25}} = \frac{\sqrt{5 \times 7 \times 11}}{4\sqrt{6}} = \frac{\sqrt{385}}{4\sqrt{6}} \ .$$

As a check, we can find that

$$16 + 9 - 24\cos\beta = |AC|^2 = 4 + 1 + 4\cos\beta \ ,$$

whence $\cos\beta = 5/7$ and $|AC| = \sqrt{\frac{55}{7}}$. Thus, $2R\sin\beta = |AC|$, so that

$$R = \frac{\sqrt{55/7}}{2\sqrt{24/49}} = \frac{\sqrt{5 \times 7 \times 11}}{4\sqrt{6}} = \frac{\sqrt{385}}{4\sqrt{6}} \ .$$

554. Determine all real pairs (x, y) that satisfy the system of equations:

$$3\sqrt[3]{x^2y^5} = 4(y^2 - x^2)$$

$$5\sqrt[3]{x^4y} = y^2 + x^2 \ .$$

Solution. Multiply the two equations to obtain

$$\begin{aligned} 15x^2y^2 = 4(y^4 - x^4) &\Leftrightarrow 0 = 4x^4 + 15x^2y^2 - 4y^4 = (4x^2 - y^2)(x^2 + 4y^2) \\ &\Leftrightarrow y^2 = 4x^2 \Leftrightarrow y = \pm 2x \ . \end{aligned}$$

Substituting $y = 2x$ into the first equation yields that

$$3\sqrt[3]{2^5x^2x^5} = 12x^2 \Longrightarrow 2^5 \times 3^3 \times x^7 = 2^6 \times 3^3 \times x^6 \Longrightarrow x = 0 \text{ or } x = 2 \ .$$

Similarly, substituting $y = -2x$ into the first equation yields the additional solution $x = -2$. There are three solutions to the system, namely, $(x, y) = (0, 0), (2, 4), (-2, 4)$. All check out.

555. Let ABC be a triangle, all of whose angles do not exceed $90°$. The points K on side AB, M on side AC and N on side BC are such that $KM \perp AC$ and $KN \perp BC$. Prove that the area $[ABC]$ of triangle ABC is at least 4 times as great as the area $[KMN]$ of triangle KMN, *i.e.*, $[ABC] \geq 4[KMN]$. When does equality hold?

Solution 1. Let $b = |AC|$, $a = |BC|$, $m = |KM|$, $n = |KN|$ and $\theta = \angle ACB$, so that $\angle MKN = 180° - \theta$. Then, by the arithmetic-geometric means inequality,

$$[ABC] = [AKC] + [AKB] = \frac{1}{2}(bm + an) \geq \sqrt{abmn} \ .$$

Therefore

$$\begin{aligned} [ABC]^2 &\geq abmn \geq abmn \sin^2\theta \\ &\geq (ab\sin\theta)(mn\sin(180° - \theta)) \\ &= 2[ABC] \cdot 2[KMN] \ , \end{aligned}$$

whence $[ABC] \geq 4[KMN]$, as desired.

Equality holds if and only if $bm = an$ and $\sin\theta = 1$, if and only if $a : b = m : n$ and $\theta = 90°$, if and only if triangle ABC is right and similar to triangles AKM and KBN. In this case, $KN \| AC$ and $KM \| BC$ and the linear dimensions of KMN are half those of CAB; thus, $AC = 2NK$, $BC = 2MK$ and K is the midpoint of AB.

Solution 2. Let the angles at A, B and C in triangle ABC be respectively α, β, γ and the sides of this triangle be, conventionally, a, b, c. Suppose that $m = |KM|$, $n = |KN|$, and $x = |AK|$, so that $m = x\sin\alpha$ and $n = (c - x)\sin\beta$.

Then $[ABC] = \frac{1}{2}ab\sin\gamma$ and

$$[KMN] = \frac{1}{2}mn\sin(180° - \gamma) = \frac{1}{2}x(c - x)\sin\alpha\sin\beta\sin\gamma \ .$$

Thus

$$\begin{aligned} \frac{[KMN]}{[ABC]} &= \frac{x(c - x)\sin\alpha\sin\beta}{ab} = x(c - x)\left(\frac{\sin\alpha}{a}\right)\left(\frac{\sin\beta}{b}\right) \\ &= x(c - x)\left(\frac{1}{2R}\right)\left(\frac{1}{2R}\right) = \frac{x(c - x)}{4R^2} \ , \end{aligned}$$

where R is the circumradius of triangle ABC. By the Arithmetic-Geometric Means Inequality, $x(c - x) \leq (\frac{1}{2}[x + (c - x)])^2 = c^2/4$, so that

$$\begin{aligned} \frac{[KMN]}{[ABC]} &= \frac{x(c - x)}{4R^2} \leq \left(\frac{c^2}{4}\right)\left(\frac{1}{4R^2}\right) = \frac{1}{4}\left(\frac{c}{2R}\right)^2 \\ &= \frac{1}{4}\sin^2\gamma \leq \frac{1}{4} \ , \end{aligned}$$

as desired. Equality holds if and only if $x = c - x$ and $\sin\gamma = 1$, *i.e.*, when ABC has a right angle at C and K is the midpoint of AB.

556. Let x, y, z be positive real numbers for which $x + y + z = 4$. Prove the inequality

$$\frac{1}{2xy + xz + yz} + \frac{1}{xy + 2xz + yz} + \frac{1}{xy + xz + 2yz} \le \frac{1}{xyz} .$$

Solution. It is straightforward to establish for $a, b > 0$ that $(a + b)^{-1} \le \frac{1}{4}(a^{-1} + b^{-1})$. Therefore,

$$\begin{aligned}\frac{1}{2xy + xz + yz} &\le \frac{1}{4}\left(\frac{1}{xy + xz} + \frac{1}{xy + yz}\right) \le \frac{1}{4}\left[\frac{1}{4}\left(\frac{1}{xy} + \frac{1}{xz}\right) + \frac{1}{4}\left(\frac{1}{xy} + \frac{1}{yz}\right)\right] \\ &= \frac{1}{16}\left(\frac{2}{xy} + \frac{1}{xz} + \frac{1}{yz}\right) = \frac{1}{16}\left(\frac{2z + y + x}{xyz}\right) .\end{aligned}$$

Similarly,

$$\frac{1}{xy + 2xz + yz} \le \frac{1}{16}\left(\frac{z + 2y + x}{xyz}\right)$$

and

$$\frac{1}{xy + xz + 2yz} \le \frac{1}{16}\left(\frac{z + y + 2x}{xyz}\right) .$$

Adding the three inequalities yields that

$$\frac{1}{2xy + xz + yz} + \frac{1}{xy + 2xz + yz} + \frac{1}{xy + xz + 2yz} \le \frac{1}{16}\left(\frac{4x + 4y + 4z}{xyz}\right) = \frac{1}{xyz} .$$

Equality holds if and only if $x = y = z = 4/3$.

557. Suppose that the polynomial $f(x) = (1+x+x^2)^{1004}$ has the expansion $a_0 + a_1x + a_2x^2 + \cdots + a_{2008}x^{2008}$. Prove that $a_0 + a_2 + \cdots + a_{2008}$ is an odd integer.

Solution. Observe that

$$a_0 + a_2 + \cdots + a_{2008} = \frac{1}{2}(f(1) + f(-1)) = \frac{1}{2}(3^{1004} + 1) .$$

It remains to show that $3^{1004} + 1$ is congruent to 2 modulo 4.

558. Determine the sum

$$\sum_{m=0}^{n-1}\sum_{k=0}^{m}\binom{n}{k} .$$

Solution. Let $S_m = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m}$. Then $S_0 + S_{n-1} = S_1 + S_{n-2} = \cdots = S_{n-1} + S_0 = 2^n$, so that $S = n2^{n-1}$.

Comment. In more detail,

$$\begin{aligned}S_k + S_{n-1-k} &= \left[\binom{n}{0} + \cdots + \binom{n}{k}\right] + \left[\binom{n}{0} + \cdots + \binom{n}{n-1-k}\right] \\ &= \left[\binom{n}{0} + \cdots + \binom{n}{k}\right] + \left[\binom{n}{n} + \cdots + \binom{n}{k+1}\right] = 2^n .\end{aligned}$$

559. Let ϵ be one of the roots of the equation $x^n = 1$, where n is a positive integer. Prove that, for any polynomial $f(x) = a_0 + a_x + \cdots + a_n x^n$ with real coefficients, the sum $\sum_{k=1}^n f(1/\epsilon^k)$ is real.

Solution. If $\epsilon = 1$, the result is clear. Let $\epsilon \neq 1$; we have that $\epsilon^n = 1$.

$$\begin{aligned}\sum_{k=1}^n f(1/\epsilon^k) &= \sum_{k=1}^n \sum_{j=0}^n a_j (1/\epsilon^k)^j = \sum_{k=1}^n \sum_{j=0}^n a_j (1/\epsilon^{jk}) \\ &= \sum_{j=0}^n a_j \sum_{k=1}^n (1/\epsilon^{jk}) = na_0 + \sum_{j=2}^{n-1} a_j (1/\epsilon^j)\left(\frac{1-\epsilon^{-jn}}{1-\epsilon^{-j}}\right) + na_n \\ &= na_0 + 0 + na_n = n(a_0 + a_n) \ .\end{aligned}$$

560. Suppose that the numbers $x_1, x_2, \cdots, x_n$ all satisfy $-1 \leq x_i \leq 1$ $(1 \leq i \leq n)$ and $x_1^3 + x_2^3 + \cdots + x_n^3 = 0$. Prove that

$$x_1 + x_2 + \cdots + x_n \leq \frac{n}{3} \ .$$

Solution. Since $-1 \leq x_i \leq 1$, for $1 \leq i \leq n$, there exists θ_i with $0 \leq \theta_i \leq \pi$ such that $x_i = \cos\theta_i$. Therefore

$$\begin{aligned}\sum_{i=1}^n x_i = \sum_{i=1}^n \cos\theta_i &= \frac{1}{3}\left[4\sum_{i=1}^n \cos^3\theta_i - \sum_{i=1}^n \cos 3\theta_i\right] \\ &= -\frac{1}{3}\sum_{i=1}^n \cos 3\theta_i \leq \frac{n}{3} \ ,\end{aligned}$$

as desired.

561. Solve the equation

$$\left(\frac{1}{10}\right)^{\log_{(1/4)}(\sqrt[4]{x}-1)} - 4^{\log_{10}(\sqrt[4]{x}+5)} = 6 \ ,$$

for $x \geq 1$.

Solution. Let $a = \log_{(1/4)}(\sqrt[4]{x} - 1)$ and $b = \log_{10}(\sqrt[4]{x} + 5)$. Then $(1/4)^a = \sqrt[4]{x} - 1$ and $10^b = \sqrt[4]{x} + 5$, whence $(1/4)^a + 1 = 10^b - 5$, or

$$\left(\frac{1}{4}\right)^a - 10^b = -6 \ .$$

On the other hand, the given equation is

$$\left(\frac{1}{10}\right)^a - 4^b = 6 \ .$$

Therefore

$$\left(\frac{1}{4}\right)^a - 4^b + \left(\frac{1}{10}\right)^a - 10^b = 0$$

which is equivalent to

$$(4^{-a} - 4^b) + (10^{-a} - 10^b) = 0 \ .$$

The left side is less than 0 when $-a < b$ and greater than 0 when $-a > b$. Therefore $-a = b$ and so $10^b - 4^b = 6$. One solution of this is $b = 1$.

We show that this solution is unique. Observe that the function $f(x) = 6(1/10)^x + (4/10)^x$ decreases as x increases from 0 and takes the value 1 when $x = 1$. Since $f(x) = 1$ is equivalent to $6 = 10^x - 4^x$, we see that $x = 1$ is the only solution of the latter equation.

562. The circles $\mathfrak{C}$ and $\mathfrak{D}$ intersect at the two points A and B. A secant through A intersects $\mathfrak{C}$ at C and $\mathfrak{D}$ at D. On the segments CD, BC, BD, consider the respective points M, N, K for which $MN \| BD$ and $MK \| BC$. On the arc BC of the circle $\mathfrak{C}$ that does not contain A, choose E so that $EN \perp BC$, and on the arc BD of the circle $\mathfrak{D}$ that does not contain A, choose F so that $FK \perp BD$. Prove that angle EMF is right.

Solution. We have that $BN : NC = DM : MC = DK : KB$. Let G be the point of intersection of FK and $\mathfrak{D}$. Then $\angle BGD = \angle BAD = \angle BEC$. In triangle BGD and CEB, we have that $\angle BGD = \angle CEB$. Compare triangles BGD and CEB: $\angle BGD = \angle CEB$; GK and EN are respective altitudes; $DK : KB = BN : NC$. There is a similarilty transformation with factor $|DK|/|BN|$ that takes $B \rightarrow D$, $C \rightarrow B$, $N \rightarrow K$ and E to a point E' on the line KG. Since $\angle BGD = \angle CEB = \angle BE'D$, we must have $E' = G$. Thus triangles BGD and CEB are similar, whence $\angle EBC = \angle GDB = \angle GFB$. As a result, triangles BNE and FKB are similar.

Since $MNBK$ is a parallelogram, $\angle MNB = \angle MKB$. Thus $\angle MNE = \angle MKF$. Since $MN : KF = BK : KF = EN : NB = EN : MK$, triangles ENM and MKF are similar. Therefore $\angle NME = \angle KFM$. But $MN \perp KF$. Therefore $EM \perp FM$.

563. (a) Determine infinitely many triples (a, b, c) of integers for which a, b, c are not in arithmetic progression and $ab + 1$, $bc + 1$, $ca + 1$ are all squares.

(b) Determine infinitely many triples (a, b, c) of integers for which a, b, c are in arithemetic progression and $ab + 1$, $bc + 1$, $ca + 1$ are all squares.

(c) Determine infinitely many triples (u, v, w) of integers for which $uv - 1$, $vw - 1$, $wu - 1$ are all squares. (Can it be arranged that u, v, w are in arithmetic progression?)

Solution. (a) Here are some families of solutions that are (mostly) not in arithmetic progression, where n is an integer:

$$(0,0,n); (0,n-1,n+1); (0,2,2n(n+1)); (1,n^2-1,n^2+2n); (n-1,n+1,4n); (n,n+2,4(n+1));$$

$$(m, mn^2+2n, m(n+1)^2+2(n+1)); (f_{2(n-1)}, f_{2n}, f_{2(n+1)}) \ .$$

Here, $\{f_n\}$ is the Fibonacci sequence defined by $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for every integer n. We need to establish that $f_{2n}f_{2n+2} + 1 = f_{2n+1}^2$ and $f_{2n-2}f_{2n+2} + 1 = f_{2n}^2$ for each integer n. Since

$$f_{2n+1}^2 - f_{2n}^2 = f_{2n+2}f_{2n-1} = f_{2n+2}(f_{2n} - f_{2n-2}) = (f_{2n}f_{2n+2} + 1) - (f_{2n-2}f_{2n+2} + 1) \ ,$$

the two equations are equivalent. Note that

$$f_{2n}f_{2n+2} - f_{2n+1}^2 = f_{2n}(f_{2n+1} + f_{2n}) - f_{2n+1}^2 = f_{2n+1}(f_{2n} - f_{2n+1}) + f_{2n}^2 = -(f_{2n+1}f_{2n-1} - f_n^2) \ ;$$

a proof by induction can be devised for the first equation.

(b) (i) Some examples for (a, b, c) are $(-1, 0, 1)$, $(0, 2, 4)$, $(1, 8, 15)$, $(4, 30, 56))$, $(15, 112, 209)$. This suggests the possibility $(u_n, 2u_{n+1}, u_{n+2})$ where $u_0 = 0$, $u_1 = 1$, $u_2 = 4$ and $u_{n+1} = 4u_{n-1} - u_n$ for integral n. Since $u_{n+1} - 2u_n = 2u_n - u_{n-1}$, $u_{n-1}, 2u_n, u_{n+1}$ are in arithmetic progression.

We now prove, for each integer n,

$$2u_nu_{n+1} + 1 = (u_{n+1} - u_n)^2 \tag{1}$$

$$u_{n+2}u_n + 1 = u_{n+1}^2 \tag{2}$$

$$2u_{n+1}u_{n+2} + 1 = (u_{n+2} - u_{n+1})^2 \tag{3}$$

Properties (1) and (3) are the same. The truth of (2) is equivalent to the truth of (1), since

$$\begin{aligned}[(2u_nu_{n+1}+1)-(u_{n+1}-u_n)^2)]&+[(u_nu_{n+2}+1)-u_{n+1}^2]\\ &=u_n(2u_{n+1}-u_{n+2})+u_n(2u_{n+1}-u_n)\\ &=-u_n(u_{n+2}-4u_{n+1}+u_n)=0\ .\end{aligned}$$

We establish (2) by induction. Since

$$\begin{aligned}u_{n+2}u_n+1-u_{n+1}^2&=u_n(4u_{n+1}-u_n)+1-u_{n+1}^2\\ &=u_{n+1}(4u_n-u_{n+1})+1-u_n^2\\ &=u_{n+1}u_{n-1}+1-u_n^2\ ,\end{aligned}$$

$u_{n+2}u_n+1-u_{n+1}^2=u_2u_0+1-u_1^2=0$ for all n. The desired results follow.

(b) (ii) [A. Dhawan] Let $v^2-3u^2=1$ for some integers v and u. Then, if $(a,b,c)=(2u-v,2u,2u+v)$, then

$$\begin{aligned}ab+1&=(2u-v)2u+1=4u^2-2uv+1\\ &=u^2+(v^2-1)-2uv+1=(u-v)^2\ ;\end{aligned}$$

$$\begin{aligned}bc+1&=2u(2u+v)+1=4u^2+2uv+1\\ &=u^2+2uv+v^2-1+1=(u+v)^2\ ;\end{aligned}$$

and

$$\begin{aligned}ac+1&=(2u-v)(2u+v)+1\\ &=4u^2-v^2+1=u^2\ .\end{aligned}$$

(Note that in this solution, the roots of the square, not all positive, are also in arithmetic progression.)

The equation $v^2-3u^2=1$ is a Pell's equation with infinitely many solutions given by $(v,u)=(x_n,y_n)$, where $x_n+y_n\sqrt{3}=(2+\sqrt{3})^n$, for positive integers n.

(b) (iii) We look for solutions in which one integer is 0, Thus (a,b,c) has the form $(0,p,2p)$, where $2p^2+1=q^2$. This is a Pell's equation whose solutions are given by $(q,p)=(x_n,y_n)$ where $x_n+y_n\sqrt{2}=(3+2\sqrt{2})^n$ for positive integers n. P. Wen also identified triples $(0,p,2p)$ where $2p^2+1$ is square. Since this square is odd, it must have the form $(2y+1)^2$, so that $p^2=2y(y+1)$. Thus, p is even, say $p=2x$, and so $x^2=\frac{1}{2}y(y+1)$, which is at once a square and a triangula number. Conversely, any triangular number which is also a square gives a solution triple, so we need to know that there are infinitely many such. If $x^2=\frac{1}{2}y(y+1)$, then $8x^2+1=(2y+1)^2$, so that

$$\begin{aligned}[x(2y+1)]^2=4p^2(2y+1)^2&=\frac{4y(y+1)(2y+1)^2}{2}\\ &=\frac{1}{2}(4y^2+4y)(4y^2+4y+1)\ .\end{aligned}$$

Starting with $(x,y)=(1,1)$, we are led to $(6,8),(204,288)$ and so on, and obtain the solutions $(a,b,c)=(0,2,4),(0,12,24),(0,408,816),\cdots$.

(c) Here are some families of solutions for (u,v,w): $(1,1,n^2+1)$, $(1,n^2+1,(n+1)^2+1)$ along with $(f_{2n-1},f_{2n+1},f_{2n+3})$, where f_n is the Fibonacci sequence defined in the solution to (a). S.H. Lee found a two-parameter family:

$$(m^2+1,(m^2+1)n^2+2mn+1,(m^2+1)(n+1)^2+2m(n+1)+1)\ .$$

In this case,

$$uv-1=[(m^2+1)n+m^2]^2\ ;\qquad uw-1=[(m^2+1)(n+1)+m]^2\ ;$$

$$uw - 1 = [(m^2+1)n(n+1) + 2mn + m + 1]^2 \ .$$

J. Zung identified the triple $(1, 2, p^2+1)$ where $q^2 - 2p^2 = 1$ for some integer q. [Exercise: Check these out.]

564. Let $x_1 = 2$ and

$$x_{n+1} = \frac{2x_n}{3} + \frac{1}{3x_n}$$

for $n \geq 1$. Prove that, for all $n > 1$, $1 < x_n < 2$.

Solution 1. Since, for $n \geq 1$,

$$x_{n+1} - 1 = \frac{2x_n}{3} + \frac{1}{3x_n} - 1 = \frac{2x_n^2 - 3x_n + 1}{3x_n} = \frac{(2x_n - 1)(x_n - 1)}{3x_n} \ ,$$

it can be shown by induction that $x_n > 1$ for all $n \geq 1$.

Since, for $n \geq 1$,

$$x_n - x_{n+1} = \frac{x_n^2 - 1}{3x_n} \ ,$$

it follows that $x_{n+1} < x_n \leq 2$ for all $n \geq 1$. Thus we have the desired inequality.

Solution 2. Observe that $x_2 = 3/2$. As an induction hypothesis, suppose that $1 < x_n < 2$ for some $n \geq 2$. Then

$$x_{n+1} = \frac{x_n}{3} + \frac{1}{3}\left(x_n + \frac{1}{x_n}\right) > \frac{1}{3} + \frac{1}{3} \cdot 2 = 1$$

by the arithmetic-geometric means inequality. Also,

$$x_{n+1} = \frac{2x_n}{3} + \frac{1}{3x_n} < \frac{4}{3} + \frac{1}{3} = \frac{5}{3} < 2 \ .$$

The result follows.

Comment. An induction argument for the right inequality can be based on the observation that, when $1 < x < 2$, the quadratic $2x^2 - 6x + 1 = 2(x - \frac{3}{2})^2 - \frac{7}{2} \leq \frac{1}{2} - \frac{7}{2} = -3 < 0$, whence $2x^2 + 1 < 6x$ and $(2x_n/3) + (1/3x^n) < 2$.

Solution 3. Let $f(x) = (2x/3) + (1/3x)$. Then $f'(x) = (2/3) - (1/3x^3) \geq 1/3 > 0$ when $1 \leq x$. Hence $f(x)$ is strictly increasing on the interval $[1, 2]$ and so takes values strictly between $f(1) = 1$ and $f(2) = 3/2$ on the open interval $(0, 1)$. Since $x_2 \in (0, 1)$ and $x_{n+1} = f(x_n)$ for $n \geq 1$, the desired result can be established by induction.

565. Let ABC be an acute-angled triangle. Points A_1 and A_2 are located on side BC so that the four points are ordered B, A_1, A_2, C; similarly B_1 and B_2 are on CA in the order C, B_1, B_2, A and C_1 and C_2 on side AB in order A, C_1, C_2, B. All the angles AA_1A_2, AA_2A_1, BB_1B_2, BB_2B_1, CC_1C_2, CC_2C_1 are equal to θ. Let $\mathfrak{T}_1$ be the triangle bounded by the lines AA_1, BB_1, CC_1 and $\mathfrak{T}_2$ the triangle bounded by the lines AA_2, BB_2, CC_2. Prove that all six vertices of the triangles are concyclic.

Solution 1. Let $A_0B_0C_0$ be the triangle with $B_0C_0 \| BC$, $C_0A_0 \| CA$, $A_0B_0 \| AB$ where A, B, C are the respective midpoints of B_0C_0, C_0A_0, A_0B_0. Then the orthocentre H of triangle ABC is the circumcentre of triangle $A_0B_0C_0$.

Suppose that K is the intersection point of AA_2 and BB_2. Since the exterior angle at A_2 is equal to the interior angle at B_2, the quadrilateral A_2KB_2C is concyclic, so that $\angle BKA_2 = \angle BCA = \angle BC_0A$. Therefore, the quadrilateral AC_0BK is concyclic; the quadrilateral AC_0BH with right angles at A and B is concyclic. Thus, BC_0AKH is concyclic and so $\angle C_0KH = \angle C_0AH = 90°$.

Since $C_0A_0 \| AC$, $\angle C_0HK = \angle C_0BK = \angle BB_2C = \theta$. Therefore $|HK| = |HC_0|\cos\theta = R\cos\theta$, where R is the circumradius of triangle $A_0B_0C_0$. The same argument can be applied to the intersection point of any pairs (AA_i, BB_i), (BB_i, CC_i), (CC_i, AA_i) $(i = 1, 2)$. All the vertices lie on the circle with centre H and radius R.

Solution 2. [A. Murali] Let $AA_1 \cap BB_1 = P$, $BB_1 \cap CC_1 = Q$, $CC_1 \cap AA_1 = R$, $AA_2 \cap BB_2 = V$, $BB_2 \cap CC_2 = W$, $CC_2 \cap AA_2 = U$. We have that

$$\begin{aligned}\angle A_2CU = \angle BCC_2 &= 180^\circ - \angle ABC - \angle BC_2C \\ &= 180^\circ - \angle ABC - (180^\circ - \theta) = \theta - \angle ABC \ ,\end{aligned}$$

and

$$\begin{aligned}\angle CUA_2 &= 180^\circ - (\angle A_2CU + \angle AA_2C) \\ &= 180^\circ - (\theta - \angle ABC) - (180^\circ - \theta) = \angle ABC \ .\end{aligned}$$

Since $\angle AA_1B = \angle CA_2U$ and $\angle ABA_1 = \angle ABC = \angle A_2UC$, triangles AA_1B and CA_2U are similar. Therefore $CA_2 : A_2U = AA_1 : BA_1$, from which

$$|A_2U| = \frac{|CA_2| \times |BA_1|}{|AA_1|} \ .$$

Similarly, $\angle BPA_1 = \angle BCA$, which along with $\angle BA_1A = \angle BB_1C$ implies that triangles BA_1P and BB_1C are similar. Therefore

$$|PA_1| = \frac{|BA_1| \times |B_1C|}{|BB_1|} \ .$$

Hence,

$$\frac{|A_2U|}{|A_1P|} = \frac{|CA_2| \times |BB_1|}{|AA_1| \times |B_1C|} = \frac{|CA_2| \times |BB_1|}{|AA_2| \times |B_1C|} \ .$$

Since triangles CBB_1 and CAA_2 are similar, $CA_2 : AA_2 = CB_1 : BB_1$, from which it follows that $UA_2 = PA_1$, so that $UA_2 : AA_2 = PA_1 : AA_1$ and $PU \| A_1A_2$.

Similarly, $QV \| B_1B_2$. Therefore

$$\angle PUV = \angle A_1A_2A = \theta = \angle B_2B_1B = \angle VQP$$

and $PUQV$ is concyclic.

Since $\angle C_2UA = \angle CUA_2 = \angle ABC$ and $\angle AC_2U = \theta = \angle BC_1C$, triangles AUC_2 and CBC_1 are similar, so that

$$|UC_2| = \frac{|BC_1| \times |C_2A|}{|C_1C|} \ .$$

Since triangles BQC_1 and CAC_2 are similar,

$$|QC_1| = \frac{|BC_1| \times |AC_2|}{|CC_2|} \ .$$

Since $CC_2 = CC_1$, $UC_2 = QC_1$ so that $UQ \| BA$. Similarly $WP \| AC$. Therefore, $\angle WPQ = 180^\circ - \theta = \angle WUQ$, and $WPUQ$ is concyclic.

Since $RWPU$, $WPUQ$ and $PUQV$ are all concyclic, R and Q lie on the circle through W, P, U and W and V lie on the circle through P, U, Q. The result follows.

Solution 3. [P. Wen] Use the notation of Solution 2 and let H denote the orthocentre of triangle ABC. Since $\angle WBH = 90^\circ - \theta = \angle WCH$, the points B, W, H, C are concyclic; similarly, B, H, Q, C are concyclic. Hence B, W, H, Q, C are concyclic. Similarly, A, V, H, P, B are concyclic.

Since

$$\angle PQW = \angle BQW = \angle BCW = \angle BCC_2 = \angle BAA_1$$
$$= \angle BAP = \angle BVP = \angle PVW \ ,$$

the points P, Q, V, W are concyclic. Since $\angle BHW = \angle BCW = \angle BAP = \angle BHP$, $\angle HBW = \angle HBB_2 = \angle HBB_1 = \angle HBP$, and side BH is common, the triangles BHW and BHP are congruent, so that $BP = BW$.

Since $\angle PHW = 2\angle BHW = 2\angle PQW$, H must be the centre of the circle through P, Q, V, W, so that H is equidistant from these four points. Similarly, H is equidistant from the four points R, P, U, V and from the points Q, R, W, U. The desired result follows.

Solution 4. [P.J. Zhao] Use the notation of Solutions 2 and 3, with H the orthocentre of triangle ABC. Since the quadrilaterals BC_1RA_1, BC_1B_2C and CA_2VB_2 are concyclic, we have that

$$AR : AA_1 = AC_1 \text{''} AB = AB_2 : AC = AV : AA_2 \ .$$

Since $AA_1 = AA_2$, $AR = AV$. As AA_1A_2 is isosceles, AH bisects angle A_1AA_2 and triangles AHR amd AHV are congruent (SAS), so that $HR = HV$. Similarly, $HP = HW$ and $HQ = HU$.

Since the quadrilaterals CB_1PA_1, BC_2B_1C and BC_2UA_2 are concyclic, it follows that

$$AP : AA_1 = AB_1 : AC = AC_2 : AB = AU : AA_2 \ ,$$

whence $AP = AU$. Since triangles AHP and AHU are congruent, $HP = HU$. Similarly, $HQ = HV$ and $HR = HW$.

Thus, all six vertices of the two triangles are equidistant from H and the result follows.

Comment. J. Zung observed that a rotation about H through the angle 2θ takes the line AA_1 onto the line A_2A, the line BB_1 onto the line B_2B and the line CC_1 onto the line C_2C. To see this, note that if A_3 and A_4 are the feet of the perpendiculars dropped from H to AA_1 and AA_2 respectively, then

$$\angle A_3HA_4 = \angle A_3HA + \angle AHA_4 = \angle AA_1A_2 + \angle AA_2A_1 = 2\theta \ .$$

This rotation takes $P \rightarrow V$, $Q \rightarrow W$, $R \rightarrow U$, so that $HP = HV$, $HQ = HW$, $HR = HU$. This taken with either half of the argument of Solution 4 yields the result.

566. A deck of cards numbered 1 to n (one card for each number) is arranged in random order and placed on the table. If the card numbered k is on top, remove the kth card counted from the top and place it on top of the pile, not otherwise disturbing the order of the cards. Repeat the process. Prove that the card numbered 1 will eventually come to the top, and determine the maximum number of moves that is required to achieve this.

Solution For each card, a move must result in exactly one of the following possibilities: (i) the card remains in the same position; (ii) the card moves one position lower in the deck; (iii) the card is brought to the top of the deck.

We prove by induction the following statement: *Suppose that we have deck of m cards each with a different number, and that we follow the procedure of the problem; then after at most $2^{m-1} - 1$ moves the process will have to stop either because card 1 comes to the top or a card with a number exceeding m comes to the top.* It is straightfoward to see that the result holds for $m = 1$ and $m = 2$. Suppose that when $1 \leq m \leq r - 1$.

Let $m = r$. Since there are r cards with different numbers, there is a card u where either $u = 1$ or $u > r$. Suppose that u occurs in the kth position. Then the first $k - 1$ positions must contain card 1 or a card exceeding $k - 1$. By the induction hypothesis, in at most $2^{k-2} - 1$ moves one of the following must occur: (1) the process stops because a card numbered 1 or with a number exceeding m (possibly u) comes to the top, or (2) a card with a number between $k+1$ and m inclusive comes to the top. In the second case, one more move will cause u to go to the $(k+1)$th position. Therefore, after at most $1+2+\cdots+2^{r-3}=2^{r-2}-1$, either the process has stopped or u has been forced from the $(r-1)$th position to the rth position.

The top $r-1$ cards must contain at least one lying outside of the range $[2,r-1]$. Therefore, in at most $2^{r-2}-1$ further moves, either the process stops, because card number 1 or a card with a number exceeding r comes to the top, or else r comes to the top. In the latter case, one further move will make u come to the top. Thus, we can get a card with either the number 1 or a card exceeding m to the top in at most $(2^{r-2}-1)+(2^{r-2}-1)+1=2^{r-1}-1$ moves.

The desired result is a special case of this, where $m=n$ and the card outside of the range $[2,n]$ is the card numbered 1.

There is an initial arrangement of the cards where the maximum number of moves is attained, namely $(n,1,n-1,n-2,\cdots,3,2)$. To show this, we establish the following result:

Let $m\geq 2$. Then the sequence $(m,u,m-1,m-2,\cdots,2)$ becomes the sequence $(u,m,m-1,m-2,\cdots,2)$ in exactly $2^{m-1}-1$ moves, where u is any number.

This is true for $m=2$ $((2,u)\rightarrow(u,2))$ and $m=3$ $((3,u,2)\rightarrow(2,3,u)\rightarrow(3,2,u)\rightarrow(u,3,2))$. Assume that $m\geq 4$ and that the result holds for all values of m up to and including $k-1$. Then we can use the induction hypothesis to make changes as follows (where the number in square brackets indicates the number of moves):

$$\begin{aligned}(k,u,k-1,\cdots,2) &\rightarrow [1]\rightarrow(2,k,u,k-1,\cdots 3)\rightarrow[1]\rightarrow(k,2,u,k-1,\cdots 3)\\ &\rightarrow[1]\rightarrow(3,k,2,u,k-1,\cdots,4)\rightarrow[3]\rightarrow(k,3,2,u,k-1,\cdots,4)\\ &\rightarrow[1]\rightarrow(4,k,3,2,u,k-1,\cdots,5)\rightarrow[7]\rightarrow(k,4,3,2,u,k-1,\cdots,5)\\ &\vdots\\ &\rightarrow[1]\rightarrow(j,k,j-1,\cdots,2,u,k-1,\cdots,j+1)\\ &\rightarrow[2^{j-1}-1]\rightarrow(k,j,j-1,\cdots,2,u,k-1,\cdots,j+1)\\ &\vdots\\ &\rightarrow[1]\rightarrow(k-1,k,k-2,\cdots,3,2,u))\rightarrow[2^{k-2}-1]\rightarrow(k,k-1,k-2,\cdots,3,2,u)\\ &\rightarrow[1]\rightarrow(u,k,k-1,\cdots,3,2)\ .\end{aligned}$$

The total number of moves is

$$1+\sum_{j=2}^{k-2}[(2^{j-1}-1)+1]=1+2+\cdots+2^{k-2}=2^{k-1}-1\ .$$

In particular, when $u=1$ and $k=n$, we conclude that $(n,1,n-1,\cdots,2)$ goes to $(1,n,n-1,\cdots,2)$ in $2^{n-1}-1$ moves.

Comment. A. Abdi provided the following induction argument that the process must terminate. The result clearly holds for $n=1$. Suppose it holds for $1\leq n\leq m-1$, If card 1 never comes to the top, then the process never terminates and card 1 eventually finds its way to position $r\leq m$ and stays there. The cards below position r (if any) never move from that point on. Let X be the set of cards on top of 1 at that point whose numbers exceed r and Y the set of cards on top of 1 whose numbers do not exceed r, so that $\#X+\#Y=r-1$. Since card 1 cannot move down, the cards in X never come to the top, so it is immaterial what numbers appear on these cards. Relabel these cards with numbers from $\{2,3,4,\cdots,r\}$ that do not belong to the cards in Y, so that the numbers from 2 to r inclusive all appear on top of card 1. These cards get permuted among themselves by subsequent moves.

However, by the induction hypothesis applied to this deck of $r-1\leq m-1$ cards atop card 1 (with card r relabelled to a second card 1), we see that card r must eventually come to the top, when then will force card 1 to come to the top. This yields a contradiction of the assertion that the process can go on forever.

567. (a) Let A, B, C, D be four distinct points in a straight line. For any points X, Y on the line, let XY denote the *directed* distance between them. In other words, a positive direction is selected on the line and $XY = \pm|XY|$ according as the direction X to Y is positive or negative. Define

$$(AC, BD) = \frac{AB/BC}{AD/DC} = \frac{AB \times CD}{BC \times DA} \ .$$

Prove that $(AB, CD) + (AC, BD) = 1$.

(b) In the situation of (a), suppose in addition that $(AC, BD) = -1$. Prove that

$$\frac{1}{AC} = \frac{1}{2}\left(\frac{1}{AB} + \frac{1}{AD}\right) ,$$

and that

$$OC^2 = OB \times OD \ ,$$

where O is the midpoint of AC. Deduce from the latter that, if Q is the midpoint of BD and if the circles on diameters AC and BD intersect at P, $\angle OPQ = 90°$.

(c) Suppose that A, B, C, D are four distinct on one line and that P, Q, R, S are four distinct points on a second line. Suppose that AP, BQ, CR and DS all intersect in a common point V. Prove that $(AC, BD) = (PR, QS)$.

(d) Suppose that $ABQP$ is a quadrilateral in the plane with no two sides parallel. Let AQ and BP intersect in U, and let AP and BQ intersect in V. Suppose that VU and PQ produced meet AB at C and D respectively, and that VU meets PQ at W. Prove that

$$(AB, CD) = (PQ, WD) = -1 \ .$$

Solution. (a)

$$\begin{aligned}\frac{AC \times BD}{CB \times DA} + \frac{AB \times CD}{BC \times DA} &= \frac{(AB + BC) \times (BC + CD) - AB \times CD}{BC \times AD} \\ &= \frac{BC \times (AB + BC + CD)}{BC \times AD} = 1 \ .\end{aligned}$$

(b) $AB \times CD = BC \times AD \Longrightarrow$

$$AB \times (AD + CA) = (BA + AC) \times AD \Longrightarrow 2AB \times AD = AB \times AC + AC \times AD$$

$$\Longrightarrow \frac{1}{AC} = \frac{1}{2}\left(\frac{1}{AB} + \frac{1}{AD}\right) .$$

Since $AB = AO + OB = OC + OB$, $AD = AO + OD = OC + OD$ and $AC = 2OC$,

$$\frac{1}{OC} = \frac{1}{OB + OC} + \frac{1}{OD + OC} ,$$

from which the desired result follows. Since $OP = OC^2$, $OP^2 = OB \times OD$, so that OP is tangent to the circle of diameter BD. Hence $PQ \perp OP$ and the result follows.

Comment. For the last part, M. Sardarli noted that

$$\begin{aligned}OP^2 + PQ^2 &= OC^2 + BQ^2 = OB \times OD + BQ^2 = (OQ + QB)(OQ - QB) + BQ^2 \\ &= OQ^2 - QB^2 + BQ^2 = OQ^2 \ ,\end{aligned}$$

whence $\angle OPQ = 90°$.

(c) First observe that, of both lines lie on the same side of V, then corresponding lengths among A, B, C, D and P, Q, R, S have the same signs, while if V is between the lines, then the signs are opposite. Let a, b, c, d be the respective lengths of AV, BV, CV, DV; let $\alpha, \beta, \gamma, \delta$ be the respective angles AVB, CVD, BVC, DVA; let h be the distance from V to the line $ABCD$. Then

$$\begin{aligned}|(AC, BD)| &= \left|\frac{AB \times CD}{BC \times DA}\right| = \left|\frac{(\frac{1}{2}h \times AB) \times (\frac{1}{2}h \times CD)}{(\frac{1}{2}h \times BC) \times (\frac{1}{2}h \times DA)}\right| \\ &= \frac{[AVB] \times [CVD]}{[BTC] \times [DTA]} = \frac{(\frac{1}{2}ab \sin\alpha)(\frac{1}{2}cd \sin\beta)}{(\frac{1}{2}bc \sin\gamma)(\frac{1}{2}ad \sin\delta)} \\ &= \frac{\sin\alpha \sin\beta}{\sin\gamma \sin\delta} \ .\end{aligned}$$

Since $\angle AVB = \angle PVQ$, *etc.*, we find that $|(PR, QS)| = (\sin\alpha \sin\beta)/(\sin\gamma \sin\delta)$, and the result follows.

(d) By (c), with the role of V played respectively by V and U, we obtain that

$$(AB, CD) = (PQ, WD) = (BA, CD) = \frac{1}{(AB, CD)} \ ,$$

so that $(AB, CD)^2 = 1$. Since $(AB, CD) + (AC, BD) = 1$ and (AC, BD) can vanish only if $A = B$ or $C = D$, we must have that $(AB, CD) = -1$.

568. Let ABC be a triangle and the point D on BC be the foot of the altitude AD from A. Suppose that H lies on the segment AD and that BH and CH intersect AC and AB at E and F respectively.

Prove that $\angle FDH = \angle HDE$.

Solution 1. Suppose that $ED \| AB$. Then by Ceva's theorem,

$$1 = \frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = \frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CD|}{|DB|} \ ,$$

so that $AF = FB$. Hence F is the circumcentre of the right triangle ADB, so that $AF = DF$ and $\angle FDB = \angle FAD = \angle HDE$.

Otherwise, let AB and ED produced intersect at G. Then, in the notation of problem 567, $(AB, FG) = -1$. Therefore

$$\sin\angle ADF \sin\angle BDG = \sin\angle FDB \sin\angle ADG = \cos\angle ADF \cos\angle BDG \ .$$

Hence $\tan\angle ADF = \cot\angle BDG = \tan\angle ADE$ and $\angle ADF = \angle ADE$.

Solution 2. Suppose that DE and DF intersect the line through A parallel to BC at the points M and N respectively. Since triangles BDF and ANF are similar, as are triangles CDE and AME,

$$\frac{|AM|}{|AN|} = \frac{|AM|}{|CD|} \cdot \frac{|CD|}{|DB|} \cdot \frac{|DB|}{|AN|} = \frac{|AE|}{|EC|} \cdot \frac{|CD|}{|DB|} \cdot \frac{|BF|}{|AF|} = 1 \ ,$$

by Ceva's theorem. Therefore, $AM = AN$, so that triangles AMD and AND are congruent and $\angle ADF = \angle ADM$.

Solution 3. [R. Peng] Suppose that the points K and L are selected on BC so that $EK \perp CB$ and $FL \perp BC$. Let $X = CH \cap EK$ and $Y = BH \cap FL$. Then $FL \| EK$, so that the triangles FYH and XEH with respective heights LD and KD are similar. Therefore

$$\begin{aligned}LD : DK &= FY : EX = (FY : AH)(AH : EX) = (BF : BA)(CA : EC) \\ &= (FL : AD)(AD : EK) = FL : EK \ .\end{aligned}$$

Therefore triangles FLD and EKD are similar, so that $\angle LDF = \angle KDE$. The result follows.

Solution 4. [A. Murali] Suppose that $\angle FDH = \alpha$ and $\angle HDE = \beta$. By the Law of Sines,

$$\frac{|AF|}{\sin\alpha} = \frac{|AD|}{\sin\angle AFD}$$

and

$$\frac{|AE|}{\sin\beta} = \frac{|AD|}{\sin\angle AED} \ .$$

Therefore

$$\frac{\sin\alpha}{\sin\beta} = \frac{|AF|}{|AE|}\cdot\frac{\sin\angle AFD}{\sin\angle AED} = \frac{|AF|}{|AE|}\cdot\frac{\sin\angle BFD}{\sin\angle CED} \ .$$

Since

$$\frac{|BD|}{\sin\angle BFD} = \frac{|BF|}{\sin\angle BDF} = \frac{|BF|}{\cos\alpha}$$

and

$$\frac{|CD|}{\sin\angle CED} = \frac{|CE|}{\cos\beta} \ ,$$

it follows, using Ceva's theorem, that

$$\frac{\sin\alpha}{\sin\beta} = \frac{|AF|}{|AE|}\cdot\frac{|BD|}{|BF|}\cdot\frac{|CE|}{|CD|}\cdot\frac{\cos\alpha}{\cos\beta} = \frac{|AF|}{|BF|}\cdot\frac{|BD|}{|CD|}\cdot\frac{|CE|}{|AE|}\cdot\frac{\cos\alpha}{\cos\beta} \ .$$

Therefore $\tan\alpha = \tan\beta$ and the desired result follows.

Comment. It was intended that D be an interior point of BC. However, in the case that either B or C is obtuse, the result can be adapted.

569. Let A, W, B, U, C, V be six points in this order on a circle such that AU, BV and CW all intersect in the common point P at angles of $60°$. Prove that

$$|PA| + |PB| + |PC| = |PU| + |PV| + |PW| \ .$$

Solution 1. [A. Abdi] We first recall the result: *Suppose that DEF is an equilateral triangle and that G is a point on the minor arc EF of the circumcircle of DEF. Then* $|DG| = |EG| + |FG|$. (Select H on DG so that $EH = EG$. Since $\angle EGH = 60°$, triangle EGH is equilateral. It can be shown that triangle DEH and FEG are congruent (SAS), so that $|DG| = |DH| + |HG| = |FE| + |EG|$.)

Let O be the centre of the circle and let K, M, N be respective feet of the perpendiculars from O to AU, BV, CW. Wolog, let K be between P and A, M between P and V and N be between P and C. Since triangles PKO, PMO and PNO are right with hypotenuse PO, the points O, P, K, M, N are all equidistant from the midpoint of OP and so are concyclic.

P and M lie on opposite arcs KN so $\angle NMK = 180° - \angle NPK = 180° - \angle CPA = 60°$. Also $\angle NKM = \angle NPM = 60°$ and $\angle KNM = \angle KPM = 60°$, so that triangle KMN is equilateral and $|PM| = |PK| + |PN|$.

Hence

$$\begin{aligned}
(|AP| + |BP| + |CP|) &- (|UP| + |VP| + |WP|) \\
&= (|AK| + |PK| + |BM| - |PM| + |CN| + |PN|) \\
&\qquad - (|UK| - |PK| + |VM| + |PM| + |WN| - |PN|) \\
&= (|AK| - |UK|) + (|BM| - |VM|) + (|WN| - |PN|) + 2(|PK| - |PM| + |PN|) \\
&= 0 \ .
\end{aligned}$$

Solution 2. [P.J. Zhao] Construct equilateral triangles BCD and VWT external to P. Then $PBDC$ and $PWTV$ are concyclic quadrilaterals so that $\angle DPC = \angle DBC = 60^\circ = \angle UPC$ and $\angle TPV = \angle TWV = 60^\circ = \angle APV$. Therefore, the points D, U, P, A, T are collinear.

Since $PD = PB + PC$ and $PT = PV + PW$ (see Solution 1), $|PA| + |PB| + |PC| = |DA|$ and $|PU| + |PV| + |PW| = |UT|$.

Let O be the centre of the circle. Triangles BDO and CDO are congruent (SSS), so that DO bisects angle BDC and so is perpendicular to BC. Similarly, $OT \perp VW$.

Let BC and VW intersect UA at E and S respectively. Then

$$\begin{aligned}\angle ODP &= 90^\circ - \angle CED = 90^\circ - \angle BEP \\ &= 90^\circ - (180^\circ - 60^\circ - \angle CBP) \\ &= 90^\circ - (180^\circ - 60^\circ - \angle VWP) \\ &= 90^\circ - \angle VSP = \angle OTP \ .\end{aligned}$$

Therefore, triangle DOT is isosceles and so $OD = OT$. Also $OU = OA$ and $\angle OUT = \angle OAD$. Therefore triangles DAO and TUO are congruent (ASA) and so $DA = UT$. Hence

$$|PA| + |PB| + |PC| = |PU| + |PV| + |PW| \ .$$

Solution 3. [J. Zung] Construct the equilateral triangles BCD and WVT and adopt the notation of Solution 2. Observe that P is the Fermat point of both triangles ABC and UVW; this is the point that minimizes the sum of the distances from P to the vertices of the triangle and is characterized as that point from which the rays to the vertices meet at an angle of 120°. This point has the property, that when an external equilateral triangle is erected on one side of the triangle, the line joining the vertices of the given triangle and equilateral triangle not on the common side passes through it. In the present situation, this implies that D, U, P, A, T are collinear.

Consider the rotation with centre D through an angle of 60° that takes $B \to C$, $C \to E$, $P \to Q$. Then

$$\begin{aligned}\angle QCP &= \angle QCE + \angle ECD + \angle DCB + \angle BCP \\ &= \angle PBC + 60^\circ + 60^\circ + \angle BCP = 180^\circ \ .\end{aligned}$$

Thus, Q, C, P are collinear. Since $\angle PDQ = 60^\circ$, triangle PDQ is equilateral, so that $|PQ| = |PD|$. Therefore

$$\begin{aligned}|PA| + |PB| + |PC| &= |PA| + |CQ| + |PC| \\ &= |PA| + |PQ| = |PA| + |DP| = |DA| \ .\end{aligned}$$

Similarly, $|PU| + |PV| + |PT| = |UT|$.

Let O be the centre of the circle. Since B, W, V, C are concyclic, $\angle BCW = \angle BVW$. Since $\angle BDC + \angle BPC = 180^\circ$, then B, D, C, P are concyclic and $\angle BDP = \angle BCD$. Since the right bisector of BC passes through D and O, $\angle BDO = 30^\circ$. Hence

$$\angle ODP = 30^\circ - \angle BDP = 30^\circ - \angle BCP = 30^\circ - \angle BCW \ .$$

Similarly, $\angle OTP = 30^\circ - \angle BVW$. Therefore $\angle ODP = \angle OTP$, triangle ODT is isosceles and so $DF = FT$, where F is the foot of the perpendicular from O to UA, Since, also, $FU = FA$, it follows that

$$|DA| = |DF| + |FA| = |FT| + |FU| = |UT|$$

and the desired result obtains.

Solution 4. [P. Wen] Let the centre of the circle be at the origin, the coordinates of P be (p,q) and the respective lengths of PA, PB, PC, PU, PV, PW be a, b, c, u, v, w. Take UA to be parallel to the $x-$axis. Then

$$A \sim (p+a,q) \qquad B \sim (p-b/2, q+b\sqrt{3}/2) \qquad C = (p-c/2, q-c\sqrt{3}/2)$$
$$U \sim (p-u,q) \qquad V \sim (p+v/2, q-v\sqrt{3}/2) \qquad W = (p+w/2, q+w\sqrt{3}/2) \ .$$

Since $AO = UO$,

$$\begin{aligned}(p+a)^2 + q^2 = (p-u)^2 + q^2 &\Longrightarrow a^2 + 2ap = u^2 - 2up \\ &\Longrightarrow 0 = (a+u)(a-u+2p) \Longrightarrow u = a+2p \ .\end{aligned}$$

Since $BO = VO$,

$$\begin{aligned}(p-b/2)^2 + (q+b\sqrt{3}/2)^2 &= (p+v/2)^2 + (q-v\sqrt{3}/2)^2 \\ &\Longrightarrow b^2 - b(p-q\sqrt{3}) = v^2 + v(p-q\sqrt{3}) \\ &\Longrightarrow 0 = (b+v)(b-v-p+q\sqrt{3}) \Longrightarrow v = b-p+q\sqrt{3} \ .\end{aligned}$$

Since $CO = WO$,

$$c^2 - c(p+q\sqrt{3}) = w^2 + w(p+q\sqrt{3}) \Longrightarrow w = c-p-q\sqrt{3} \ .$$

Therefore $u+v+w = a+b+c$.

Solution 5. Let $|PA| = a$, $|PB| = b$, $|PC| = c$, $|PU| = u$, $|PV| = v$, $|PW| = w$. Let r be the radius and O the centre of the circle. Suppose that $|OP| = d$. Let A, W, B be on one side of OP and U, C, V be on the other side.

Let $\angle APO = \alpha \leq 60^\circ$. Then $\angle WPO = \alpha + 60^\circ$, $\angle BPO = \alpha + 120^\circ$. $\angle UPO = 180^\circ - \alpha$, $\angle CPO = 120^\circ - \alpha$, $\angle VPO = 60^\circ - \alpha$.

Using the Law of Cosines, we obtain that

$$\begin{aligned}r^2 &= a^2 + d^2 - 2ad\cos\alpha \\ &= w^2 + d^2 - 2wd\cos(\alpha+60^\circ) \\ &= b^2 + d^2 - 2bd\cos(\alpha+120^\circ) \\ &= u^2 + d^2 - 2ud\cos(180^\circ-\alpha) = u^2 + d^2 + 2bd\cos\alpha \\ &= c^2 + d^2 - 2cd\cos(120^\circ-\alpha) = c^2 + d^2 + 2cd\cos(\alpha+60^\circ) \\ &= v^2 + d^2 - 2vd\cos(60^\circ-\alpha) = v^2 + d^2 + 2vd\cos(\alpha+120^\circ) \ .\end{aligned}$$

Each of these equations is a quadratic of the form

$$x^2 - (2d\cos\theta)x + (d^2 - r^2) = 0 \ .$$

It has one positive and one non-positive root. Since $r^2 - d^2\sin^2\theta \geq d^2\cos^2\theta$, the positive root is

$$\frac{2d\cos\theta + \sqrt{4d^2\cos^2\theta - 4d^2 + 4r^2}}{2} = d\cos\theta + \sqrt{r^2 - d^2\sin^2\theta} \ .$$

Hence,

$$a = d\cos\alpha + \sqrt{r^2 - d^2\sin^2\alpha} \ ;$$

$$b = d\cos(\alpha+120^\circ) + \sqrt{r^2 - d^2\sin^2(\alpha+120^\circ)} \ ;$$

$$c = -d\cos(\alpha+60^\circ) + \sqrt{r^2 - d^2\sin^2(\alpha+60^\circ)} \ ;$$

$$u = -d\cos\alpha + \sqrt{r^2 - d^2\sin^2\alpha} \ ;$$

$$v = -d\cos(\alpha + 120°) + \sqrt{r^2 - d^2\sin^2(\alpha + 120°)} \ ;$$

$$w = d\cos(\alpha + 60°) + \sqrt{r^2 - d^2\sin^2(\alpha + 60°)} \ ;$$

therefore

$$\begin{aligned}(a+b+c) - (u+v+w) &= 2d[\cos\alpha + \cos(\alpha + 120°) - \cos(\alpha + 60°)] \\ &= 2d[\cos\alpha(1 + \cos 120° - \cos 60°] - \sin\alpha(\sin 120° - \sin 60°)] = 0 \ ,\end{aligned}$$

as desired.

Solution 6. [H. Spink] Let the radius of the circle be R and let $|PA| = a$, $|PB| = b$, $|PC| = c$. Since

$$|PA||PU| = |PB||PV| = |PC||PW| = k \ \ \text{(say)},$$

$|PU| = k/a$, $|PV| = k/b$, $|PW| = k/c$.

Consider triangle ABC; let $|BC| = p$, $|CA| = q$, $|AB| = r$. By the Cosine Law, $p = \sqrt{b^2 + bc + c^2}$, $q = \sqrt{c^2 + ca + a^2}$, $r = \sqrt{a^2 + ab + b^2}$. Also,

$$[ABC] = [PAB] + [PBC] + [PCA] = (\sqrt{3}/4)(ab + bc + ca) \ ,$$

so that

$$R = \frac{pqr}{4[ABC]} = \frac{\sqrt{(b^2 + bc + c^2)(c^2 + ca + a^2)(a^2 + ab + b^2)}}{(ab + bc + ca)\sqrt{3}} \ .$$

Now consider triangle UVW. We get a similar expression for R with a, b, c replaced by k/a, k/b, k/c. Noting that, for example, $|VW| = (k/bc)\sqrt{b^2 + bc + c^2}$, and that $[UVW] = \frac{1}{4}\sqrt{3}k^2(a + b + c)/abc = (k^2(a + b + c)\sqrt{3})/(4abc)$, we find that

$$R = \left(\frac{k^3}{a^3b^3c^3}\right)\left(\frac{abc}{k^2(a + b + c)\sqrt{3}}\right)\sqrt{(b^2 + bc + c^2)(c^2 + ca + a^2)(a^2 + ab + b^2)} \ .$$

Comparing the two expressions for R yields that

$$\frac{1}{ab + bc + ca} = \frac{k}{abc(a + b + c)} \Rightarrow k\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = a + b + c \ .$$

Therefore

$$|PU| + |PV| + |PW| = \frac{k}{a} + \frac{k}{b} + \frac{k}{c} = a + b + c = |PA| + |PB| + |PC| \ .$$

Comment. Several solvers tried the strategy of comparing the equation for two related positions, either with the situation where the second position put P at the centre of the circle, where the result is obvious, or moved P along one of the lines, say UA to a new position. In both case, the fact that the difference in the lengths of two parallel chords was split evenly to the two half chords played a role, as did the perpendiculars to the chords for one position of P from the other position of P.

570. Let a be an integer. Consider the diophantine equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a}{xyz}$$

where x, y, z are integers for which the greatest common divisor of xyz and a is 1.

(a) Determine all integers a for which there are infinitely many solutions to the equation that satisfy the condition.

(b) Determine an infinite set of integers a for which there are solutions to the equation for which the condition is satisfied and x, y, z are all positive.

Solution. [J. Zung] (a) The given equation is equivalent to $xy + yz + zx = a$, where $xyz \neq 0$. Observe that, if x, y and z are odd, then $xy + yz + zx$ is odd. Hence, when a is even, at least one of x, y and z is even and there are no solutions satisfying the coprimality condition.

Suppose that a is odd. Then all of its prime divisors exceed 2, so that a has no divisor save 1 in common with $ka - 1$ and $ka - 2$ for each integer k, so that

$$(x, y, z) = (ka - 1, -ka + 2, a + (ka - 1)(ka - 2))$$

is a solution of the desired type.

(b) Let p_i be the ith prime, for each positive integer i and let $a = p_1 p_2 \cdots p_n - 1$. Then $xy + yz + zx = a$ has at least n solutions in positive integers, namely,

$$(x, y, z) = \left(1, p_i - 1, \frac{a+1}{p_i} - 1\right)$$

for $1 \leq i \leq n$. Let q be a prime divisor of a, Then q must exceed each p_i $(1 \leq i \leq n)$ and so cannot divide $p_i - 1$. Hence q cannot divide $a - (p_i - 1)$ and so not divide $(a - 1)/(p_i) - 1 = (a + 1 - p_i)/p_i$. Thus, for each of these solutions, gcd $(xyz, a) = 1$.

Comment. Another family of solutions for (a) is given by $(x, y, z) = (-k, k+1, a+k(k+1))$ for suitably chosen k. Let m be any number of the form $\prod(p-1)^r$, where p is a prime divisor of a and r is a positive integer. (With a being odd, all such p are odd.) Since $2^{p-1} \equiv 1 \bmod p$ for any odd prime p, we can take $k = 2^m$. Then $k - 1$ is divisible by p for each prime divisor of a, so that k and $k+1$ cannot be divisible by p.

A. Abdi identified the solutions $(x, y, z) = (m, a + m^2 - m, 1 - m)$. When a is odd and a is a divisor of $m + 1$, then the greatest common divisor of a and xyz is 1.

An approach for (b) is to take $f(x, y, z) = xy + yz + zx$ and let a be any positive integer. Suppose that $a + 1 = (u + 1)(v + 1)$ for some positive integers u and v. Then $f(1, u, v) = u + v + uv = a$. If u and v are coprime, then a must be coprime to both u and v.

When a is even, no solution satisfying the requirements of the problem can be found in this way as both u and v must be even. However, if $a = 2m + 1$ is odd, we always have at least one positive solution, since $f(1, 1, m) = a$.

We might observe that the given equation is equivalent to $xy + yz + za = a$ or $a + z^2 = (x + z)(y + z)$, when $xyz \neq 0$. Thus, we can obtain infinitely many solutions to the equation by the following procedure. Given a, let z be arbitrary. Suppose $a + z^2 = uv$; then the equation is satified by $(x, y, z) = (u - z, v - z, z)$. We thus need to choose u and v carefully to adhere to the divisibility requirement. In (a), we take $u = 1$ and $v = a + z^2$; in (b), we take values of a so that u and v are large enough to make x and y both positive.

There are other ways of finding infinitely many a for which a positive solution exists. Suppose that $b \geq 4$ and $a = b^2 - 5$ and $z = 2$. Then $a + z^2 = b^2 - 1 = (b - 1)(b + 1)$. Therefore $(x, y, z) = (b - 3, b - 1, 2)$ satisfies the equation. Suppose that b is even and that p is a prime divisor of a. Then p is odd and $b^2 \equiv 5 \pmod{p}$. Since

$$(b - 3)(b - 1) = b^2 - 4b + 3 \equiv 8 - 4b = 4(2 - b)$$

modulo p and since p cannot divide two of the consecutive integers $b - 1$, $b - 2$ and $b - 3$, it follows that $(b - 3)(b - 1) \not\equiv 0$ (modulo p), so that the greatest common divisor of xyz and a is 1 as desired.

571. Let ABC be a triangle and U, V, W points, not vertices, on the respective sides BC, CA, AB, for which the segments AU, BV, CW intersect in a common point O. Prove that

$$\frac{|OU|}{|AU|} + \frac{|OV|}{|BV|} + \frac{|OW|}{|CW|} = 1 ,$$

and

$$\frac{|AO|}{|OU|} \cdot \frac{|BO|}{|OV|} \cdot \frac{|CO|}{|OW|} = \frac{|AO|}{|OU|} + \frac{|BO|}{|OV|} + \frac{|CO|}{|OW|} + 2 .$$

Solution 1. Let F and G be points of BC for which $OF \| AB$ and $OG \| AC$. Then triangles BOG and BVC are similar, so that $|OV|/|BV| = |GC|/|BC|$. Similarly, $|OW|/|CW| = |BF|/|BC|$. Since triangles OFG and ABC are similar, $|OU|/|AU| = |FG|/|BC|$. Since $|BC| = |BF| + |FG| + |GC|$, adding the three equations yields that

$$\frac{|OU|}{|AU|} + \frac{|OV|}{|BV|} + \frac{|OW|}{|CW|} = 1 .$$

Let $x = |AO|/|OU|$, $y = |BO|/|BV|$ and $z = |CO|/|CW|$. Then $|AU|/|OU| = 1+x$, $|BV|/|OV| = 1+y$ and $|CW|/|OW| = 1+z$, so that the foregoing equation can be rewritten

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1 .$$

Multiplying this equation by $(1+x)(1+y)(1+z)$ and simplifying yields that $xyz = x+y+z+2$ as desired.

Solution 2. Observe that $OU : AU = [OBC] : [ABC]$, $OV : BV = [OCA] : [ABC]$ and $OW : CW = [OAB] : [ABC]$ (where [] denotes area). Adding the three ratios gives the first result. If $|OU|/|AU| = u$, $|OV|/|BV| = v$ and $|OW|/|CW| = w$, then the left side of the second equation is

$$\begin{aligned}\left(\frac{1}{u}-1\right)\left(\frac{1}{v}-1\right)\left(\frac{1}{w}-1\right) &= \frac{1}{uvw}(1-u)(1-v)(1-w) \\ &= \frac{1-(u+v+w)+(uv+vw+wu)-uvw}{uvw} \\ &= 0 + \frac{1}{u} + \frac{1}{v} + \frac{1}{w} - 1 \\ &= \left(\left(\frac{1}{u}-1\right)\left(\frac{1}{v}-1\right)\left(\frac{1}{w}-1\right) + 2 .\end{aligned}$$

Solution 3. [D. Hidru] We can assign weights v and w to points B and C so that $w : v = CV : BV$ and a weight u to A so that $v : u = AW : BW$. The centre of gravity of v and w is at U and of v and u is at W. The centre of gravity of u, v, w is at O, the common point of AU and CW, so that the centre of gravity of u and w lies at V, the common point of BO and AC. Since O is the centre of gravity of weight u at A and $v+w$ at U, $AO : OU = (b+c) : a \Rightarrow OU : AU = a : (a+b+c)$. Similarly, $BO : OV = (a+c) : b \Rightarrow OV : BV = b : a+b+c$ and $CO : OW = (a+b) : c \Rightarrow OW : CW = c : (a+b+c)$.

The left side of the first equation is

$$\frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c} = 1 ,$$

while the left side of the second equation is

$$\begin{aligned}\left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right)\left(\frac{a+b}{c}\right) &= \frac{1}{abc}[2abc + (a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2)] \\ &= 2 + \frac{ab(a+b)}{abc} + \frac{bc(b+c)}{abc} + \frac{ca(c+a)}{abc} = 2 + \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} .\end{aligned}$$

The desired results hold.

572. Let $ABCD$ be a convex quadrilateral that is not a parallelogram. On the sides AB, BC, CD, DA, construct isosceles triangles KAB, MBC, LCD, NDA exterior to the quadrilateral $ABCD$ such that the angles K, M, L, N are right. Suppose that O is the midpoint of BD. Prove that one of the triangles MON and LOK is a $90°$ rotation of the other around O.

What happens when $ABCD$ is a parallelogram?

Solution 1. [A. Abdi] We establish a lemma: *Let X, Y be points external to triangle PQR such that triangle PXQ and PYR are isosceles with right angles at X and Y. Let Z be the midpoint of QR. Then the segments YZ and XZ are equal and perpendicular.*

To prove this, let S and T be the respective midpoints of PQ and PR. Note that $SZ \| PR$ and $TZ \| PQ$. Then

$$\angle XSZ = 90° + \angle QSZ = 90° + \angle QPR = 90° + \angle ZTR = \angle ZTY \ ,$$

$SZ = TR = TY$ and $XS = QS = ZT$. Therefore triangles XSZ and ZTY are congruent (SAS), so that $XZ = ZY$.

Triangle XSZ is transformed to triangle ZTY by the composite of a $90°$ rotation about S that takes $S \rightarrow S$, $X \rightarrow Q$, $Z \rightarrow Z'$, and a translation in the direction of PQ that takes $S \rightarrow T$, $Q \rightarrow Z$, $Z' \rightarrow Y$. Hence $YZ \perp XZ$.

Apply this to the situation at hand. With regard to triangle ABD, we find that OK is equal and perpendicular to ON. With respect to triangle BCD, we find that OL is equal and perpendicular to OM. Therefore a $90°$ rotation about O takes $K \rightarrow N$ and $L \rightarrow M$, and the result follows.

Solution 2. A $90°$ rotation about M that takes B to C takes O to a point O' and C to a point C'. A $90°$ degree rotation about L that takes D to C takes O to a point O'' and C to a point C''. We have that

$$|CO'| = |BO| = |DO| = |CO''|$$

and $CO' \perp BO$, $CO'' \perp DO$. Therefore $O' = O''$. Since $MO = MO'$, $MO \perp MO'$, $LO = LO''$ and $LO \perp LO''$, OMO' and $OLO'' = OLO'$ are two isosceles right triangles with a common hypotenuse OO'. Therefore $LO = MO$ and $\angle LOM = 90°$.

Similarly, $OK = ON$ and $OK \perp ON$, from which the result follows.

If $ABCD$ is a parallelogram, then O is the centre of a half turn that interchanges B and D, A and C, M and N, as well as L and K, so that MON and LOK are both straight lines.

573. A point O inside the hexagon $ABCDEF$ satisfies the conditions $\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle EOF = 60°$, $OA > OC > OE$ and $OB > OD > OF$. Prove that $|AB| + |CD| + |EF| < |BC| + |DE| + |FA|$.

Solution. Let XY and ZY be two rays that meet at Y at an angle of $60°$. On the ray YX, locate points P, Q, R so that $|YP| = |OA|$, $|YQ| = |OC|$ and $|YR| = |OE|$. Similarly, on the ray YZ, locate points U, V, W so that $|YU| = |OB|$, $|YV| = |OD|$ and $|YW| = |OF|$. Then we have (by SAS congruency) that $|AB| = |PU|$, $|BC| = |UQ|$, $|CD| = |QV|$, $|DE| = |VR|$, $|EF| = |RW|$ and $|FA| = |WP|$.

Suppose that PW intersects QU and RV in the respective points S and T. Then

$$\begin{aligned} |AB| + |CD| + |EF| &= |PU| + |QV| + |RW| \\ &< (|PS| + |SU|) + (|QS + |ST| + |TV|) + (|RT| + |TW|) \\ &= |PS| + |QU| + |ST| + |RV| + |TW| = |QU| + |RV| + |PW| \\ &= |BC| + |DE| + |FA| \ . \end{aligned}$$

574. A fair coin is tossed at most n times. The tossing stops before n tosses if there is a run of an odd number of heads followed by a tail. Determine the expected number of tosses.

Solution 1. Let E_n be the expected number of tosses, when the maximum number of tosses is n. When $n = 1, 2$, there must be n tosses, so that $E_1 = 1$ and $E_2 = 2$. (When $n = 3$, the only occurrence with fewer than three tosses is HT, which occurs with probability $\frac{1}{4}$, so that $E_3 = 11/4$. Similarly, $E_4 = 27/8$.)

Let $n \geq 3$, and consider the case of a maximum of n tosses. There are three mutually exclusive events:
(1) The tossing begins with a tail, with probability 1/2. The expected number of tosses is then $1 + E_{n-1}$;
(2) The tossing begins with two heads, with probability 1/4. The expected number of tosses is then $2+E_{n-2}$;
(3) The tossing begins with a head followed by a tail, with probability 1/4, at which point the tossing stops.

Thus,

$$E_n = \frac{1}{2}(1 + E_{n-1}) + \frac{1}{4}(2 + E_{n-2}) + \frac{1}{4}2 = \frac{3}{2} + \frac{1}{2}E_{n-1} + \frac{1}{4}E_{n-2} \ .$$

One solution of the recursion is $E_n \equiv 6$ for all n. This is not the solution we are seeking. Set $X_n = 2^n(E_n - 6)$. Then $X_n = X_{n-1} + X_{n-2}$, for $n \geq 3$. The initial conditions are $X_1 = -10 = -2f_5$ and $X_2 = -16 = -2f_6$, where $\{f_n : n \geq 1\} = \{1, 1, 2, 3, 5, 8, \cdots\}$ is the Fibonacci sequence with $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$. Therefore

$$E_n = 6 + \frac{X_n}{2^n} = 6 - \frac{f_{n+4}}{2^{n-2}} \ .$$

Comment. More basically, the argument setting up the recursion for E_n is the following. Let p_k be the probability that k tosses are made in which the first odd number sequence of heads is concluded by the last toss, which is a tail and q_k the probability that k tosses are made without having to stop earlier at the termination by a tail of an odd number of heads. Then, for each $n \geq 1$, $E_n = \sum_{k=1}^{n-1} kp_k + nq_n$, where $\sum_{k=1}^{n-1} p_k + q_n = 1$. Then

$$\begin{aligned}
E_n &= \frac{1}{2}\left[\sum_{k=1}^{n-2}(1+k)p_k + (1+n-1)q_{n-1}\right] + \frac{1}{4}\left[\sum_{k=1}^{n-3}(2+k)p_k + (2+n-2)q_{n-2}\right] + \frac{1}{4}[2] \\
&= \frac{1}{2}\left[\left(\sum_{k=1}^{n-2} p_k\right) + q_{n-1} + \sum_{k=1}^{n-2} kp_k + (n-1)q_{n-1}\right] \\
&\qquad + \frac{1}{4}\left[2\left(\sum_{k=1}^{n-3} p_k + q_{n-2}\right) + \sum_{k=1}^{n-3} kp_k + (n-2)q_{n-2}\right] + \frac{1}{2} \\
&= \frac{1}{2}[1 + E_{n-1}] + \frac{1}{4}[2 + E_{n-2}] + \frac{1}{2} \ .
\end{aligned}$$

Solution 2. [J. Zung] As in Solution 1, we can derive $E_n = \frac{1}{2}E_{n-1} + \frac{1}{4}E_{n-2} + \frac{3}{2}$ for $n \geq 2$, where $E_0 = 0$. Let

$$f(x) = \sum_{n=1}^{\infty} E_n x^n$$

be the generating function for E_n. Then

$$\begin{aligned}
f(x) &= x + \sum_{n=2}^{\infty} E_n x^n \\
&= x + \frac{1}{2}\sum_{n=2}^{\infty} E_{n-1}x^n + \frac{1}{4}\sum_{n=2}^{\infty} E_{n-2}x^n + \frac{3}{2}\sum_{n=2}^{\infty} x^n \\
&= x + \frac{x}{2}f(x) + \frac{x^2}{4}f(x) + \frac{3x^2}{2(1-x)} \\
&= \frac{x}{2}f(x) + \frac{x^2}{4}f(x) + \frac{x(x+2)}{2(1-x)} \ .
\end{aligned}$$

Therefore

$$\begin{aligned} f(x) &= \frac{x(x+2)}{2(1-x)(1-\frac{1}{2}x-\frac{1}{4}x^2)} \\ &= \frac{x(x+2)}{2(1-x)(1-\alpha x)(1-\beta x)} \\ &= \frac{C}{1-x} + \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \\ &= \sum_{n=0}^{\infty} (C + A\alpha^n + B\beta^n)x^n \ , \end{aligned}$$

where $\alpha + \beta = \frac{1}{2}$, $\alpha\beta = -\frac{1}{4}$, so that $\alpha = \frac{1}{4}(1+\sqrt{5})$, $\beta = \frac{1}{4}(1-\sqrt{5})$, and where A, B, C satisfy

$$\frac{1}{2}x(x+2) = A(1-x)(1-\beta x) + B(1-x)(1-\alpha x) + C(1-\alpha x)(1-\beta x) \ .$$

Comparing leading and constant coefficients leads to

$$\frac{1}{2} = A\beta + B\alpha + C\alpha\beta = A\beta + B\alpha - \frac{1}{4}C$$

and

$$0 = A + B + C \ .$$

Substituting $x = 1$ leads to

$$\frac{3}{2} = C(1-\alpha)(1-\beta) = C\left(1 - \frac{1}{2} - \frac{1}{4}\right) = \frac{C}{4} \ .$$

Therefore $C = 6$ and $2 = A\beta + B\alpha = A\beta - (6+A)\alpha$. Thus we find that

$$A = -\frac{1}{5}(15 + 7\sqrt{5})$$

and

$$B = -\frac{1}{5}(15 - 7\sqrt{5}) \ .$$

Hence,

$$f(x) = \sum_{n=0}^{\infty} \left[6 - \frac{1}{5}(15+7\sqrt{5})\left(\frac{1+\sqrt{5}}{4}\right)^n - \frac{1}{5}(15-7\sqrt{5})\left(\frac{1-\sqrt{5}}{4}\right)^n\right] x^n \ .$$

Therefore

$$E_n = 6 - \frac{1}{5}(15+7\sqrt{5})\left(\frac{1+\sqrt{5}}{4}\right)^n - \frac{1}{5}(15-7\sqrt{5})\left(\frac{1-\sqrt{5}}{4}\right)^n \ .$$

575. A partition of the positive integer n is a set of positive integers (repetitions allowed) whose sum is n. For example, the partitions of 4 are (4), (3,1), (2,2), (2,1,1), (1,1,1,1); of 5 are (5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1); and of 6 are (6), (5,1), (4,2), (3,3), (4,1,1), (3,2,1), (2,2,2), (3,1,1,1), (2,2,1,1), (2,1,1,1), (1,1,1,1,1,1).

Let $f(n)$ be the number of 2's that occur in all partitions of n and $g(n)$ the number of times a number occurs exactly once in a partition. For example, $f(4) = 3$, $f(5) = 4$, $f(6) = 8$, $g(4) = 4$, $g(5) = 8$ and $g(6) = 11$. Prove that, for $n \geq 2$, $f(n) = g(n-1)$.

Solution 1. Define the function $p(n)$ to be 0 when $n < 0$, 1 when $n = 0$ and the number of partitions of n when $n > 0$. For example, $p(4) = 5$, $p(5) = 7$ and $p(6) = 11$. For each positive integer i, $p(n-2i)$ is the number of partitions of n that have at least i occurrences of 2. Suppose that a partition has exactly k occurrences of 2; then this partition is counted once among $p(n-2i)$ for $1 \le i \le k$. Therefore

$$f(n) = p(n-2) + p(n-4) + p(n-6) + \cdots = \sum_{i=1}^{\infty} p(n-2i) \ .$$

The total number of partitions of n in which a number j occurs as a singleton is $p(n-j) - p(n-2j)$. Hence

$$\begin{aligned} g(n) &= [p(n-1) - p(n-2)] + [p(n-2) - p(n-4)] + [p(n-3) - p(n-6)] + \cdots \\ &= p(n-1) + p(n-3) + p(n-5) + \cdots = \sum_{j=1}^{\infty} p(n+1-2j) \ . \end{aligned}$$

It follows that $f(n) = g(n-1)$.

Solution 2. [A. Abdi] Let $u(n,i)$ be the number of partitions of n for which the number 2 appears exactly i times, and let $v(n,j)$ be the number of partitions of n that have exactly one occurrence of j. Then

$$f(n) = \sum_{i=1}^{n} iu(n,i)$$

and

$$g(n) = \sum_{j=1}^{n} v(n,j) \ .$$

Since the number of partitions with exactly i occurrences of 2 is equal to the number with at least i minus the number with at least $i+1$ occurrences, we have that $u(n,i) = p(n-2i) - p(n-2i-2)$, whence

$$\begin{aligned} f(n) &= \sum_{i=1}^{n} i(p(n-2i) - p(n-2(i+1)) \\ &= p(n-2) - p(n-4) + 2p(n-4) - 2p(n-6) + 3p(n-6) - p(n-8) + \cdots \\ &= \sum_{i=1}^{n} p(n-2i) \ . \end{aligned}$$

Since the number of partitions with exactly one occurrence of j is equal to the number with at least one occurrence minus the number with at least two occurrences, we have that $v(n,j) = p(n-j) - p(n-2j)$, whence

$$\begin{aligned} g(n) &= [p(n-1) - p(n-2)] + [p(n-2) - p(n-4)] + [p(n-3) - p(n-6) + \cdots \\ &= \sum_{j=1}^{n} p(n+1-2j) \ . \end{aligned}$$

The desired result follows.

Solution 3. The generating function for $p(n)$ is

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n} \ .$$

The number of partitions of n with exactly k occurrences of 2 is the coefficients of x^n in $x^{2k}(1-x^2)P(x)$. Therefore the total number of 2s in all partitions of n is the coefficient of x^n in

$$\sum_{k=1}^{\infty} kx^{2k}(1-x^2)P(x) = \left(\frac{x^2}{(1-x^2)^2} \right)(1-x^2)P(x) = \frac{x^2}{1-x^2} P(x) \ .$$

The number of partitions of $n-1$ containing k as a singleton is the coefficient of x^n in $x^k(1-x^k)P(x)$. Therefore the number of singletons over all partitions of $n-1$ is the coefficient of x^n in

$$x\sum_{k=1}^{\infty} x^k(1-x^k)P(x) = x\left(\frac{x}{1-x} - \frac{x^2}{1-x^2}\right)P(x) = \frac{x^2}{1-x^2}P(x) \ .$$

The result follows.

576. (a) Let $a \geq b > c$ be the radii of three circles each of which is tangent to a common line and is tangent externally to the other two circles. Determine c in terms of a and b.

(b) Let a, b, c, d be the radii of four circles each of which is tangent to the other three. Determine a relationship among a, b, c, d

Solution. (a) Let the centres of the three circles with radii a, b, c be respectively A, B, C and let the points of tangency with the line be respectively U, V, W. Then

$$|UV| = \sqrt{(a+b)^2 - (a-b)^2} = 2\sqrt{ab} \ ,$$

$$|UW| = 2\sqrt{ac} \qquad \text{and} \qquad |VW| = 2\sqrt{bc} \ .$$

Hence $\sqrt{ab} = \sqrt{ab} + \sqrt{bc}$, so that

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \ ,$$

or

$$c = \frac{ab}{a+b+2\sqrt{ab}} \ .$$

(b) Let the centres of the four circles with radii a, b, c, d be respectively A, B, C, D, and let $\mathbf{u} = \overrightarrow{DA}$, $\mathbf{v} = \overrightarrow{DB}$ and $\mathbf{w} = \overrightarrow{DC}$ denote the vectors from the centre D to the other three centres. Up to labelling of the circles, there are two possible configurations to consider. The first is that the circle of centre D contains the other three circles, in which case $\mathbf{u}\cdot\mathbf{u} = (d-a)^2 = (a-d)^2$, $\mathbf{v}\cdot\mathbf{v} = (d-b)^2 = (b-d)^2$, $\mathbf{w}\cdot\mathbf{w} = (d-c)^2 = (c-d)^2$. The second is that in which the circle of centre D is exterior to and surrounded by the three other circles. In this case, $\mathbf{u}\cdot\mathbf{u} = (d+a)^2 = (a+d)^2$, $\mathbf{v}\cdot\mathbf{v} = (b+d)^2$ and $\mathbf{w}\cdot\mathbf{w} = (c+d)^2$. We can comprise these two cases into one by adopting the convention that the in the first case, with the largest circle containing the others, the radius of the largest circle is assigned a negative value. Let $p = 1/a$, $q = 1/b$, $r = 1/c$ and $s = 1/d$ be the curvatures of the circle, where s can be positive or negative depending on the configuration. For both configurations, we have that

$$(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = |AB|^2 = (a+b)^2 \ ,$$

$$(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = |BC|^2 = (b+c)^2 \ ,$$

$$(\mathbf{w}-\mathbf{u})\cdot(\mathbf{w}-\mathbf{u}) = |AC|^2 = (c+a)^2 \ .$$

The three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ reside in two-dimensional space; therefore, they are linearly dependent. This means that there are constants, α, β, γ, not all zero, for which

$$\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = 0 \ .$$

Taking the inner product of each equation with the three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, we find that the following system of three equations in α, β, γ has a nontrivial solution:

$$(\mathbf{u}\cdot\mathbf{u})\alpha + (\mathbf{u}\cdot\mathbf{v})\beta + (\mathbf{u}\cdot\mathbf{w})\gamma = 0 \ ,$$

$$(\mathbf{u}\cdot\mathbf{v})\alpha + (\mathbf{v}\cdot\mathbf{v})\beta + (\mathbf{v}\cdot\mathbf{w})\gamma = 0 \ ,$$
$$(\mathbf{u}\cdot\mathbf{w})\alpha + (\mathbf{v}\cdot\mathbf{w})\beta + (\mathbf{w}\cdot\mathbf{w})\gamma = 0 \ .$$

From the first two equations, we find that

$$\alpha : \beta : \gamma = [(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w}) - (\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})] : [(\mathbf{u}\cdot\mathbf{w})(\mathbf{u}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{w}] : [(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{v})^2] \ .$$

Plugging this into the third equation yields that

$$2(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})(\mathbf{u}\cdot\mathbf{w}) + (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w}) = (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{w})^2 + (\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})^2 + (\mathbf{w}\cdot\mathbf{w})(\mathbf{u}\cdot\mathbf{v})^2 \ . \qquad (1)$$

Recall that $\mathbf{u}\cdot\mathbf{u} = (a+d)^2$, $\mathbf{v}\cdot\mathbf{v} = (b+d)^2$ and $\mathbf{w}\cdot\mathbf{w} = (c+d)^2$. Since

$$(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = |AB|^2 = (a+b)^2 \ ,$$

$$\mathbf{u}\cdot\mathbf{v} = \frac{1}{2}[(a+d)^2 + (b+d)^2 - (a+b)^2] = d^2 + ad + bd - ab = (a+d)(b+d) - 2ab \ .$$

Similarly,

$$\mathbf{u}\cdot\mathbf{w} = (a+d)(c+d) - 2ac \ ,$$

and

$$\mathbf{v}\cdot\mathbf{w} = (b+d)(c+d) - 2bc \ .$$

Expressing these in terms of p, q, r, s, we find that

$$\mathbf{u}\cdot\mathbf{u} = \frac{(p+s)^2}{p^2s^2} \ , \qquad \mathbf{v}\cdot\mathbf{v} = \frac{(q+s)^2}{q^2s^2} \ , \qquad \mathbf{w}\cdot\mathbf{w} = \frac{(r+s)^2}{r^2s^2} \ ,$$

$$\mathbf{u}\cdot\mathbf{v} = \frac{(p+s)(q+s) - 2s^2}{pqs^2} \ ,$$

$$\mathbf{v}\cdot\mathbf{w} = \frac{(q+s)(r+s) - 2s^2}{qrs^2} \ ,$$

$$\mathbf{w}\cdot\mathbf{u} = \frac{(r+s)(p+s) - 2s^2}{prs^2} \ .$$

The left side of (1), multiplied by $p^2q^2r^2s^6$, is equal to

$$\begin{aligned} 2(p+s)^2(q+s)^2)r+s)^2 - 4s^2(p+s)(q+s)(r+s)[(p+s)+(q+s)+(r+s)] \\ + 8s^4[(p+s)(q+s)+(q+s)(r+s)+(r+s)(p+s)] \\ - 16s^6 + (p+s)^2(q+s)^2(r+s)^2 \ ; \end{aligned}$$

the right side of (1), multiplied by $p^2q^2r^2s^6$ is equal to

$$\begin{aligned} (p+s)^2[(q+s)^2(r+s)^2 - 4s^2(q+s)(r+s) + 4s^4] \\ + (q+s)^2[(r+s)^2(p+s)^2 - 4s^2(r+s)(p+s) + 4s^4] \\ + (r+s)^2[(p+s)^2(q+s)^2 - 4s^2(p+s)(q+s) + 4s^4] \ . \end{aligned}$$

Equating the two sides, removing common terms and dividing by $4s^4$ yields the equation

$$2[(p+s)(q+s)+(q+s)(r+s)+(r+s)(p+s)] - 4s^2 = (p+s)^2 + (q+s)^2 + (r+s)^2$$

$$\Longrightarrow 2(pq+qr+rp) + 4s(p+q+r) + 2s^2 = p^2 + q^2 + r^2 + 2s(p+q+r) + 3s^2$$

$$\implies 2(pq+qr+rp+ps+qs+rs) = p^2+q^2+r^2+s^2$$
$$\implies (p+q+r+s)^2 = 2(p^2+q^2+r^2+s^2)$$
$$\implies \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2 = 2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right) .$$

Solution 2. [J. Zung] Suppose that the centres and respective radii of the four circles are A, B, C, D and a, b, c, d. Consider the case that the circle with centre D is surrounded by the other three circles. Let $p = 1/a$, $q = 1/b$, $r = 1/c$ and $s = 1/d$. Let the incircles of the triangles ABC, ABD, ACD and BCD be centred at D', C', B' and A' and their radii be d', c', b' and a' respectively. Let $p' = 1/a'$, $q' = 1/b'$, $r' = 1/c'$ and $s' = 1/d'$.

Observe that the circle of centre D' intersects the circles of centres A and B at their point of tangency, the circles of centres B and C at their point of tangency and the circles of centres A and C at their point of tangency. (Why?) The sides of triangle ABC are $a+b$, $a+c$, $b+c$, the semi-perimeter is $a+b+c$ and the inradius is d'. Equating two determinations of the area of triangle $[ABC]$ yields the equation

$$d'(a+b+c) = \sqrt{abc(a+b+c)} ,$$

whence

$$d' = \sqrt{\frac{abc}{a+b+c}} \implies \frac{1}{d'^2} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$$

and

$$s'^2 = pq + qr + rp .$$

Similarly, we obtain that

$$p'^2 = qr + rs + sq ; \qquad q'^2 = pr + rs + sp ; \qquad r'^2 = pq + qs + sp .$$

Adding these four equations together yields that

$$p'^2 + q'^2 + r'^2 + s'^2 = (p+q+r+s)^2 - (p^2+q^2+r^2+s^2) . \tag{1}$$

Observe that $A'B'$ is tangent to the circle of centre D, *etc.*, so that this circle is the incentre of triangle $A'B'C'$. Equating two expressions for the area of triangle $A'B'C'$ yields the equation

$$s^2 = p'q' + q'r' + r's' .$$

The circle with centre A and radius a is the escribed circle opposite D' and tangent to $B'C'$ of triangle $B'C'D'$, whose sides are of lengths $d'-c'$, $d'-b'$ and $b'+c'$ and whose semiperimeter is d'. Thus, $[B'C'D'] = a(d'-b'-c') = \sqrt{(d'-b'-b')d'b'c'}$ so that

$$\frac{1}{a^2} = \frac{1}{b'c'} - \frac{1}{b'd'} - \frac{1}{c'd'} \implies p^2 = q'r' - q's' - r's' .$$

Similarly,

$$q^2 = p'r' - p's' - q's' \quad \text{and} \quad r^2 = p'q' - p's' - q's' .$$

Therefore

$$p^2 + q^2 + r^2 + s^2 = (p'+q'+r'-s')^2 - (p'^2+q'^2+r'^2+s'^2) . \tag{2}$$

Comparing equations (1) and (2), we conclude that

$$p' + q' + r' - s' = p + q + r + s . \tag{3}$$

(Why is the left side positive?) Call the common value σ. Then, using either (1) or (2),

$$\begin{aligned}\sigma^2 &= (p+q+r+s)^2 = (p^2+q^2+r^2+s^2) + (p'^2+q'^2+r'^2+s'^2) \\ &= (p+q+r)^2 - 2(pq+qr+rp) + s^2 + (p'+q'+r')^2 - 2(p'q'+q'r'+r's') + s'^2 \\ &= (\sigma - s)^2 - 2s'^2 + s^2 + (\sigma + s')^2 - 2s^2 + s'^2 \\ &= 2\sigma^2 - 2\sigma(s-s') ,\end{aligned}$$

whence $\sigma = 2(s-s')$. Similarly, it can be shown that $\sigma = 2(a+a') = 2(b+b') = 2(c+c')$. Since, from (3),

$$2\sigma[(p-p') + (q-q') + (r-r') + (s+s')] = 0 ,$$

we can make the various substitutions for σ to obtain that

$$p^2+q^2+r^2+s^2 = p'^2+q'^2+r'^2+s'^2 = (p+q+r+s)^2 - (p^2+q^2+r^2+s^2 .$$

from which

$$2(p^2+q^2+r^2+s^2) = (p+q+r+s)^2 .$$

A similar analysis can be made to the same result when the circles with centres A, B and C are contained within the circle of centre D. When one of the circles, say of centre D, is replaced by a straight line, then $s = 0$ and the condition becomes $2(p^2+q^2+r^2) = (p+q+r)^2$.

Comment. In the special case that three of the circles have radius 1, say $p = q = r = 1$, then it can be checked directly that the radius of the inner fourth circle is $\frac{1}{3}(2\sqrt{3}-3)$ and of the outer fourth circle is $-\frac{1}{3}(2\sqrt{3}+3)$. The quadratic equation to be solved for s is $s^2 - 6s - 3 = 0$, whose roots are $s = 3 \pm 2\sqrt{3} = 3(-3 \pm 2\sqrt{3})^{-1}$.

Part (a) can be considered a special case of (b), where the fourth circle has infinite radius and curvature 0. In this case, $s = 0$, and the condition in (a) reads

$$\begin{aligned}\sqrt{r} = \sqrt{p} + \sqrt{r} &\Longrightarrow r - p - q = 2\sqrt{pq} \\ &\Longrightarrow p^2+q^2+r^2 - 2pr - 2qr + 2pq = 4pq \\ &= 2(p^2+q^2+r^2) = (p+q+r)^2 ,\end{aligned}$$

The result (b) is quite ancient. Known as the Descartes' Circle Theorem, it was disclosed by him in a letter to Princess Elisabeth of Bohemia in November, 1643, although his proof is unclear. It was rediscovered in 1842 by Philip Beecroft and again in 1936 by Frederick Soddy, a physicist and Nobel prizewinner, who published notes in *Nature* magazine (137 (1936), 1021; 139 (1939), 62). This result is discussed by H.S.M. Coxeter on pages 11-16 of his *Introduction to geometry, second edition* (Wiley, 1961, 1969). Other treatments and extensions appear in the *American Mathematical Monthly* (Vandeghen, 71 (1964), 176-179; Alexander, 74 (1967), 128-140; Pedoe, 74 (1967), 627-640; Coxeter, 75 (1968), 5-15; Brown, 76 (1969), 661-663; Lagarias, 109 (2002), 338-361). The result can be extended to spheres of higher dimension.

A. Murali defined $\alpha = \angle BDC$, $\beta = \angle ADC$ and $\gamma = \angle ADB$. Then

$$\cos\alpha = \frac{(b+d)^2 + (c+d)^2 - (b+c)^2}{2(b+d)(c+d)}$$

with analogous expressions for $\cos\beta$ and $\cos\gamma$. Since $\alpha + \beta + \gamma = 360°$, one can deduce from $\sin\beta\sin\gamma = \cos\beta\cos\gamma - \cos\alpha$ (by squaring) that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 + 2\cos\alpha\cos\beta\cos\gamma .$$

After what appears to be a horrendous computation, this leads to the condition

$$[(ab+bc+ca)^2 - 4abc(a+b+c)]d^2 + 2(abc)(ab+bc+ca)d - a^2b^2c^2 = 0 ,$$

which yields two values of d corresponding to the two possible circles for given circles with centres A, B, C.

577. $ABCDEF$ is a regular hexagon of area 1. Determine the area of the region inside the hexagon thst belongs to none of the triangles ABC, BCD, CDE, DEF, EFA and FAB.

Solution 1. Let O be the centre of the hexagon. The hexagon is the union of three nonoverlapping congruent rhombi, $ABCO$, $CDEO$, $EFAO$, each of area $\frac{1}{3}$. Each rhombus is the union of two congruent triangles, each of area $\frac{1}{6}$. In particular, $[ABC] = \frac{1}{2}$.

Let BD and AC intersect at P, and BF and AC at Q. By reflection about BE, we see that $BP = BQ$, triangle BPQ is equilateral and $\angle BPQ = 60°$. Since triangle BPC is isosceles (use symmetry) and $\angle BPC = 120°$, $CP = PB = PQ = BQ = QA$. Therefore $[BPC] = [BPQ] = [BQA] = \frac{1}{3}[ABC] = \frac{1}{18}$.

The union of triangles ABC, BCD, CDE, DEF, EFA, FAB is comprised of twelve nonoverlapping triangles congruent to either of the triangles BPC or BPQ, as so has area $\frac{2}{3}$. Therefore the area of the prescribed region inside the hexagon is $\frac{1}{3}$.

Solution 2. Let O be the centre of the hexagon. Since triangle ACE is the union of triangles OAC, OCE, OEA, and since $[OAC] = [BAC]$, $[OCE] = [DCE]$, $[OEA] = [FEA]$, it follows that $[ACE] = \frac{1}{2}[ABCDEF] = \frac{1}{2}$. As in Solution 1, we determine that $[BPQ] = [DUT] = [FRS] = \frac{1}{18}$, where $U = BD \cap CE$, $T = CE \cap DF$, $S = DF \cap EA$, $R = AE \cap BF$. Hence the area of the inner region is $\frac{1}{2} - (3 \times \frac{1}{18}) = \frac{1}{3}$.

Solution 3. [T. Bappi] The inner figure is a regular hexagon. To see this, consider a rotation of 60° about the centre of the given hexagon that takes

$$A \to F \to E \to D \to C \to B \to A .$$

Then

$$AE \to FD \to EC \to DB \to CA \to BF \to AE ,$$

so that the rotation takes the intersections of each adjacent pair of these chords to the intersection of the next adjacent pair, and thus each vertex of the inner figure to the next.

Let AC intersect BD and BF at P and Q respectively. Then PQ is a side of the inner hexagon and triangle BPQ is equilateral. By the Law of Sines,

$$PQ : BC = BP : BC = \sin \angle 30° : \sin \angle 120° = 1/2 : \sqrt{3}/2 ,$$

so that the inner hexagon is similar to the given hexagon with factor $1/\sqrt{3}$. Hence its area must be $1/3$.

Solution 4. [D. Hidru] Let BD and BF intersect AC at P and Q respectively. Using a symmetry argument, we can establish that $CP = BP = BQ = QA$ and that triangle BDF is equilateral. [Do this.] Since $\angle PBQ = 60°$, triangle BPQ is equilateral and $CP = PQ = QA$. Therefore, $[BCP] = [BPQ] = [BQA]$. The diagonals of hexagon $ABCDEF$ trisect the sides of triangle BDF, so that this triangle can be partitioned into nine equilateral triangles congruent to triangle BPQ, six of which make up the inner region whose area is to be found.

The hexagon $ABCDEF$ is the nonoverlapping union of 18 triangles of equal area, six of which are congruent to triangle BCP and twelve of which are congruent to triangle BPQ. It follows that the inner region has area $1/3$.

Solution 5. [J. Zung] By symmetry, the inner region is a regular hexagon. Let R and r be the respective inradii of the given and inner hexagons, and let O be their common centre. Triangle OAB and OBC are equilateral and $OB \perp AC$; hence OB and AC right bisect each other. Therefore, OAB is an equilateral triangle with side length equal to $2r$ and altitude R, so that $R/2r = \sqrt{3}/2$ and $r/R = 1/\sqrt{3}$. Thus, the inner and outer regular hexagons are similar with factor $1/\sqrt{3}$ and the area of the inner figure is $1/3$.

Comment. In the original statement of the problem, triangle DEF was omitted by mistake from the statement. In this case, the region whose area was to be found is the union of $PQRSTU$ and one of the twelve small triangles; the answer is $1/3 + 1/18 = 7/18$.

578. $ABEF$ is a parallelogram; C is a point on the diagonal AE and D a point on the diagonal BF for which $CD \| AB$. The segments CF and EB intersect at P; the segments ED and AF intersect at Q. Prove that $PQ \| AB$.

Solution. Consider the shear that fixes A and B and shifts E in a parallel direction to E' so that $E'B \perp AB$. This shear preserves parallelism and takes $F \rightarrow F'$, $C \rightarrow C'$, $D \rightarrow D'$, $P \rightarrow P'$, $Q \rightarrow Q'$, so that $ABE'F'$ is a rectangle. A reflection about the right bisector of AB takes $E' \leftrightarrow F'$, $C' \leftrightarrow D'$, and so $P" \leftrightarrow Q'$. Hence $PQ \| P'Q' \| AB$.

579. Solve, for real x, y, z the equation

$$\frac{y^2 + z^2 - x^2}{2yz} + \frac{z^2 + x^2 - y^2}{2zx} + \frac{x^2 + y^2 - z^2}{2xy} = 1 \ .$$

Solution 1. Note that none of x, y, z can vanish. We have that

$$\begin{aligned} 0 &= \frac{y^2 + z^2 - x^2}{2yz} = \frac{z^2 + x^2 - y^2}{2zx} + \frac{x^2 + y^2 - z^2}{2xy} - 1 \\ &= \frac{xy^2 + xz^2 - x^3 + yz^2 + x^2y - y^3 + x^2z + y^2z - z^3 - 2xyz}{2xyz} \\ &= \frac{(x + y - z)(xy + z^2) + (x^2 + y^2 - xy)z - (x^2 + y^2 - xy)(x + y)}{2xyz} \\ &= \frac{(x + y - z)(z^2 - (x - y)^2)}{2xyz} = \frac{(x + y - z)(z + x - y)(z + y - x)}{2xyz} \ , \end{aligned}$$

whereupon (x, y, z) is a solution if and only if one of the conditions $x + y = z$, $y + z = x$ and $z + x = y$ is satisfied.

Solution 2. We must have $xyz \neq 0$ for the equation to be defined. Suppose that a, b, c are such that $y^2 + z^2 - x^2 = 2ayz$, $z^2 + x^2 - y^2 = 2bzx$, $x^2 + y^2 - z^2 = 2cxy$. Then $a + b + c = 1$. Adding pairs of the three equations yields that

$$\begin{aligned} 2z^2 &= 2z(ay + bx) \ , \\ 2y^2 &= 2y(az + cx) \ , \\ 2x^2 &= 2x(bz + cy) \ . \end{aligned}$$

Hence

$$\begin{aligned} bx + ay - z &= 0 \ , \\ cx - y + az &= 0 \ , \\ -x + cy + bz &= 0 \ . \end{aligned}$$

From the first two equations, we find that

$$x : y : z = (a^2 - 1) : (-c - ab) : (-b - ac) \ .$$

Plugging this into the third equation yields that

$$1 - a^2 - c^2 - abc - b^2 - abc = 0 \Longrightarrow a^2 + b^2 + c^2 = 1 - 2abc$$

$$\Longrightarrow 1 - 2(ab + bc + ca) = (a + b + c)^2 - 2(ab + bc + bc) = 1 - 2abc$$

$$\Longrightarrow ab + bc + ca = abc \Longrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 \ ,$$

the last implication holding only if a, b, c are all nonzero.

But if any of a, b, c vanish, then two of them must vanish. Suppose that $a = b = 0$, $c = 1$. Then $z^2 = x^2 - y^2 = y^2 - x^2 = (x - y)^2$. This is impossible as $z \neq 0$.

Therefore

$$\frac{a+b}{ab} = \frac{1}{a} + \frac{1}{b} = 1 - \frac{1}{c} = \frac{c-1}{c} = \frac{-(a+b)}{c} \ .$$

Therefore, either $a + b = 0$ or $ab = -c$. Similarly, either $b + c = 0$ or $bc = -a$, and either $c + a = 0$ or $ca = -b$. It is not possible for all of $a + b = 0$, $b + c = 0$ and $c + a = 0$ to occur.

Suppose wolog, $ab = -c$. If $b + c = 0$, then $a = 1$ and $ac = -b$. The condition $a = 1$ implies that $x^2 = (y - z)^2$, whence either $x + y = z$ or $x + z = y$ (which leads to $c = 1$ or $b = 1$).

If $ab = -c$, $bc = -a$, $ca = -b$, then $(abc)^2 = -abc$, so that $a^2 = b^2 = c^2 = -abc = 1$, whence $(a, b, c) = (1, 1, -1), (1, -1, 1), (-1, 1, 1)$.

In any case, two of a, b, c equal 1 and one of them equals -1. If, say $(a, b, c) = (1, 1, -1)$, then $x^2 - (y - z)^2 = y^2 - (z - x)^2 = z^2 - (x + y)^2 = 0$, whence

$$0 = (x - y + z)(x + y - z) = (y - z + x)(y + z - x) = (z - x - y)(x + y + z) \ .$$

The solutions $x + y + z = 0$ is not possible; otherwise

$$\frac{y^2 + z^2 - x^2}{2yz} + \frac{z^2 + x^2 - y^2}{2zx} + \frac{x^2 + y^2 - z^2}{2xy} = \frac{-2yz}{2yz} + \frac{-2zx}{2zx} + \frac{-2xy}{2xy} = -3 \ .$$

Therefore $x + y - z = 0$. Similarly, if $(a, b, c) = (1, -1, 1)$, then $z + x - y = 0$, and if $(a, b, c) = (-1, 1, 1)$, then $y + z - x$. it is readily checked that these solutions work.

Comments. N. Lvov reworked the equation to the equivalent

$$(x + y + z)[x(y + z - x) + y(z + x - y) + z(x + y - z)] = 8xyz$$

which in turn is equivalent to

$$(x + y + z)[4(xy + yz + zx) - (x + y + z)^2] = 8xyz \ .$$

If we take x, y, z to the roots of the cubic equation $t^3 - at^2 + bt - c = 0$ and we find that the condition implies that $8c = a[4b - a^2]$. It remains to show that this condition on the coefficients characterizes those cubic equations for which one root is the sum of the other two. There does not seem to be a convenient way of doing this without essentially having to start from scratch and go through the workings of the given solutions. As an exercise, you might wish to show that the coefficients of any cubic equation for which one root is the sum of the other two satisfies the condition on c; the converse seems not so straightforward.

Many solvers laid out their solutions as a sucession of equations obtained from manipulating their predecessors. It is extremely important to indicate that each equation in the sequence is logically equivalent to its predecessor, for you must be assured that the solutions of the last one are exactly the solutions of the equations that have to be solved. Failure to do this gives an incomplete solution. However, this sort of layout is to be avoided where possible. It is better, as in Solution 1, to set the equation up so that you can simply manipulate one side into a form that allows you to read off the solutions. This makes for a shorter, clearer and more efficient solution. Always edit your solution to put it into the best form.

580. Two numbers m and n are two perfect squares with four decimal digits. Each digit of m is obtained by increasing the corresponding digit of n be a fixed positive integer d. What are the possible values of the pair (m, n).

Solution. Let

$$n = y^2 = p \times 10^3 + q \times 10^2 + r \times 10 + s$$

and

$$m = x^2 = (p + d) \times 10^3 + (q + d) \times 10^2 + (r + d) \times 10 + (s + d) \ ,$$

where $1 \le p < p + d \le 9$, $0 \le q < q + d \le 9$, $0 \le r < r + d \le 9$, $0 \le s < s + d \le 9$. Then

$$(x + y)(x - y) = x^2 - y^2 = d \times 1111 = d \times 11 \times 101 \ .$$

Since $10^3 \le n < m < 10^4$, $32 \le y < x \le 99$, it follows that $x + y < 200$ and $x - y \le 67$. Since the prime 101 must be a factor of either $x + y$ or $x - y$ and since each multiple of 101 exceeds 200, we must have that $x + y = 101$ and $x - y = 11d$. Since x and y must have opposite parity, d must be odd.

Since $64 \le 2y = 101 - 11d$, $11d \le 37$, so that $d \le 3$. Therefore, either $d = 1$ or $d = 3$. The case $d = 1$ leads to $x + y = 101$ and $x - y = 11$, so that $(x, y) = (56, 45)$ and $(m, n) = (3136, 2025)$. The case $d = 3$ leads to $x + y = 101$ and $x - y = 33$, so that $(x, y) = (67, 34)$ and $(m, n) = (4489, 1156)$.

Thus, there are two possibilities for (m, n): $(3136, 2025)$, $(4489, 1156)$.

581. Let $n \ge 4$. The integers from 1 to n inclusive are arranged in some order around a circle. A pair (a, b) is called *acceptable* if $a < b$, a and b are not in adjacent positions around the circle and at least one of the arcs joining a and b contains only numbers that are less than both a and b. Prove that the number of acceptable pairs is equal to $n - 3$.

Solution 1. We prove the result by induction. Let $n = 4$. If 2 and 4 are not adjacent, then $(2, 4)$ is acceptable. If 2 and 4 are adjacent, then 1 must be between 3 and one of 2 and 4, in which case $(2, 3)$ or $(3, 4)$ is the only acceptable pair.

Suppose that $n \ge 5$, that the result holds for $n - 1$ numbers and that a configuration of the numbers 1 to n, inclusive is given. The number 1 must lie between two immediate neighbours u and v that are non-adjacent. Thus, the pair (u, v) is acceptable.

Now remove the number 1 and replace each remaining number r by $r' = r - 1$ to obtain a configuration of $n - 1$ numbers. We show that (r', s') is acceptable in the latter configuration if and only if (r, s) is acceptable in the given configuration.

If (r', s') is acceptable, then r' and s' are not adjacent and there is an arc of smaller numbers between them. The addition of 1 to these numbers and the insertion of 1 will not change either characteristic for (r, s). On the other hand, if $(r, s) \ne (u, v)$ is acceptable in the original configuration , then r and s are not adjacent and each arc connecting them must contain some number other than 1; one of these arcs, at least, contains only numbers less than both r and s. In the final configuration, r' and s' continue to be non-adjacent and a corresponding arc contains only numbers less than both of them.

By the induction hypothesis, there are $(n - 1) - 3 = n - 4$ acceptable pairs in the latter configuration, and so, with the inclusion of (u, v), there are $(n - 4) + 1 = n - 3$ acceptable pairs in the given configuration.

Solution 2. We formulate the more general result that, if $n \ge 3$ and any n distinct real numbers are arranged in a circle and acceptability of pairs is defined as in the problem, then there are precisely $n - 3$ acceptable pairs. This is equivalent to the given problem, since there is an order-preserving one-one mapping from these numbers to $\{1, 2, \cdots, n\}$ that takes the kth largest of them to k.

We use induction. As in the previous solution, we see that it is true for $n = 3$ and $n = 4$. Let $n \geq 5$ and suppose that the largest three numbers are u, v, w. At least one of these three pairs is non-adjacent; otherwise, if w is adjacent to both u and v, then w is between u and v; since u and v are separated on both sides by at least one number, they are non-adjacent. This pair is acceptable, since a larger number can appear on at most one of the arcs connecting them.

Suppose that this acceptable pair is (u, v). Since all the numbers in at least one of the arcs connecting them are smaller, there is no acceptable pair (a, b) with a and b on different arcs joing u and v.

Consider two "circles" of numbers consisting of the $k \geq 3$ numbers of one arc A determined by (u, v) including u and v, and the $n + 2 - k$ numbers of the other arc B determined by (u, v) including u and v.

The set A contains exactly $k - 3$ acceptable pairs and the set B $n - 1 - k$ acceptable pairs, by the induction hypothesis. Each of these pairs is acceptable in the original circle of n numbers since none of the acceptable arcs includes u and v. Therefore, the original circle has $1 + (k-3) + (n-1-k) = n-3$ acceptable pairs.

Solution 3. [C. Bruggeman] Suppose that $1 \leq k \leq n - 3$. Examine numbers counterclockwise from k until the first number a that exceeds k is reached; the examine numbers clockwise from k until the first number b that exceeds k is reached. Every number of the arc containing k between a and b is less than both a and b. Since there are at least three numbers exceeding k, at least one of them must be between a and b outside the arc containing k, so that a and b are not adjacent. Hence (a, b) is an acceptable pair.

We now prove that every acceptable pair is obtained exactly once in this way. Suppose that (a, b) is an acceptable pair with at least one of a and b not equal to $n - 1$ and n. Then, as one of the arcs between a and b must contain a number h bigger than at least one of them, the other arc must contain only numbers smaller than both of them. Let the largest such number be m. The m must engender the pair (a, b) by the foregoing process. Suppose that $k \leq n - 3$ is some other number other than a, b and m. Then m must lie on the arc between a and b opposite h between a and m or between m and b, or on the arc between a and b opposite m between a and h or between h and b; in each case, the pair engendered by k cannot be (a, b).

The only case remaining is $(n - 1, n)$ which may or may not be acceptable. If $(n - 1, n)$ is acceptable, then one arc connecting it must contain $n - 2$; by an argument similar to that in the last paragraph, no other element in this arc can engender $(n - 1, n)$. However, the largest element m in the other arc does not exceed $n - 3$ and it is the sole element that engenders $(n - 1, n)$.

Thus, there is a one-one correspondence between the numbers $1, 2, \cdots, n - 3$ and acceptable pairs; the desired result follows.

Comment. A. Abdi claims that the acceptable pair determine diagonals yielding a triangulation of the $n-$gon determined by the positions of the n numbers. Is this true?

582. Suppose that f is a real-valued function defined on the closed unit interval $[0, 1]$ for which $f(0) = f(1) = 0$ and $|f(x) - f(y)| < |x - y|$ when $0 \leq x < y \leq 1$. Prove that $|f(x) - f(y)| < \frac{1}{2}$ for all $x, y \in [0, 1]$. Can the number $\frac{1}{2}$ in the inequality be replaced by a smaller number and still result in a true proposition?

Solution 1. Suppose that $0 \leq x < y \leq 1$. If $y - x < \frac{1}{2}$, the result holds trivially. Suppose that $y - x \geq \frac{1}{2}$. Then

$$\begin{aligned} |f(y) - f(x)| &\leq |f(1) - f(y)| + |f(x) - f(0)| \\ &< (1 - y) + x = 1 - (y - x) \leq \frac{1}{2} , \end{aligned}$$

as desired.

The coefficient $\frac{1}{2}$ cannot be replaced by anything smaller. Suppose that $0 < \lambda < 1$; define

$$f_\lambda = \begin{cases} \lambda x & \text{if } 0 \leq x \leq \frac{1}{2} \\ \lambda(1 - x) & \text{if } \frac{1}{2} < x \leq 1 . \end{cases}$$

We show that f_λ has the desired property. However, note that $f_\lambda(\frac{1}{2}) - f_\lambda(0) = \frac{\lambda}{2}$, so that by our choice of λ, we can make the right side arbitrarily close to $\frac{1}{2}$.

If $0 \le x < y \le \frac{1}{2}$, then
$$|f(x) - f(y)| = \lambda|x - y| < |x - y| \ .$$
If $\frac{1}{2} \le x < y \le 1$, then
$$|f(x) - f(y)| = \lambda|(1 - x) - (1 - y)| = \lambda|x - y| < |x - y| \ .$$

Finally, suppose that $0 \le x < \frac{1}{2} < y \le 1$. Then
$$|f_\lambda(x) - f_\lambda(y)| = \lambda|x - (1 - y)| = \lambda|(x + y) - 1| \ .$$
If $x + y \ge 1$, then
$$|(x + y) - 1| = (x + y) - 1 = (y - x) - (1 - 2x) < y - x \ ;$$
if $x + y \le 1$, then
$$|(x + y) - 1| = 1 - (x + y) = (y - x) - (2y - 1) < y - x \ .$$
In either case
$$|f_\lambda(x) - f_\lambda(y)| < \lambda(y - x) < y - x = |x - y| \ .$$

Solution 2. [J. Zung] Let $0 \le x < y \le 1$. Then $|f(x)| \le x$, $|f(y) - f(x)| < y - x$ and $|f(y)| = |f(1) - f(y)| \le 1 - y$. Therefore, adding these three inequalities gives us that
$$\begin{aligned} 2|f(y) - f(x)| &= |f(y) - f(x)| + |f(y) - f(x)| \\ &\le |f(y)| + |f(x)| + |f(y) - f(x)| < (1 - y) + x + (y - x) = 1 \ , \end{aligned}$$
so that $|f(y) - f(x)| < \frac{1}{2}$.

Solution 3. Since $|f(x) - f(y)| < |x - y|$, f must be continuous on $[0, 1]$. [Provide an $\epsilon - \delta$ argument for this.] Therefore it assumes its maximum value M at a point $v \in [0, 1]$ and its minimum value m at a point $u \in [0, 1]$. We have that
$$0 \le M = f(v) = f(v) - f(0) < v \le 1$$
and
$$0 \ge m = f(u) = f(u) - f(0) > -u \ge 1 \ ,$$
since $|f(u) - f(0)| < |u - 0| = u$. Thus $|m| < u \le 1$ and $M < v \le 1$.

Suppose that $v < u$. Then
$$\begin{aligned} 2(M - m) &= M - m + (f(v) - f(u)) \\ &= f(v) + (f(1) - f(u)) + |f(u) - f(v)| \\ &< v + (1 - u) + (u - v) = 1 \ . \end{aligned}$$
Suppose that $u < v$. Then
$$\begin{aligned} 2(M - m) &= M - m + (f(v) - f(u)) \\ &= |f(1) - f(v)| + |f(u)| + |f(v) - f(u)| \\ &< (1 - v) + u + (v - u) = 1 \ . \end{aligned}$$
In either case, $M - m < \frac{1}{2}$. If $x, y \in [0, 1]$, then $f(x)$ and $f(y)$ both lie in $[m, M]$ and so $|f(x) - f(y)| \le M - m < \frac{1}{2}$.

Comments. The hard part of this problem was the last past, replacing the term $\frac{1}{2}$. To get a clear idea of what is being asked, reformulate the result explicitly that you want to examine: *Let f satisfy $f(0) = f(1) = 0$ and $|f(x) - f(y)| < |x - y|$ for all x, y. Prove that $|f(x) - f(y)| < \lambda$, where λ is a parameter less than $\frac{1}{2}$.* Either this result is true or it is false. If true, then you try to modify the proof given in the earlier part of the solution to see if you can get one that works. However, if it is false, you can demonstrate this by producing a *counterexample*, that is, a function f for which the hypothesis is true, but the conclusion is false. Such a counterexample would satisfy these conditions: (1) $f(0) = f(1) = 0$; (2) $|f(x) - f(y)| < |x - y|$ for $x, y \in [0, 1]$; (3) **there exists** $u, v \in [0, 1]$ for which $|f(u) - f(v)| > \lambda$. Note that the conclusion in the problem has the form "for all x, y" while its contradition has the form "there exists x, y".

Many solvers tried to set the solution to the first part of the problem up as a contradiction. This is completely unnecessary and complicated the solution. In many cases, the contradiction could be avoided by deleting the contradiction hypothesis and let the conclusion stand. While it is natural in solving a problem to look at it from a contradiction point of view, when you write up your solution, see if you can edit it to avoid the contradiction. The way you think about solving a problem is not always the best or most efficient way to write up the solution.

583. Suppose that $ABCD$ is a convex quadrilateral, and that the respective midpoints of AB, BC, CD, DA are K, L, M, N. Let O be the intersection point of KM and LN. Thus $ABCD$ is partitioned into four quadrilaterals. Prove that the sum of the areas of two of these that do not have a common side is equal to the sum of the areas of the other two, to wit

$$[AKON] + [CMOL] = [BLOK] + [DNOM] .$$

Solution 1. Using the fact that triangles with equal bases and heights have the same area, we have that $[AKO] = [BKO]$, $[BLO] = [CLO]$, $[CMO] = [DMO]$ and $[DNO] = [ANO]$. Therefore

$$\begin{aligned}[AKON] + [CMOL] &= [AKO] + [ANO] + [CLO] + [CMO] \\ &= [BKO] + [BLO] + [DNO] + [DMO] = [BLOK] + [DNOM] .\end{aligned}$$

Solution 2. Observe that $KN \| BD \| LM$ and $KL \| AC \| NM$, so that $KLMN$ is a parallelogram and O is the intersection of its diagonals. Therefore KM and LN bisect each other, and $[KOL] = [LOM] = [MON] = [NOK]$.

Since triangle BKL and BAC are similar with factor $\frac{1}{2}$, $[BKL] = \frac{1}{4}[BAC]$. Similarly, $[DMN] = \frac{1}{4}[DAC]$, whence $[BKL] + [DMN] = \frac{1}{4}[ABCD]$. Likewise, $[CLM] + [ANK] = \frac{1}{4}[ABCD]$.

Therefore

$$\begin{aligned}[AKON] + [CMOL] &= [ANK] + [NOK] + [CLM] + [LPM] = [ANK] + [CLM] + [NOK] + [LPM] \\ &= \frac{1}{4}[ABCD] + [KOL] + [MON] = [BKL] + [KOL] + [DMN] + [MON] \\ &= [BLOK] + [DNOM] .\end{aligned}$$

584. Let n be an integer exceeding 2 and suppose that $x_1, x_2, \cdots, x_n$ are real numbers for which $\sum_{i=1}^n x_i = 0$ and $\sum_{i=1}^n x_i^2 = n$. Prove that there are two numbers among the x_i whose product does not exceed -1.

Solution. We can supppose that the x_i are ordered in increasing sequence and that there is a positive integer k with $x_1 \leq x_2 \leq \cdots \leq x_k \leq 0 \leq x_{k+1} \leq \cdots \leq x_n$. Then, noting that $-x_1 \geq 0$, we have that

$$\sum_{i=1}^k x_i^2 \leq \sum_{i=1}^k x_1 x_i = -x_1(x_{k+1} + x_{k+2} + \cdots + x_n) \leq -(n-k)x_1 x_n$$

and

$$\sum_{i=k+1}^{n} x_i^2 \leq \sum_{i=k+1}^{n} x_n x_i = -x_n(x_1 + x_2 + \cdots + x_k) \leq -kx_1x_n \ .$$

Finally, $n = \sum_{i=1}^{n} x_i^2 \leq -nx_1x_n$; thus $x_1x_n \leq -1$.

585. Calculate the number

$$a = \lfloor \sqrt{n-1} + \sqrt{n} + \sqrt{n+1} \rfloor^2 \ ,$$

where $\lfloor x \rfloor$ denotes the largest integer than does not exceed x and n is a positive integer exceeding 1.

Solution. It does not appear that there is a neat expression for this. One can obtain without too much trouble the inequality

$$3\sqrt{n-1} < \sqrt{n-1} + \sqrt{n} + \sqrt{n+1} < 3\sqrt{n} \ ,$$

from which we can find that when $k^2 + 1 \leq n \leq k^2 + (2k/3)$, then $\sqrt{a} = 3k$, when $k^2 + (2k/3) + (10/9) < n \leq k^2 + (4k/3) + (1/3)$, then $\sqrt{a} = 3k + 1$, and when $k^2 + 4k + (13/9) < n \leq (k+1)^2$, then $\sqrt{a} = 3k + 2$. However, this leaves the difficulty of getting the right expression for the gaps between the various ranges of n.

586. The function defined on the set $\mathbf{C^*}$ of all nonzero complex numbers satisfies the equation

$$f(z)f(iz) = z^2 \ ,$$

for all $z \in \mathbf{C}^*$. Prove that the function $f(z)$ is odd, *i,e.*, $f(-z) = -f(z)$ for all $z \in \mathbf{C}^*$. Give an example of a function that satisfies this condition.

Solution. Note that $f(z) \neq 0$ for all $x \in \mathbf{C}^*$. Replacing z by iz leads to $f(iz)f(-z) = -z^2$, from which we have that

$$f(z)f(iz) + f(iz)f(-z) = 0 \Longrightarrow f(z) + f(-z) = 0 \ .$$

Therefore the function is odd.

An example is given by $f(z) = (-1 + i)z/\sqrt{2}$.

587. Solve the equation

$$\tan 2x \tan\left(2x + \frac{\pi}{3}\right) \tan\left(2x + \frac{2\pi}{3}\right) = \sqrt{3} \ .$$

Solution. Using the standard trigonometric identities for $\sin A \sin B$, $\cos A \cos B$, $\cos 2A$ and $\sin 2A$, we have that

$$\begin{aligned}
\sqrt{3} &= \tan 2x \left(\frac{\sin(2x + (\pi/3)) \sin(2x + (2\pi/3))}{\cos(2x + (\pi/3)) \cos(2x + (2\pi/3))} \right) \\
&= \tan 2x \left(\frac{\cos(\pi/3) - \cos(4x + \pi)}{\cos(\pi/3) + \cos(4x + \pi)} \right) \\
&= \tan 2x \left(\frac{1 + 2\cos 4x}{1 - 2\cos 4x} \right) = \tan 2x \left(\frac{1 + 2(2\cos^2 2x - 1)}{1 - 2(1 - 2\sin^2 2x)} \right) \\
&= \left(\frac{\sin 2x}{\cos 2x} \right) \left(\frac{4\cos^2 2x - 1}{4\sin^2 2x - 1} \right) = \frac{2\sin 4x \cos 2x - \sin 2x}{2\sin 4x \sin 2x - \cos 2x} \\
&= \frac{\sin 6x + \sin 2x - \sin 2x}{\cos 2x - \cos 6x - \cos 2x} = \frac{\sin 6x}{-\cos 6x} = -\tan 6x \ .
\end{aligned}$$

Therefore $x = -10° + k \cdot 30°$ for some integer k.

588. Let the function $f(x)$ be defined for $0 \leq x \leq \pi/3$ by

$$f(x) = \sec\left(\frac{\pi}{6} - x\right) + \sec\left(\frac{\pi}{6} + x\right) .$$

Determine the set of values (its image or range) assumed by the function.

Solution. Making use of the inequality $(1/a) + (1/b) \geq 2/\sqrt{ab}$ for $a, b > 0$, we find that

$$f(x) \geq \frac{2}{\sqrt{\cos((\pi/6) - x)\cos((\pi/6) + x)}} \geq \frac{2}{\sqrt{(1/4) + ((\cos 2x)/2)}} .$$

Since $0 \leq x \leq \pi/3$ implies that $-\frac{1}{2} \leq \cos 2x \leq 1$, it follows that

$$0 \leq \sqrt{\frac{1}{4} + \frac{\cos 2x}{2}} \leq \frac{\sqrt{3}}{2} ,$$

and

$$f(x) \geq \frac{4}{\sqrt{3}} .$$

Since $f(x)$ is continuous, $f(0) = 4/\sqrt{3}$ and $f(x)$ grows without bound when x approaches $\pi/3$, the image of f on $[0, \pi/3)$ is $[4/\sqrt{3}.\infty)$.

589. In a circle, A is a variable point and B and C are fixed points. The internal bisector of the angle BAC intersects the circle at D and the line BC at G; the external bisector of the angle BAC intersects the circle at E and the line BC at F. Find the locus of the intersection of the lines DF and EG.

Solution. Suppose without loss of generality that $AB > AC$. If M is the midpoint of BC, since $BG : GC = AB : AC$, $BG > GC$ so that G lies between M and C and A lies between E and F. Let P be the intersection of DF and EG.

Observe that D is the midpoint of the arc BC and that $AD \perp EF$. Therefore DA is an altitude of triangle DEF and DE is a diameter of the circle. Therefore DE must pass through M, and so $FM \perp DE$, *i.e.*, FM is an altitude of triangle DEF. The intersection of these two altitudes, G, is the orthocentre of triangle ABC and so $EG \perp DF$. Thus, $\angle EPD = 90°$, so that P must lie on the given circle.

Conversely, let P be a point on the given circle. Wolog, we may assume that P lies between D, the midpoint of arc BC and C. Let DE be the diameter of the circle that right bisects BC. Suppose that DP produced intersects BC produced at F and that EF intersects the circle at A. This is the point A that produced the point P as described in the problem. Thus, the locus is indeed the given circle with the exception of the points B and C.

590. Let $SABC$ be a regular tetrahedron. The points M, N, P belong to the edges SA, SB and SC respectively such that $MN = NP = PM$. Prove that the planes MNP and ABC are parallel.

Solution. Let $|SM| = a$, $|SN| = b$ and $|SP| = c$. From the Law of Cosines, we have that $|MN|^2 = a^2 + b^2 - ab$, *etc.*, whence $a^2 + b^2 - ab = b^2 + c^2 - bc = c^2 + a^2 - ac = 0$. This implies that $a = b = c$ [prove it], so that $SM : SA = SN : SB = SP : SC$ and the result follows.