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Notes: Given a triangle, extend two nonadjacent sides. The circle tangent to these two sides and to the third side of the triangle is called an *excircle*, or sometimes an *escribed circle*. The centre of the circle is called the *excentre* and lies on the angle bisector of the opposite angle and the bisectors of the external angles formed by the extended sides with the third side. Every triangle has three excircles along with their excentres.

The *incircle* of a polygon is a circle inscribed inside of the polygon that is tangent to all of the sides of a polygon. While every triangle has an incircle, this is not true of all polygons.

The *greatest common divisor* of two integers m, n, denoted by *gcd* (m, n) is the largest positive integer which divides (evenly) both m and n. The *least common multiple* of two integers m, n, denoted by *lcm* (m, n) is the smallest positive integer which is divisible by both m and n.

Let n be a positive integer. It can be written uniquely as a sum of powers of 2, *i.e.* in the form

$$n = \epsilon_k \cdot 2^k + \epsilon_{k-1} \cdot 2^{k-1} + \cdots + \epsilon_1 \cdot 2 + \epsilon_0$$

where each ϵ_i takes one of the values 0 and 1. This is known as the *binary representation* of n and is denoted $(\epsilon_k, \epsilon_{k-1}, \cdots, \epsilon_0)_2$. The numbers ϵ_i are known as the *(binary) digits* of n.

The *circumcircle* of a triangle is the centre of the circle that passes through the three vertices of the triangle; the *incentre* of a triangle is centre of the circle within the triangle that is tangent to the three sides; the *orthocentre* of a triangle is the intersection point of its three altitudes.

The function f defined on the real numbers and taking real values is *increasing* if and only if, for $x < y$, $f(x) \leq f(y)$.

423. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y + 1) \leq (x + 1)^2$, then $y(y - 1) \leq x^2$.

424. Simplify

$$\frac{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} - 2}{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} + 2}$$

to a fraction whose numerator and denominator are of the form $u\sqrt{v}$ with u and v each linear polynomials. For which values of x is the equation valid?

425. Let $\{x_1, x_2, \cdots, x_n, \cdots\}$ be a sequence of nonzero real numbers. Show that the sequence is an arithmetic progression if and only if, for each integer $n \geq 2$,

$$\frac{1}{x_1 x_2} + \frac{1}{x_2 x_3} + \cdots + \frac{1}{x_{n-1} x_n} = \frac{n-1}{x_1 x_n} \ .$$

426. (a) The following paper-folding method is proposed for trisecting an acute angle.

(1) transfer the angle to a rectangular sheet so that its vertex is at one corner P of the sheet with one ray along the edge PY; let the angle be XPY;

(2) fold up PY over QZ to fall on RW, so that $PQ = QR$ and $PY \| QZ \| RW$, with QZ between PY and RW;

(3) fold across a line AC with A on the sheet and C on the edge PY so that P falls on a point P' on QZ and R on a point R' on PX;

(4) suppose that the fold AC intersects the fold QZ at B and carries Q to Q'; make a fold along BQ'.

It is claimed that the fold BQ' passes through P and trisects angle XPY.

Explain why the fold described in (3) is possible. Does the method work? Why?

(b) What happens with a right angle?

(c) Can the method be adapted for an obtuse angle?

427. The radius of the inscribed circle and the radii of the three escribed circles of a triangle are consecutive terms of a geometric progression. Determine the largest angle of the triangle.

428. **a**, **b** and **c** are three lines in space. Neither **a** nor **b** is perpendicular to **c**. Points P and Q vary on **a** and **b**, respectively, so that PQ is perpendicular to **c**. The plane through P perpendicular to **b** meets **c** at R, and the plane through Q perpendicular to **a** meets **c** at S. Prove that RS is of constant length.

429. Prove that

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\binom{kn}{n} = (-1)^{n+1}n^n \ .$$

430. Let triangle ABC be such that its excircle tangent to the segment AB is also tangent to the circle whose diameter is the segment BC. If the lengths of the sides BC, CA and AB of the triangle form, in this order, an arithmetic sequence, find the measure of the angle ACB.

431. Prove the following trigonometric identity, for any natural number n:

$$\sin\frac{\pi}{4n+2}\cdot\sin\frac{3\pi}{4n+2}\cdot\sin\frac{5\pi}{4n+2}\cdots\sin\frac{(2n-1)\pi}{4n+2} = \frac{1}{2^n} \ .$$

432. Find the exact value of:

(a)

$$\sqrt{\frac{1}{6}+\frac{\sqrt{5}}{18}} - \sqrt{\frac{1}{6}-\frac{\sqrt{5}}{18}} \ ;$$

(b)

$$\sqrt{1+\frac{2}{5}}\cdot\sqrt{1+\frac{2}{6}}\cdot\sqrt{1+\frac{2}{7}}\cdot\sqrt{1+\frac{2}{8}}\cdots\sqrt{1+\frac{2}{57}}\cdot\sqrt{1+\frac{2}{58}} \ .$$

433. Prove that the equation

$$x^2 + 2y^2 + 98z^2 = 77777\ldots777$$

does not have a solution in integers, where the right side has 2006 digits, all equal to 7.

434. Find all natural numbers n for which $2^n + n^{2004}$ is equal to a prime number.

435. A circle with centre I is the incircle of the convex quadrilateral $ABCD$. The diagonals AC and BD intersect at the point E. Prove that, if the midpoints of the segments AD, BC and IE are collinear, then $AB = CD$.

436. In the Euro-African volleyball tournament, there were nine more teams participating from Europe than from Africa. In total, the European won nine times as many points as were won by all of the African teams. In this tournamet, each team played exactly once against each other team; there were no ties; the winner of a game gets 1 point, the loser 0. What is the greatest possible score of the best African team?

437. Let a, b, c be the side lengths and m_a, m_b, m_c the lengths of their respective medians, of an arbitrary triangle ABC. Show that

$$\frac{3}{4} < \frac{m_a + m_b + m_c}{a + b + c} < 1 \ .$$

Furthermore, show that one cannot find a smaller interval to bound the ratio.

438. Determine all sets (x, y, z) of real numbers for which

$$x + y = 2 \qquad \text{and} \qquad xy - z^2 = 1 \ .$$

439. A natural number n, less than or equal to 500, has the property that when one chooses a number m randomly among $\{1, 2, 3, \cdots, 500\}$, the probability that m divides n (*i.e.*, n/m is an integer) is $1/100$. Find the largest such n.

440. You are to choose 10 distinct numbers from $\{1, 2, 3, \cdots, 2006\}$. Show that you can choose such numbers with a sum greater than 10039 in more ways than you can choose such numbers with a sum less than 10030.

441. Prove that, no matter how 15 points are placed inside a circle of radius 2 (including the boundary), there exists a circle of radius 1 (including the boundary) containing at least 3 of the 15 points.

442. Prove that the regular tetrahedron has minimum diameter among all tetrahedra that circumscribe a given sphere. (The diameter of a tetrahedron is the length of its longest edge.)

443. For $n \geq 3$, show that $n - 1$ straight lines are sufficient to go through the interior of every square of an $n \times n$ chessboard. Are $n - 1$ lines necessary?

444. (a) Suppose that a 6×6 square grid of unit squares (chessboard) is tiled by 1×2 rectangles (dominoes). Prove that it can be decomposed into two rectangles, tiled by disjoint subsets of the dominoes.

(b) Is the same thing true for an 8×8 array?

(c) Is the same thing true for a 6×8 array?

445. Two parabolas have parallel axes and intersect in two points. Prove that their common chord bisects the segments whose endpoints are the points of contact of their common tangent.

446. Suppose that you have a 3×3 grid of squares. A *line* is a set of three squares in the same row, the same column or the same diagonal; thus, there are eight lines.

Two players A and B play a game. They take alternate turns, A putting a 0 in any unoccupied square of the grid and B putting a 1. The first player is A, and the game cannot go on for more than nine moves. (The play is similar to noughts-and-crosses, or tictactoe.) A move is *legitimate* if it does not result in two lines of squares being filled in with different sums. The winner is the last player to make a legitimate move.

(For example, if there are three 0s down the diagonal, then B can place a 1 in any vacant square provided it completes no other line, for then the sum would differ from the diagonal sum. If there are two zeros at the top of the main diagonal and two ones at the left of the bottom line, then the lower right square cannot be filled by either player, as it would result in two lines with different sums.)

(a) What is the maximum number of legitimate moves possible in a game?

(b) What is the minimum number of legitimate moves possible in a game that would not leave a legitimate move available for the next player?

(c) Which player has a winning strategy? Explain.

447. A high school student asked to solve the surd equation

$$\sqrt{3x-2} - \sqrt{2x-3} = 1$$

gave the following answer: *Squaring both sides leads to*

$$3x - 2 - 2x - 3 = 1$$

so $x = 6$. The answer is, in fact, correct.

Show that there are infinitely many real quadruples (a, b, c, d) for which this method leads to a correct solution of the surd equation

$$\sqrt{ax-b} - \sqrt{cx-d} = 1 \ .$$

448. A criminal, having escaped from prison, travelled for 10 hours before his escape was detected. He was then pursued and gained upon at 3 miles per hour. When his pursuers had been 8 hours on the way, they met an express (train) going in the opposite direction at the same rate as themselves, which had met the criminal 2 hours and 24 minutes earlier. In what time from the beginning of the pursuit will the criminal be overtaken? [from *The high school algebra* by Robertson and Birchard, approved for Ontario schools in 1886]

449. Let $S = \{x : x > -1\}$. Determine all functions from S to S which both

(a) satisfies the equation $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all $x, y \in S$, and

(b) $f(x)/x$ is strictly increasing or strictly decreasing on each of the two intervals $\{x : -1 < x < 0\}$ and $\{x : x > 0\}$.

450. The 4-sectors of an angle are the three lines through its vertex that partition the angle into four equal parts; adjacent 4-sectors of two angles that share a side consist of the 4-sector through each vertex that is closest to the other vertex.

Prove that adjacent 4-sectors of the angles of a parallelogram meet in the vertices of a square if and only if the parallelogram has four equal sides.

451. Let a and b be positive integers and let $u = a + b$ and $v = lcm\ (a, b)$. Prove that

$$gcd\ (u, v) = gcd\ (a, b)\ .$$

452. (a) Let m be a positive integer. Show that there exists a positive integer k for which the set

$$\{k + 1, k + 2, \ldots, 2k\}$$

contains exactly m numbers whose binary representation has exactly three digits equal to 1.

(b) Determine all intgers m for which there is exactly one such integer k.

453. Let A, B be two points on a circle, and let AP and BQ be two rays of equal length that are tangent to the circle that are directed counterclockwise from their tangency points. Prove that the line AB intersects the segment PQ at its midpoint.

454. Let ABC be a non-isosceles triangle with circumcentre O, incentre I and orthocentre H. Prove that the angle OIH exceeds 90°.

455. Let $ABCDE$ be a pentagon for which the position of the base AB and the lengths of the five sides are fixed. Find the locus of the point D for all such pentagons for which the angles at C and E are equal.

456. Let $n+1$ cups, labelled in order with the numbers $0, 1, 2, \cdots, n$, be given. Suppose that $n+1$ tokens, one bearing each of the numbers $0, 1, 2, \cdots, n$ are distributed randomly into the cups, so that each cup contains exactly one token.

We perform a sequence of moves. At each move, determine the smallest number k for which the cup with label k has a token with label m not equal to k. Necessarily, $k < m$. Remove this token; move all the tokens in cups labelled $k+1, k+2, \cdots, m$ to the respective cups labelled $k, k+1, m-1$; drop the token with label m into the cup with label m. Repeat.

Prove that the process terminates with each token in its own cup (token k in cup k for each k) in not more that $2^n - 1$ moves. Determine when it takes exactly $2^n - 1$ moves.

457. Suppose that $u_1 > u_2 > u_3 > \cdots$ and that there are infinitely many indices n for which $u_n \geq 1/n$. Prove that there exists a positive integer N for which

$$u_1 + u_2 + u_3 + \cdots + u_N > 2006 .$$

458. Let ABC be a triangle. Let A_1 be the reflected image of A with axis BC, B_1 the reflected image of B with axis CA and C_1 the reflected image of C with axis AB. Determine the possible sets of angles of triangle ABC for which $A_1B_1C_1$ is equilateral.

459. At an International Conference, there were exactly 2006 participants. The organizers observed that: (1) among any three participants, there were two who spoke the same language; and (2) every participant spoke at most 5 languages. Prove that there is a group of at least 202 participants who speak the same language.

460. Given two natural numbers x and y for which

$$3x^2 + x = 4y^2 + y ,$$

prove that their positive difference is a perfect square. Determine a nontrivial solution of this equation.

461. Suppose that x and y are integers for which $x^2 + y^2 \neq 0$. Determine the minimum value of the function

$$f(x, y) \equiv |5x^2 + 11xy - 5y^2| .$$

462. For any positive real numbers a, b, c, d, establish the inequality

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+d}} + \sqrt{\frac{c}{d+a}} + \sqrt{\frac{d}{a+b}} > 2 .$$

463. In *Squareland*, a newly-created country in the shape of a square with side length of 1000 km, there are 51 cities. The country can afford to build at most 11000 km of roads. Is it always possible, within this limit, to design a road map that provides a connection between any two cities in the country?

464. A square is partitioned into non-overlapping rectangles. Consider the circumcircles of all the rectangles. Prove that, if the sum of the areas of all these circles is equal to the area of the circumcircle of the square, then all the rectangles must be squares, too.

465. For what positive real numbers a is

$$\sqrt[3]{2+\sqrt{a}}+\sqrt[3]{2-\sqrt{a}}$$

an integer?

466. For a positive integer m, let $\overline{m}$ denote the sum of the digits of m. Find all pairs of positive integers $(m.n)$ with $m < n$ for which $(\overline{m})^2 = n$ and $(\overline{n})^2 = m$.

467. For which positive integers n does there exist a set of n distinct positive integers such that

(a) each member of the set divides the sum of all members of the set, *and*

(b) none of its proper subsets with two or more elements satisfies the condition in (a)?

468. Let a and b be positive real numbers satisfying $a + b \geq (a-b)^2$. Prove that

$$x^a(1-x)^b + x^b(1-x)^a \leq \frac{1}{2^{a+b-1}}$$

for $0 \leq x \leq 1$, with equality if and only if $x = \frac{1}{2}$.

469. Solve for t in terms of a, b in the equation

$$\sqrt{\frac{t^3+a^3}{t+a}}+\sqrt{\frac{t^3+b^3}{t+b}}=\sqrt{\frac{a^3-b^3}{a-b}}$$

where $0 < a < b$.

470. Let ABC, ACP and BCQ be nonoverlapping triangles in the plane with angles CAP and CBQ right. Let M be the foot of the perpendicular from C to AB. Prove that lines AQ, BP and CM are concurrent if and only if $\angle BCQ = \angle ACP$.

471. Let I and O denote the incentre and the circumcentre, respectively, of triangle ABC. Assume that triangle ABC is not equilateral. Prove that $\angle AIO \leq 90^\circ$ if and only if $2BC \leq AB + CA$, with equality holding only simultaneously.

472. Find all integers x for which

$$(4-x)^{4-x} + (5-x)^{5-x} + 10 = 4^x + 5^x \ .$$

473. Let $ABCD$ be a quadrilateral; let M and N be the respective midpoint of AB and BC; let P be the point of interesection of AN and BD, and Q be the point of intersection of DM amd AC. Suppose the $3BP = BD$ and $3AQ = AC$. Prove that $ABCD$ is a parallelogram.

474. Solve the equation for positive real x:

$$(2^{\log_5 x} + 3)^{\log_5 2} = x - 3 \ .$$

475. Let z_1, z_2, z_3, z_4 be distinct complex numbers for which $|z_1| = |z_2| = |z_3| = |z_4|$. Suppose that there is a real number $t \neq 1$ for which

$$|tz_1 + z_2 + z_3 + z_4| = |z_1 + tz_2 + z_3 + z_4| = |z_1 + z_2 + tz_3 + z_4| \ .$$

Show that, in the complex plane, z_1, z_2, z_3, z_4 lie at the vertices of a rectangle.

476. Let p be a positive real number and let $|x_0| \leq 2p$. For $n \geq 1$, define

$$x_n = 3x_{n-1} - \frac{1}{p^2}x_{n-1}^3 \ .$$

Determine x_n as a function of n and x_0.

477. Let S consist of all real numbers of the form $a + b\sqrt{2}$, where a and b are integers. Find all functions that map S into the set $\mathbf{R}$ of reals such that (1) f is increasing, and (2) $f(x+y) = f(x) + f(y)$ for all x, y in S.

478. Solve the equation

$$\sqrt{2+\sqrt{2+\sqrt{2+x}}} + \sqrt{3}\sqrt{2-\sqrt{2+\sqrt{2+x}}} = 2x$$

for $x \geq 0$

Solutions

423. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

Solution 1. The statement is true. The result holds when $y \leq 1$ since $y(y-1) \leq 0 \leq x^2$. Assume $y \geq 1$, so that $\sqrt{y(y-1)} - 1 > 0$. We have that $4y(y+1) < (2y+1)^2$, so that $2\sqrt{y(y+1)} < 2y+1$. Hence

$$y(y-1) < y^2 + y - 2\sqrt{y(y+1)} + 1 = (\sqrt{y(y+1)} - 1)^2 \ .$$

Since $y(y+1) \leq (x+1)^2$ and $|x+1| \leq |x| + 1$, it follows that

$$y(y+1) \leq (|x|+1)^2 \Longrightarrow \sqrt{y(y+1)} - 1 \leq |x|$$

whence

$$y(y-1) < |x|^2 = x^2 \ .$$

Thus the assertion holds.

Solution 2. [D. Dziabenko] The statement holds. If $0 \leq y \leq 1$, then $y(y-1) \leq 0 \leq x^2$. Assume henceforth that $y > 1$. If $x + \frac{1}{2} \leq y$, then

$$y(y-1) = y(y+1) - 2y \leq (x+1)^2 - 2\left(x + \frac{1}{2}\right) = x^2 \ .$$

If $x + \frac{1}{2} > y$, then $x > y - \frac{1}{2} > 0$, whence

$$x^2 > \left(y - \frac{1}{2}\right)^2 = y(y-1) + \frac{1}{4} > y(y-1) \ .$$

Solution 3. [G. Ghosn] The result holds. Let $y > 0$. The region in the cartesian plane defined by $y(y+1) \leq (x+1)^2$ lies between the $x-$axis and the upper branch of the hyperbola with equation

$$\left(y + \frac{1}{2}\right)^2 - (x+1)^2 = \frac{1}{4} \ .$$

The region in the cartesian plane defined by $y(y-1) \leq x^2$ lies between the $x-$axis and the upper branch of the hyperbola with equation

$$\left(y - \frac{1}{2}\right)^2 - x^2 = \frac{1}{4}.$$

The second hyperbola lies above the first when $x = -1$. Thus, if it can be shown that the two hyperbolas do not intersect, then the top branch of the second hyperbola lies above top branch of the first, and the result will follow.

But solving $y(y+1) = (x+1)^2$ and $y(y-1) = x^2$ leads to $y = x+\frac{1}{2}$ and ultimately to $(x+\frac{1}{2})^2-(x+\frac{1}{2}) = x^2$ which has no solution (being equivalent to $-1/4 = 0$).

424. Simplify

$$\frac{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} - 2}{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} + 2}$$

to a fraction whose numerator and denominator are of the form $u\sqrt{v}$ with u and v each linear polynomials. For which values of x is the equation valid?

Solution. For a real expression, we require that $x^2 \geq 2$. Observe that $x^3 - 3x + 2 = (x-1)^2(x+2)$ and that $x^3 - 3x - 2 = (x+1)^2(x-2)$. Thus, the denominator vanishes when $x = -2$, and we must exclude this value. Suppose, first, that $x \geq 2$. Then

$$\begin{aligned}\frac{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} - 2}{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} + 2} &= \frac{(x+1)^2(x-2) + (x^2-1)\sqrt{x^2-4}}{(x-1)^2(x+2) + (x^2-1)\sqrt{x^2-4}} \\ &= \frac{[(x+1)\sqrt{x-2}][(x+1)\sqrt{x-2} + (x-1)\sqrt{x+2}]}{[(x-1)\sqrt{x+2}][(x-1)\sqrt{x+2} + (x+1)\sqrt{x-2}]} \\ &= \frac{(x+1)\sqrt{x-2}}{(x-1)\sqrt{x+2}} \ .\end{aligned}$$

Now suppose that $x \leq -2$. Then

$$\begin{aligned}\frac{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} - 2}{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} + 2} &== \frac{-(x+1)^2(2-x) + (x^2-1)\sqrt{(2-x)(-2-x)}}{-(x-1)^2(-2-x) + (x^2-1)\sqrt{(2-x)(-2-x)}} \\ &= \frac{[(x+1)\sqrt{2-x}][-(x+1)\sqrt{2-x} + (x-1)\sqrt{-2-x}]}{[(1-x)\sqrt{-2-x}][(x-1)\sqrt{-2-x} - (x+1)\sqrt{2-x}]} \\ &= \frac{(x+1)\sqrt{2-x}}{(1-x)\sqrt{-2-x}} \ .\end{aligned}$$

Comment. Most solvers neglected to ensure that the quantities under the radical were nonnegative. This is the sort of "easy" question where many marks can be lost because of inattention to detail.

425. Let $\{x_1, x_2, \cdots, x_n, \cdots\}$ be a sequence of nonzero real numbers. Show that the sequence is an arithmetic progression if and only if, for each integer $n \geq 2$,

$$\frac{1}{x_1x_2} + \frac{1}{x_2x_3} + \cdots + \frac{1}{x_{n-1}x_n} = \frac{n-1}{x_1x_n} \ .$$

Solution. A constant sequence is an arithmetic progression and clearly satisfies the condition. Suppose that the sequence is a nonconstant arithmetic progression with common difference d. Then, for each positive index i, we have that

$$\frac{d}{x_ix_{i+1}} = \frac{x_{i+1} - x_i}{x_ix_{i+1}} = \frac{1}{x_i} - \frac{1}{x_{i+1}} \ .$$

Hence, for each $n \geq 2$,

$$\begin{aligned}d\sum_{i=1}^{n-1} \frac{1}{x_ix_{i+1}} &= \sum_{i=1}^{n-1}\left(\frac{1}{x_i} - \frac{1}{x_{i+1}}\right) \\ &= \left(\frac{1}{x_1} - \frac{1}{x_n}\right) = \frac{x_n - x_1}{x_1x_n} = \frac{(n-1)d}{x_1x_n} \ ,\end{aligned}$$

from which the condition follows.

On the other hand, suppose that the condition holds. Let $d = x_2 - x_1$, and suppose that we have established that $x_n - x_1 = (n-1)d$ (this is true for $n = 2$). Then, we have that

$$\frac{n}{x_1 x_{n+1}} - \frac{n-1}{x_1 x_n} = \frac{1}{x_n x_{n+1}}$$

so that

$$n x_n - (n-1)x_{n+1} = x_1 \Longrightarrow (n-1)(x_{n+1} - x_1) = n(x_n - x_1) = n(n-1)d$$

from which $x_{n+1} - x_1 = nd$. It follows, by induction, that the $\{x_n\}$ is an arithmetic progression.

426. (a) The following paper-folding method is proposed for trisecting an acute angle.

(1) transfer the angle to a rectangular sheet so that its vertex is at one corner P of the sheet with one ray along the edge PY; let the angle be XPY;

(2) fold up PY over QZ to fall on RW, so that $PQ = QR$ and $PY \| QZ \| RW$, with QZ between PY and RW;

(3) fold across a line AC with A on the sheet and C on the edge PY so that P falls on a point P' on QZ and R on a point R' on PX;

(4) suppose that the fold AC intersects the fold QZ at B and carries Q to Q'; make a fold along BQ'.

It is claimed that the fold BQ' passes through P and trisects angle XPY.

Explain why the fold described in (3) is possible. Does the method work? Why?

(b) What happens with a right angle?

(c) Can the method be adapted for an obtuse angle?

Solution. [F. Barekat] Let $\angle XPY = 3\theta$. Select B on QZ so that $\angle PBR = 4\theta$ and draw the circle with centre B that passes through P and R. Suppose that this circle intersects QZ at P' and XP at R'. Since $\angle PBR = 4\theta$, $\angle P'PY = \angle PP'Q = \frac{1}{2}\angle PP'R = \theta$. Also, $\angle P'RR' = \angle XPP' = \angle XPY - \angle P'PY = 2\theta$, so that $\angle R'RW = \angle P'RR' - \angle WRP' = \theta$.

Hence PP' and RR' are parallel chords in the circle, and their right bisectors pass through B and defines the required fold to interchange P and P', and R and R'. Since $BP = BP'$, $\angle BPY = \angle QBP = \angle BPP' + \angle BP'P = 2\angle BP'P = 2\theta$, so that $\angle BPR' = \theta$ and BP trisects the angle.

The fold (reflection) fixes B and interchanges P and P', and Q and Q'. Since P', B, Q are collinear, so are P, B, Q'. Hence the line through B and Q' also passes through P and so trisects the angle.

(b) When $\angle XPY = 90°$, then X lies on PR and R and R' coincide. We have that $PR = P'R = P'P$, so that triangle $PP'R$ is equilateral and RC is an altitude. Hence Q' is the midpoint of $P'R$ and $\angle Q'PP' = 30° = \angle P'PX$.

(c) One way to trisect an obtuse angle is to trisect its supplement, and subtract the result from $60°$..

427. The radius of the inscribed circle and the radii of the three escribed circles of a triangle are consecutive terms of a geometric progression. Determine the largest angle of the triangle.

Solution 1. [F. Barekat] Let r be the inradius and r_a, r_b, r_c the respective exradii of the circles touching the sides a, b, c. Let s be the semiperimeter and Δ the area of the triangle. Recall that

$$\Delta = rs = r_a(s-a) = r_b(s-b) = r_c(s-c)$$

[derive this], from which

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

so that r is the smallest term in the geometric progression. Suppose that $A \leq B \leq C$. Then

$$r_a = s\tan(A/2) \leq r_b = s\tan(B/2) \leq r_c = s\tan(C/2) \ .$$

Hence there is a number $t > 1$ for which $r_a = tr$, $r_b = t^2 r$, $r_c = t^3 r$ and $s - b = t(s - c)$, $s - a = t^2(s - c)$, $s = t^3 s$. By Heron's formula,

$$\begin{aligned} t^3 r(s-c) &= r_c(s-c) = rs = \Delta \\ &= \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{t^6(s-c)^4} = t^3(s-c)^2 \ . \end{aligned}$$

Hence $r = s - c$, and so $\tan(C/2) = r/(s - c) = 1$. Thus $C = 90°$.

Solution 2. [D. Dziabenko] Define the symbols as in the first solution. The equation $r_a^{-1}+r_b^{-1}+r_c^{-1} = r^{-1}$ leads to $t^3 = t^2 + t + 1$. Now

$$\begin{aligned} \Delta &= \frac{1}{2}ab\sin C = \frac{1}{2}[(s-b+s-c)(s-a+s-c)\sin C \\ &= \frac{1}{2}[(t+1)(t^2+1)(s-c)^2\sin C] = t^3(s-c)^2\sin C \ . \end{aligned}$$

On the other hand, by Heron's formula, we find that

$$\Delta = \sqrt{t^6(s-c)^4} = t^3(s-c)^2 \ .$$

Comparing the two expressions leads to $\sin C = 1$, so that $C = 90°$.

Solution 3. [G. Ghosn; A. Remorov] With $r \leq r_a \leq r_b \leq r_c$ as in the previous solutions and from the inverse proportionality of $r : r_a : r_b : r_c$ and $s : (s-a) : (s-b) : (s-c)$, we have that $(s-a)(s-b) = s(s-c)$, whence

$$ab = (a + b - c)s \Longrightarrow 2ab = (a + b)^2 - c^2 \Longrightarrow c^2 = a^2 + b^2 \ .$$

Hence the triangle is right, and its largest angle, C, is $90°$.

428. **a**, **b** and **c** are three lines in space. Neither **a** nor **b** is perpendicular to **c**. Points P and Q vary on **a** and **b**, respectively, so that PQ is perpendicular to **c**. The plane through P perpendicular to **b** meets **c** at R, and the plane through Q perpendicular to **a** meets **c** at S. Prove that RS is of constant length.

Solution. Let the point P on line **a** be given by $\mathbf{p} + s\mathbf{u}$ and the point Q on line **b** be given by $\mathbf{q} + t\mathbf{v}$, where $\mathbf{p}$, $\mathbf{q}$, $\mathbf{u}$ and $\mathbf{v}$ are fixed vectors and s, t parameters determining the points. Let $\mathbf{r}$ be a point on **c** and $\mathbf{w}$ be the direction vector for **c**. Wolog, we can normalize $\mathbf{u}$ and $\mathbf{v}$ so that $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 1$.

By hypothesis, $[(\mathbf{p} + s\mathbf{u}) - (\mathbf{q} + t\mathbf{v}] \cdot \mathbf{w} = 0$, whence $s - t = (\mathbf{q} - \mathbf{p}) \cdot \mathbf{w}$. Let R be given by $\mathbf{r} + m\mathbf{w}$ and S by $\mathbf{r} + n\mathbf{w}$. Then

$$[(\mathbf{p} + s\mathbf{u}) - (\mathbf{r} + m\mathbf{w})] \cdot \mathbf{v} = 0$$

so that $m = (\mathbf{p} - \mathbf{r}) \cdot \mathbf{v} + s(\mathbf{u} \cdot \mathbf{v})$. Also

$$[(\mathbf{q} + t\mathbf{v}) - (\mathbf{r} + m\mathbf{w})] \cdot \mathbf{u} = 0$$

so that $n = (\mathbf{q} - \mathbf{r}) \cdot \mathbf{u} + t(\mathbf{v} \cdot \mathbf{u})$. Hence

$$m - n = (\mathbf{p} - \mathbf{r}).\mathbf{v} - (\mathbf{q} - \mathbf{r}).\mathbf{u} + (\mathbf{u} \cdot \mathbf{v})[(\mathbf{q} - \mathbf{p}).\mathbf{w}$$

is independent of s and t and the result follows.

429. Prove that

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \binom{kn}{n} = (-1)^{n+1} n^n \ .$$

Solution 1. Let $f(x) = (1-(1+x)^n)^n$. Since $(1-(1+x)^n) = -nx + x^2 g(x)$, for some polynomial $g(x)$, we have that

$$f(x) = (-1)^n n^n x^n + x^{n+1} h(x)$$

for some polymonial $h(x)$. On the other hand,

$$\begin{aligned} f(x) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+x)^{nk} \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{j=0}^{nk} \binom{nk}{j} x^j \\ &= \sum_{j=0}^{n^2} \left[\sum_{k=\lceil j/n \rceil}^{n} (-1)^k \binom{n}{k} \binom{nk}{j} \right] x^j \ . \end{aligned}$$

Comparing the coefficients of x^n in the two expressions for $f(x)$ yields that $(-1)^n n^n = \sum_{k=1}^{n} (-1)^k \binom{n}{k}\binom{nk}{n}$, from which the desired result follows.

Solution 2. Consider a set of n^2 distinct objects arranged in a $n \times n$ square array. There are n^n ways of choosing n of them so that one is chosen from each row. We count this in a different way, using the Principle of Inclusion-Exclusion. Let $f(r)$ be the number of ways of selecting the n objects so that they come from at most r distinct rows. There are $\binom{n}{r}$ ways of selecting the r rows containing the objects, and rn objects to choose from. Hence $f(r) = \binom{n}{r}\binom{rn}{n}$. [Note that this doublecounts choices involving fewer than r rows.]

There are $f(n) = \binom{n^2}{n}$ ways of choosing n objects from the array without restriction. But this includes the $f(n-1)$ selections where they are drawn from at most $n-1$ rows. But then $f(n) - f(n-1)$ subtracts off those from $n-2$ rows twice, so we need to add $f(n-2)$ back. But then, in $f(n) - f(n-1) + f(n-2)$, we have added in each selection from $n-3$ rows $\binom{3}{3}$ times in $f(n)$, subtracted in $\binom{3}{2}$ times in $f(n-1)$, added it back $\binom{3}{1}$ times. So we need to add it back. Continuing in this way, we find that

$$n^n = \sum_{k=0}^{n} (-1)^{n-k} f(k) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{kn}{k}$$

from which the desired result follows.

Solution 4. [G. Ghosn] Let $P(x) = (x^n - 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^{kn} (-1)^{n-k}$. We calculate the nth derivative of the two expressions.

Recall Leibniz' Rule that $D^n(QR) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k}(Q) D^k(R)$. Taking $Q(x) = (x-1)^n$ and $R(x) = (x^{n-1} + x^{n-2} + \cdots + 1)^n$, we have that

$$\begin{aligned} D^n[(x^n - 1)^n] &= D^n[(x-1)^n (x^{n-1} + \cdots + x + 1)^n] \\ &= n!(x^{n-1} + \cdots + x + 1)^n + (x-1)S(x) \ , \end{aligned}$$

for some polynomial $S(x)$. When $x = 1$, this takes the value $n! n^n$.

On the other hand,

$$\begin{aligned} D^n \left[\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^{kn} \right] &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (kn)(kn-1)\cdots(kn-n+1) x^{kn-n} \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{(kn)!}{(kn-n)!} x^{kn-n} \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{kn}{n} n! x^{kn-n} \ . \end{aligned}$$

When $x = 1$, this takes the value $n! \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{kn}{n}$. The desired result follows.

430. Let triangle ABC be such that its excircle tangent to the segment AB is also tangent to the circle whose diameter is the segment BC. If the lengths of the sides BC, CA and AB of the triangle form, in this order, an arithmetic sequence, find the measure of the angle ACB.

Solution. Let M be the midpoint of BC, E the centre of the excircle tangent to AB and r its radius, T, S and K the respective points of tangency of this excircle to CB, CA and AB respectively, and P the point of tangency of the two circles of the problem. Following convention, let a, b, c be the sidelengths of the sides of triangle ABC and s its semiperimeter. We have that $|CS| = |CT| = s$, $|AK| = |AS| = s - b$ and $|BK| = |BT| = s - a$. Then

$$[ABC] = [ACE] + [BCE] - [ABE] = \frac{1}{2}rb + \frac{1}{2}ra - \frac{1}{2}rc = r(s-c) \ .$$

Consider the right triangle MET. We have that $|ME| = |MP| + |PE| = \frac{a}{2} + r$, $|MT| = |MB| + |BC| = \frac{a}{2} + s - a = s - \frac{a}{2}$ and $|ET| = r$, so that

$$\left(r + \frac{a}{2}\right)^2 = r^2 + \left(s - \frac{a}{2}\right)^2 \Longrightarrow ra = s(s-a) \ .$$

Using Heron's formula, we find that

$$r(s-c) = [ABC] = \sqrt{s(s-a)(s-b)(s-c)} \ ,$$

so that

$$\frac{s(s-a)(s-c)}{a} = \sqrt{s(s-a)(s-b)(s-c)} \Longrightarrow s(s-a)(s-c) = a^2(s-b) \ .$$

Since a, b, c are in arithmetic progression, for some number d, $a = b - d$, $c = b + d$ and $2s = 3b$, so that

$$\begin{aligned}
\frac{3b}{2}\left(\frac{b}{2} - d\right)\left(\frac{b}{2} + d\right) &= (b-d)^2\left(\frac{b}{2}\right) \\
&\Longrightarrow 3b(b^2 - 4d^2) = 4b(b-d)^2 \\
&\Longrightarrow 3b^2 - 12d^2 = 4b^2 - 8bd + 4d^2 \\
&\Longrightarrow 0 = b^2 - 8bd + 16d^2 = (b - 4d)^2 \ .
\end{aligned}$$

Hence $(a, b, c) = (3d, 4d, 5d)$ so that triangle ABC is right and $\angle C = 90°$.

431. Prove the following trigonometric identity, for any natural number n:

$$\sin \frac{\pi}{4n+2} \cdot \sin \frac{3\pi}{4n+2} \cdot \sin \frac{5\pi}{4n+2} \cdots \sin \frac{(2n-1)\pi}{4n+2} = \frac{1}{2^n} \ .$$

Solution 1. [F. Barekat] Let $\theta = \pi/(4n+2)$. For each integer m, we have that

$$\sin m\theta = \frac{\sin 2m\theta}{2\cos m\theta} = \frac{\sin 2m\theta}{2\sin(\frac{\pi}{2} - m\theta)} \ .$$

When $m = 2k - 1$ $(1 \leq k \leq n)$,

$$\frac{\pi}{2} - m\theta = \frac{\pi}{2}\left[1 - \frac{2k-1}{2n+1}\right] = \frac{\pi}{2}\left[\frac{2(n+1-k)}{2n+1}\right] = 2(n+1-k)\theta \ .$$

Hence

$$\begin{aligned}
\prod_{k=1}^{n} \sin \frac{(2k-1)\pi}{4n+2} &= \prod_{k=1}^{n} \sin(2k-1)\theta = \prod_{k=1}^{n} \frac{\sin 2(2k-1)\theta}{2\sin 2(n+1-k)\theta} \\
&= \frac{\prod_{k=1}^{n} \sin 2(2k-1)\theta}{\prod_{k=1}^{n} 2\sin 2(n+1-k)\theta} \\
&= \frac{1}{2^n} \frac{\prod_{k=1}^{n} \sin 2(2k-1)\theta}{\prod_{k=1}^{n} \sin 2k\theta} = \frac{1}{2^n} \prod_{k=1}^{n} \frac{\sin 2(2k-1)\theta}{\sin 2k\theta} \\
&= \frac{1}{2^n} \prod \left\{ \frac{\sin 2i\theta}{\sin 2i\theta} : i \enspace \text{odd}, 1 \le i \le n \right\} \cdot \prod \left\{ \frac{\sin 2(2n+i-j)\theta}{\sin 2j\theta} : j \enspace \text{even}, 2 \le j \le n \right\} .
\end{aligned}$$

Since $2(2n+1-j)\theta + 2j\theta = (4n+2)\theta = \pi$, all fractions in the products are equal to unity, and the required value is $1/2^n$.

Solution 2. We first illustrate the argument for $n = 3$. Consider a regular heptagon $ABCDEFG$ with $|AB| = a$, $|AC| = b$ and $|AD| = c$, Let M be the midpoint of DE so that AM right bisects DE as well as the parallel diagonals CF and BG. Being one-quarter of the angle subtended by a side at the centre of the circumcircle, $\angle DAM = \pi/14$. Since $\frac{1}{2}|BC| = |AB| \sin \angle BAM$, $\frac{1}{2}b = a \sin \frac{5\pi}{14}$. Since $\frac{1}{2}|CF| = |AC| \sin \angle CAM$, $\frac{1}{2}c = b \sin \frac{3\pi}{14}$. Since $\frac{1}{2}|DE| = |AD| \sin \angle DAM$, $\frac{1}{2}a = c \sin \frac{\pi}{14}$. Multiplying these equations yields that

$$\frac{1}{2^3} = \sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} .$$

In general, consider a regular $(2n+1)-$gon $AB_1B_2 \cdots B_nC_n \cdots C_2C_1$, with M the midpoint of B_nC_n. Observe that the segments B_kC_k $(1 \le k \le n)$ are all parallel. Suppose that $\theta = \pi/(4n+2)$. Then, for $1 \le k \le n$,

$$\angle B_kAM = \frac{[2n-(2k-1)]\pi}{4n+2} = [2n-(2k-1)]\theta = [2(n+1-k)-1]\theta .$$

Therefore,

$$\frac{1}{2}|B_kC_k| = |AB_k| \sin[2(n+1-k)-1]\theta .$$

The chord AB_k has $k-1$ vertices of the polygon on one side and $2n-k$ vertices on the other side, while the chord B_kC_k has $2k-1$ vertices of the polygon on one side and $2(n-k)$ vertices on the other. Hence $|AB_k| = |B_jC_j|$ where $j = \frac{1}{2}[2n+1-k]$ when k is odd and $j = \frac{1}{2}k$ when k is even. The sequence $\{|AB_k|\}$ is a permutation of the sequence $\{|B_kC_k|\}$. Hence

$$\begin{aligned}
\frac{1}{2^n} \prod_{k=1}^{n} |AB_k| &= \frac{1}{2^n} \prod_{k=1}^{n} |B_kC_k| = \prod_{k=1}^{n} \frac{1}{2}|B_kC_k| \\
&= \prod_{k=1}^{n} |AB_k| \sin[2(n+1-k)-1]\theta = \prod_{k=1}^{n} |AB_k| \prod_{k=1}^{n} \sin(2k-1)\theta ,
\end{aligned}$$

from which the result follows.

432. Find the exact value of:

(a)

$$\sqrt{\frac{1}{6} + \frac{\sqrt{5}}{18}} - \sqrt{\frac{1}{6} - \frac{\sqrt{5}}{18}} ;$$

(b)

$$\sqrt{1 + \frac{2}{5}} \cdot \sqrt{1 + \frac{2}{6}} \cdot \sqrt{1 + \frac{2}{7}} \cdot \sqrt{1 + \frac{2}{8}} \cdots \sqrt{1 + \frac{2}{57}} \cdot \sqrt{1 + \frac{2}{58}} .$$

Solution. (a)

$$\sqrt{\frac{1}{6}+\frac{\sqrt{5}}{18}} = \frac{1}{6}\sqrt{6+2\sqrt{5}} = \frac{1}{6}(1+\sqrt{5})$$

and

$$\sqrt{\frac{1}{6}-\frac{\sqrt{5}}{18}} = \frac{1}{6}\sqrt{6-2\sqrt{5}} = \frac{1}{6}(\sqrt{5}-1)\ ,$$

so that the difference is equal to $1/3$.

(b) For each n, $\sqrt{1+(2/n)} = \sqrt{(n+2)/n}$, so that the product is equal to

$$\sqrt{\frac{7}{5}\cdot\frac{8}{6}\cdot\frac{9}{7}\cdots\frac{59}{57}\cdot\frac{60}{58}} = \sqrt{\frac{59}{5}\cdot\frac{60}{6}} = \sqrt{118}\ .$$

433. Prove that the equation

$$x^2+2y^2+98z^2 = 77777\ldots777$$

does not have a solution in integers, where the right side has 2006 digits, all equal to 7.

Solution. Since, modulo 7, squares have the values 0, 1, 2, 4, $x^2+2y^2 \equiv 0 \pmod 7$ implies that $x \equiv y \equiv 0 \pmod 7$, whence $x = 7u$ and $y = 7v$ for some integers u and v. Hence

$$0 \equiv 7u^2+14v^2+14z^2 = 11111\ldots111$$

(mod 7). However $111111 = 7 \times 15873 \equiv 0 \pmod 7$, and $2006 = 6 \times 334 + 2$. Thus,

$$11111\ldots111 = 11 + 111111(10^2+10^8+\cdots+10^{2000}) \equiv 11 \equiv 4$$

(mod 7), and we arrive at a contradiction.

434. Find all natural numbers n for which $2^n + n^{2004}$ is equal to a prime number.

Solution. Let $N = 2^n + n^{2004}$. If n is even, then N is even and composite. Let n be odd and not a multiple of 3. Then $N \equiv 2^n + (n^2)^{1003} \equiv 2+1 \equiv 0 \pmod 3$. When $n = 1$, $N = 3$ and is prime, while when $n > 1$, N exceeds 3 and is composite.

Finally, let n be a multiple of 3. Then $n = 3k$ for some integer k and $N = (2^k)^3 + (n^{668})^3$ is properly divisible by $2^k + n^{668}$. Hence N is prime exactly when $n = 1$.

435. A circle with centre I is the incircle of the convex quadrilateral $ABCD$. The diagonals AC and BD intersect at the point E. Prove that, if the midpoints of the segments AD, BC and IE are collinear, then $AB = CD$.

Solution. Let M be the midpoint of AD and N the midpoint of BC. Since the mindpoints of AD, BC and IE are collinear, I and E are on opposite sides of MN. Wolog, let I lie inside $AMNB$ and E lie inside $DMNC$.

We have that

$$\begin{aligned}[MIN] &= [AMNB]-[ABI]-[AMI]-[BNI]\\ &= [AMNB]-[ABI]-\frac{1}{2}([ADI]+[BCI])\\ &= [AMNB]-\frac{1}{2}[ABI]-\frac{1}{2}([ABI]+[ADI]+[BCI])\\ &= [AMNB]-\frac{1}{2}[ABI]-\frac{1}{2}([ABCD]-[CDI])\ .\end{aligned}$$

Similarly,

$$[MEN] = [DMNC] - \frac{1}{2}[DCE] - \frac{1}{2}([ABCD] - [ABE]) \ .$$

Since MN bisects IE, I and E are equidistant from MN and $[MIN] = [MEN]$. Now

$$[AMNB] = [AMN] + [ABN] = \frac{1}{2}[AND] + \frac{1}{2}[ABC]$$

and

$$[DMNC] = [MDN] + [DNC] = \frac{1}{2}[AND] + \frac{1}{2}[DBC] \ .$$

Hence

$$[AND] + [ABC] - [ABI] - [ABCD] + [CDI] = [AND] + [DBC] - [DEC] - [ABCD] + [ABE] \ ,$$

whence

$$[ABC] + [CDI] + [DEC] = [DBC] + [ABE] + [ABI] \ .$$

Since

$$[ABC] + [DEC] = [ABCDE] = [DBC] + [ABE] \ ,$$

$[CDI] = [ABI]$. But I is equidistant from CD and AB whence $AB = CD$,

436. In the Euro-African volleyball tournament, there were nine more teams participating from Europe than from Africa. In total, the European won nine times as many points as were won by all of the African teams. In this tournamet, each team played exactly once against each other team; there were no ties; the winner of a game gets 1 point, the loser 0. What is the greatest possible score of the best African team?

Solution. Let a be the number of teams from Africa, so that $a+9$ is the number of teams from Europe. Supose that there were k African wins over Europeans. Then the total number of points taken by the African teams is $\binom{a}{2} + k = \frac{1}{2}a(a-1) + k$, while the Europeans won

$$\binom{a+9}{2} + [a(a+9) - k] = \frac{(a+9)(3a+8)}{2} - k \ .$$

By the given conditions,

$$\frac{(a+9)(3a+8)}{2} - k = 9\left[\frac{a(a-1)}{2} + k\right] \ ,$$

which simplifies to $3a^2 - 22a + (10k - 36) = 0$. This is a quadratic equation in a with discriminant equal to $916 - 120k = 4(229 - 30k)$. There are integer values of a satisfying the quadratic only if $229 - 30k$ is square with $k > 0$. Thus, $k = 2$ or $k = 6$.

When $k = 2$, $0 = 3a^2 - 22a - 16 = (a-8)(3a+2)$ so $a = 8$. In this case, there are 8 African teams, and any of these teams can get at most $7 + 2 = 9$ points, which can occur when one African team vanquishes all the other African teams as well as two European teams.

When $k = 6$, $0 = 3a^2 - 22a + 24 = (a-6)(3a-4)$ so $a = 6$. In this case, there are 6 African teams, and any of these teams can get at most $5 + 6 = 11$ points, which can occur when one African team vanquishes all the other African teams as well as six European teams.

Thus the greatest possible score for an African team is 11.

437. Let a, b, c be the side lengths and m_a, m_b, m_c the lengths of their respective medians, of an arbitrary triangle ABC. Show that

$$\frac{3}{4} < \frac{m_a + m_b + m_c}{a + b + c} < 1 \ .$$

Furthermore, show that one cannot find a smaller interval to bound the ratio.

Solution. We use the property that the intersection of the medians trisects the medians. From the triangle inequality, we obtain

$$a < \frac{2}{3}m_b + \frac{2}{3}m_c \qquad b < \frac{2}{3}m_c + \frac{2}{3}m_a \qquad c < \frac{2}{3}m_a + \frac{2}{3}m_b \ .$$

Summing these inequalities and manipulating gives the left inequality.

For the right inequality, extend a median, say m_b to the same length on the other side of AC. Joining A and C to the endpoint D of the extended median gives a parallellogram $ABCD$ with diagonal AD of length $2m_b$ and sides a and c. Do the same with the other two medians. From the triangle inequality, we have that

$$2m_b < a + c \qquad 2m_c < a + b \qquad 2m_a < b + c \ .$$

Summing the inequalities will lead to the right inequality.

To see that the inequality cannot be improved, consider the isosceles triangle ABC with sides AB and AC of length 1 and angle A equal to 2θ, where $0 < \theta < 90°$. Then

$$(a, b, c) = (2\sin\theta, 1, 1)$$

and

$$(m_a, m_b, m_c) = (\cos\theta, f(\theta), f(\theta))$$

where $f(\theta) = \frac{1}{2}\sqrt{5 - 4\cos 2\theta}$. Observe that

$$\lim_{\theta\to 0} f(\theta) = \frac{1}{2} \qquad \text{and} \qquad \lim_{\theta\to 90°} f(\theta) = \frac{3}{2} \ .$$

We have that

$$\frac{m_a + m_b + m_c}{a + b + c} = \frac{\cos\theta + 2f(\theta)}{2(1 + \sin\theta)} \ .$$

When θ is close to $0°$, this ratio is close to 1, and when θ is close to $90°$, the ratio is close to $3/4$.

438. Determine all sets (x, y, z) of real numbers for which

$$x + y = 2 \qquad \text{and} \qquad xy - z^2 = 1 \ .$$

Solution. From the second equation, $z^2 = xy - 1 = x(2 - x) - 1 = -(x - 1)^2$. Since squares are nonnegative, we must have $z = 0 = (x - 1)$, so that $(x, y, z) = (1, 1, 0)$.

439. A natural number n, less than or equal to 500, has the property that when one chooses a number m randomly among $\{1, 2, 3, \cdots, 500\}$, the probability that m divides n (*i.e.*, n/m is an integer) is $1/100$. Find the largest such n.

Solution. The number n must have 5 divisors. If the prime factorization of n is $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, then n has $(\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_k + 1)$ divisors. To obtain five divisors, n must have the form p^4 for some prime p. The largest such n is therefore $n = 3^4 = 81$, as $5^4 > 500$.

440. You are to choose 10 distinct numbers from $\{1, 2, 3, \cdots, 2006\}$. Show that you can choose such numbers with a sum greater than 10039 in more ways than you can choose such numbers with a sum less than 10030.

Solution. Let M be the set $\{1, 2, 3, \cdots, 2006\}$, let S be the set of 10−tuples of distinct elements of M with sum less than 10030, and L be the set of 10−tuples of distinct elements of M with sum greater than 10039. Define a function f on S by

$$f(a_1, a_2, \cdots, a_{10}) = (2007 - a_1, 2007 - a_2, \cdots, 2007 - a_{10}) \ .$$

Each $f(a_1, a_2, \cdots, a_{10})$ consists of ten distinct numbers in M and the sum of the numbers is

$$10 \cdot 2007 - \sum_{i=1}^{10} a_i > 27000 - 10030 = 10040 \ .$$

Hence the range of f is a subset of L. Since $(999, 1000, 1001, 1002, 1003, 1005, 1006, 1007, 1008, 1009)$ is in L, but not in the range of f, the range of f is a proper subset of L. As f is injective and the sets are finite, it follows that S has fewer elements than L.

441. Prove that, no matter how 15 points are placed inside a circle of radius 2 (including the boundary), there exists a circle of radius 1 (including the boundary) containing at least 3 of the 15 points.

Solution. We will cover the circle of radius 2 entirely with seven circles of radius 1. By the Pigeonhole Principle, at least one of these circles will contain at least three of the fifteen points.

Construct the circle of radius 1 concentric with the circle of radius 2; denote the centre of these circles by O. For each of the six $60°$−sectors, construct a circle as follows. Let A and B be on the inner circle with $\angle AOB = 60°$, and let OA and OB produced meet the outer circle at C and D respectively; let E be the midpoint of CD. The following triangles are equilaterial with side length 1: BOA, EAC, EBA, EDB. The inner circle and the circle with centre E and radius 1 together cover the sector COD.

442. Prove that the regular tetrahedron has minimum diameter among all tetrahedra that circumscribe a given sphere. (The diameter of a tetrahedron is the length of its longest edge.)

Solution. Let T be the tetrahedron with volume V and surface area S; suppose that r is the radius of the sphere inscribed within T. Let one vertex of T be at the origin and let the other three vertices and the centre of the sphere be given by the position vectors A, B, C and P, respectively. Then $P = \alpha A + \beta B + \gamma C$, with $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma < 1$.

Suppose that $A \times B$ points into T. Then, since P can be written as the orthogonal sum of a vector in the plane of A and B and a vector of length r perpendicular to this plane from the point of tangency of the insphere,

$$r|A \times B| = P \cdot (A \times B) = \gamma C \cdot (A \times B) = 6\gamma V \ ,$$

and likewise

$$r|B \times C| = P \cdot (B \times C) = 6\alpha V \ ,$$

$$r|C \times A| = P \cdot (C \times A) = 6\beta V \ ,$$

$$r|(C - A) \times (B - A)| = (P - A) \cdot ((C - A) \times (B - A)) \ .$$

Since $P - A = (\alpha + \beta + \gamma - 1)A + \beta(B - A) + \gamma(C - A)$,

$$\begin{aligned}(P - A) \cdot ((C - A) \times (B - A)) &= (\alpha + \beta + \gamma - 1)A \cdot ((C - A) \times (B - A)) \\ &= (1 - \alpha - \beta - \gamma)(O - A) \cdot ((C - A) \times (B - A) \\ &= 6(1 - \alpha - \beta - \gamma)V \ .\end{aligned}$$

The total surface area S is equal to

$$\frac{1}{2}[|A \times B| + |B \times C| + |C \times A| + |(C - A) \times (B - A)|] = 3V/r$$

so that $r = 3V/A$. The triangle of given perimeter with maximum area is equilateral and (it is possible to show that) the tetrahedron of given surface area with maximum volume is regular. The desired result follows immediately from the last formula.

443. For $n \geq 3$, show that $n-1$ straight lines are sufficient to go through the interior of every square of an $n \times n$ chessboard. Are $n-1$ lines necessary?

Solution. Let the corners of the board be $(-n, n)$, $(0, n)$, $(-n, 0)$, $(0, 0)$. Draw $n-2$ lines L_i with slope $1/2$, the first intersecting the base line at $(-9/2, 0)$, and the rest spaced so that the vertical distance between the lines is $3/2$ units. L_i has equation $y = \frac{1}{2}x + \frac{9}{4} + \frac{3i}{2}$, for $i = 0, 1, 2, \cdots, n-3$. (L_{i+1} is obtained by shifting L_i one unit to the left and then one unit up.) L_1 goes through the points $(-9/2, 0)$, $(-4, 1/4)$, $(-3, 3/4)$, $(-5/2, 1)$, $(-2, 5/4)$, $(-1, 7/4)$, $(-1/2, 2)$, $(0, 9/4)$. L_{n-3} goes through the point $(-n + 9/2, n)$. (Thus, the arrangement is symmetric under 180 degree rotation.) Leaving out the two rightmost squares in the lowest row an the two leftmost squares in the top row, every unit square has an interior point on one or other of the lines. The $(n-1)$th line can be drawn to pass through the interiors of the four squares.

To prove necessity, we need an inductive argument. Let there be m rows and n columns with $m \leq n$, and let $f(m, n)$ be the minimum number of lines needed to cover an $m \times n$ board, in the sense that some interior point of every unit square is on one or another of the lines. Since any line going from one side to the other with ends in adjacent rows crosses both squares in a column exactly once, $f(1, n) = 1$, $f(2, n) = 2$, $f(n-1, n) \leq n-1$. It can be shown that $f(3, 4) = 3$, $f(4, 5) = 4$, $f(5, 6) = 5$. Thus, $f(3, n) = 3$, $f(4, n) = 4$, $f(5, n) = 5$. If $f(n-1, n) = n-1$, then $f(m, n)$ is determined for all m, n: $f(n-k, n) = n-k$, if $0 < k < n$, $n \geq 3$; $f(n, n) = n-1$.

444. (a) Suppose that a 6×6 square grid of unit squares (chessboard) is tiled by 1×2 rectangles (dominoes). Prove that it can be decomposed into two rectangles, tiled by disjoint subsets of the dominoes.

(b) Is the same thing true for an 8×8 array?

(c) Is the same thing true for a 6×8 array?

Solution. (a) There are 18 dominoes and 10 interior lines in the grid. For the decomposition not to occur, each of the lines must be straddled by at least one domino. We argue that, in fact, at least two dominos must straddle each line. Since no domino can straddle more than one line, this would require 20 dominos and so yield a contradiction.

Each interior line has six segments. For a line next to the side of the square grid, an adjacent domino between it and the side must either cross one segment or be adjacent to two segments. Since the number of segments is even, evenly many dominos must cross a segment. For the next line in, an adjacent domino must be adjacent to two segments, be adjacent to one segment and cross the previous line, or cross one segment. Since the number of dominoes straddling the previous line is even, there must be evenly many that cross the segment. In this way, we can work our way from one line to the next.

Comment. F. Barekat had the following argument. Consider a subrectangle determined by one interior line. It contains an even number of unit squares. Since each domino covers two unit squares, there must be an even number of unit squares belonging to dominoes that straddle the internal line.

(b) Number the squares in the grid by pairs ij of digits where the square is in the ith row and jth column. Here is a tiling with dominos in which each interior line is straddled and no decomposition into subrectangles is possible:

$$(11-12), (13-14), (15-16), (17-27), (18-28), (21-31), (22-32), (23-33),$$
$$(24-34), (25-26), (35-45), (36-46), (37-38), (41-42), (43-44), (47-57),$$
$$(48-58), (51-61), (52-53), (54-55), (56-66), (62-72), (63-73), (64-74),$$
$$(65-75), (67-68), (71-81), (76-77), (78-88), (82-83), (84-85), (86-87)\ .$$

(c) [J. Schneider; C. Sun] Using a similar notation as in (b), we have the example for which no decomposition into subrectangles is possible. Note that there are 12 interior lines and 24 dominoes, so that, for each example, each interior line is straddled by exactly two dominoes.

$$(11-21),(12-13),(14-15),(16-26),(17-18),(22-32),(23-24),(25,35),$$

$$(27-28),(31-41),(33-34),(36-37),(38-48),(42-43),(44-54),(45-46),$$

$$(47-57),(51-52),(53-63),(55-56),(58-68),(61-62),(64-65),(66-67)\ .$$

Here are two coverings that exhibit symmetry due respectively to P. Chen and K. Huynh. The first is

$$(11-21),(12-22),(13-14),(15-16),(17-18),(23-33),(24-25),(26-27),$$

$$(28-38),(31-32),(34-44),(35-45),(36-37),(41-51),(42-43),(46-56),$$

$$(47-48),(52-53),(54-55),(57-67),(58-68),(61-62),(63-64),(65-66)\ .$$

The second is

$$(11-21),(12-13),(14-15),(16-17),(18-28),(22-32),(23-24),(25-26),$$

$$(27-37),(31-41),(33-34),(35-36),(38-48),(42-43),(44-54),(45-55),$$

$$(46-47),(51-52),(53-63),(56-66),(57-58),(61-62),(64-65),(67-78)\ .$$

445. Two parabolas have parallel axes and intersect in two points. Prove that their common chord bisects the segments whose endpoints are the points of contact of their common tangent.

Solution 1. Wolog, we may assume that the parabolas have the equations $y = ax^2$ and $y = b(x-1)^2 + c$. The common chord has equation

$$a[b(x-1)^2 + c - y] - b[ax^2 - y] = 0\ ,$$

or

$$(a-b)y + 2abx - a(b+c) = 0\ . \tag{1}$$

Consider a point (u, au^2) on the first parabola. The tangent at this point has equation $y = 2aux - au^2$. The abscissa of the intersection point of this tangent with the parabola of equation $y = b(x-1)^2 + c$ is given by the question

$$bx^2 - 2(b+au)x + (au^2 + b + c) = 0\ .$$

This has coincident roots if and only if

$$(b+au)^2 = b(au^2 + b + c) \Longleftrightarrow a(a-b)u^2 + 2abu - bc = 0\ . \tag{2}$$

In this situation, the coincident roots are $x = 1 + (au)/b$ and the point of contact of the common tangent with the second parabola is

$$\left(1 + \frac{au}{b}, \frac{a^2u^2}{b} + c\right)\ .$$

The midpoint of the segment joining the two contact points is

$$\left(\frac{b + au + bu}{2b}, \frac{abu^2 + a^2u^2 + bc}{2b}\right)\ .$$

Plugging this into the left side of (1) and using (2) yields that

$$\begin{aligned}[1/(2b)][(a-b)a(a+b)u^2 + (a-b)bc + 2ab^2 + 2ab(a+b)u - 2ab(b+c)]\\ = [(a+b)/(2b)][a(a-b)u^2 + 2abu - bc] = 0 \ .\end{aligned}$$

Thus, the coordinates of the midpoint of the segment satisfy (1) and the result follows.

Solution 2. [A. Feizmohammadi] Let the two parabolas have equations $y = ax(x-u)$ and $y = bx(x-v)$. Since the two parabolas must open the same way for the situation to occur, wolog, we may suppose that $a, b > 0$. The parabolas intersect at the points $(0,0)$ and $((au-bv)/(a-b), (ab(au-bv)(u-v)/(a-b)^2)$, and the common chord has equation $(a-b)y - ab(u-v)x = 0$.

Let $y = mx+k$ be the equation of the common tangent. Then both of the equations $ax^2-(au+m)x-k = 0$ and $bx^2 - (bv+m)x - k = 0$ have double roots. Therefore $(au+m)^2 + 4ak = (bv+m)^2 + 4bk = 0$, from which (by eliminating k),

$$ab(au^2 - bv^2) + 2ab(u-v)m + (b-a)m^2 = 0 \ .$$

The common tangent of equation $y = mx + k$ touches the first parabola at

$$\left(\frac{au+m}{2a}, \frac{m^2 - a^2u^2}{4a}\right)$$

and the second parabola at

$$\left(\frac{bv+m}{2b}, \frac{m^2 - b^2v^2}{4b}\right) \ .$$

The midpoint of the segment joining these two points is

$$\left(\frac{ab(u+v) + (a+b)m}{4ab}, \frac{(a+b)m^2 - ab(au^2+bv^2)}{8ab}\right) \ .$$

Using these coordinates as the values of x and y, we find that

$$\begin{aligned}8ab[(a-b)y - ab(u-v)x] &= (a-b)[(a+b)m^2 - ab(au^2+bv^2)] - 2a^2b^2(u-v)(u+v)\\ &\quad - 2ab(a+b)(u-v)m\\ &= (a^2-b^2)m^2 - 2ab(a+b)(u-v)m\\ &\quad - [(a-b)a^2bu^2 + (a-b)ab^2v^2 + 2a^2b^2u^2 - 2a^2b^2v^2]\\ &= (a^2-b^2)m^2 - 2ab(a+b)(u-v)m - [a^3bu^2 + a^2b^2u^2 - a^2b^2v^2 - ab^3v^2]\\ &= (a+b)[(a-b)m^2 - 2ab(u-v)m] - [ab(a(a+b)u^2 - ab(b(a+b)v^2]\\ &= (a+b)[(a-b)m^2 - 2ab(u-v)m - ab(au^2 - bv^2)] = 0 \ .\end{aligned}$$

Solution 3. [J. Kileel] We may assume that both parabolas have vertical axes and that one has equation $y = x^2$. The second has an equation of the form $y = ax^2 + bx + c$, where $a > 0$ and $a \neq 1$. (The latter ensures two points of intersection.)

The equation of the chord through the points (d, d^2) and (e, e^2) is $y = (d+e)x - de$. The abscissae x_1 and x_2 of the intersection points of the two parabolas are the roots of the quadratic $(a-1)x^2 + bx + c = 0$, so that

$$x_1 + x_2 = \frac{-b}{a-1} \qquad \text{and} \qquad x_1x_2 = \frac{c}{a-1} \ .$$

The line passing through the points (x_1, x_1^2) and (x_2, x_2^2) is

$$(1-a)y = bx + c \ . \tag{1}$$

The equation of a line tangent to the first parabola at (u, u^2) is

$$y = 2ux - u^2 \ , \tag{2}$$

and to the second parabola at $(v, av^2 + bv + c)$ is

$$y = (2av + b)x + (c - av^2) \ . \tag{3}$$

For the common tangent, these two equations are identical, whence

$$2u = 2av + b \tag{4}$$

and

$$u^2 = av^2 - c \ . \tag{5}$$

Eliminating v from (4) and (5) yields

$$4(1-a)u^2 - 4bu - 4ac + b^2 = 0 \ . \tag{6}$$

The chord and common tangent intersect at the point

$$\left(\frac{au^2 - u^2 - c}{2au - 2u + b}, -\frac{2uc + bu^2}{2au - 2u + b} \right) \ .$$

(Solve (1) and (2).) The midpoint of the segment of the common tangent joining the two points of tangency has, by (4), the abscissa,

$$\frac{u+v}{2} = \frac{2au + 2u - b}{4a} \ .$$

Now

$$\begin{aligned} \frac{au^2 - u^2 - c}{2au - 2u + b} &- \frac{2au + 2u - b}{4a} \\ &= \frac{4a^2u^2 - 4au^2 - 4ac - 4a^2u^2 + (2u-b)^2}{4a(2au - 2u + b} \\ &= \frac{4(1-a)u^2 - 4bu - 4ac + b^2}{4a(2au - 2u + b)} = 0 \ , \end{aligned}$$

by (6). Hence the abscissae of the intersection point and the midpoint of the common tangent are equal and the result follows.

Comment. This can be solved using projective geometry, as it holds for any conic. To capture the idea of midpoint, we use harmonic range involving points at infinity.

446. Suppose that you have a 3×3 grid of squares. A *line* is a set of three squares in the same row, the same column or the same diagonal; thus, there are eight lines.

Two players A and B play a game. They take alternate turns, A putting a 0 in any unoccupied square of the grid and B putting a 1. The first player is A, and the game cannot go on for more than nine moves. (The play is similar to noughts-and-crosses, or tictactoe.) A move is *legitimate* if it does not result in two lines of squares being filled in with different sums. The winner is the last player to make a legitimate move.

(For example, if there are three 0s down the diagonal, then B can place a 1 in any vacant square provided it completes no other line, for then the sum would differ from the diagonal sum. If there are two zeros at the top of the main diagonal and two ones at the left of the bottom line, then the lower right square cannot be filled by either player, as it would result in two lines with different sums.)

(a) What is the maximum number of legitimate moves possible in a game?

(b) What is the minimum number of legitimate moves possible in a game that would not leave a legitimate move available for the next player?

(c) Which player has a winning strategy? Explain.

Solution. (a) A game cannot continue to nine moves. Otherwise, the line sum must be three times the value on the centre square of the grid (why?) and so must be 0 or 3. But some line must contain both zeros and ones, yielding a contradiction. [An alternative argument is that, if the array is filled, not all the rows can have the same numbers of 0s and 1s, and therefore cannot have the same sums.] However, an 8-move grid is possible, in which one player selects the corner squares and the other the squares in the middle of the edges.

(b) After three or fewer moves have occurred, there are at most three additional squares that would complete a line and the next player can avoid all of these. Consider any game after four moves have occurred and it is A's turn to play a zero. Suppose, first of all, that no lines have been filled with numbers. The only way an inaccessible square can occur is if it is the intersection of two lines each having the other two squares filled in. This can happen in at most one way. So A would have at least four possible squares to fill in. On the other hand, if three of the first four moves complete a line, the fourth number can bar at most three squares for A in the three lines determined by the fourth number and one of the other three. Thus, A would have at least two possible positions to fill. Thus, a game must go to at least five moves.

A five-move game can be obtained when A has placed three 0's down the left column and B has 1 in the centre square and another square of the middle column. Each remaining position is closed to B as it would complete a line whose sum is not 0. Other configurations where no further move is possible are the one in which the top row starts with two 1s and the middle row has three 0s, and where the top row has three 0s and the bottom row has a 1 at each end.

(c) A has a winning strategy. A places 0 in the middle square. For the next four moves, he plays symmetrically, completing the line through the centre initiated by B. After the fifth move, up to rotation and reflection, there are four possible configurations (where * denotes a vacant square, and the rows are listed from left to right):

$$\alpha : (1, 1, *{/}*, 0, *{/}*, 0, 0)$$

$$\beta : (1, *, 1{/}*, 0, *{/}0, *, 0)$$

$$\gamma : (1, *, *{/}0, 0, 1{/}*, *, 0)$$

$$\delta : (*, 1, *{/}0, 0, 1{/}*, 0, *)$$

In the case of α, A can respond symmetrically to B if B plays in the middle row, and can achieve

$$(1, 1, *{/}*, 0, 0{/}1, 0, 0)$$

if B plays in the bottom row. B cannot move. In the case of β, B has only one move and A can counter by a move in the middle row. As for γ and δ, A can respond respond symetrically to B and then B has no further move.

447. A high school student asked to solve the surd equation

$$\sqrt{3x-2} - \sqrt{2x-3} = 1$$

gave the following answer: *Squaring both sides leads to*

$$3x - 2 - 2x - 3 = 1$$

so $x = 6$. The answer is, in fact, correct.

Show that there are infinitely many real quadruples (a, b, c, d) for which this method leads to a correct solution of the surd equation

$$\sqrt{ax - b} - \sqrt{cx - d} = 1 \ .$$

Solution 1. Solving the general equation properly leads to

$$\sqrt{ax - b} - \sqrt{cx - d} = 1 \Longrightarrow ax - b = 1 + cx - d - 2\sqrt{cx - d}$$

$$\Longrightarrow (a - c)x = (b + 1 - d) - 2\sqrt{cx - d} \ .$$

To make the manipulation simpler, specialize to $a = c + 1$ and $d = b + 1$. Then the equation becomes

$$x^2 = 4(cx - d) \Longrightarrow 0 = x^2 - 4cx + 4d \ .$$

Using the student's "method" to solve the same equation gives $ax - b - cx - d = 1$ which yields $x = (1 + b + d)/(a - c) = 2d$. So, for the "method" to work, we need

$$0 = 4d^2 - 8cd + 4d = 4d(d - 2c + 1)$$

which can be achieved by making $2c = d + 1$. So we can take

$$(a, b, c, d) = (t + 1, 2(t - 1), t, 2t - 1)$$

for some real t. The original problem corresponds to $t = 2$.

The equation

$$\sqrt{(t + 1)x - 2(t - 1)} - \sqrt{tx - (2t - 1)} = 1$$

is satisfied by $x = 2$ and $x = 4t - 2$. The first solution works for all values of t, while the second is valid if and only if $t \geq \frac{1}{2}$. The equation $(t + 1)x - 2(t - 1) - tx - (2t - 1) = 1$ is equivalent to $x = 4t - 2$.

Solution 2. [G. Goldstein] *Analysis.* We want to solve simultaneously the equations

$$\sqrt{ax - b} - \sqrt{cx - d} = 1 \tag{1}$$

and

$$ax - b - cx - d = 1 \ . \tag{2}$$

From (1), we find that

$$ax - b = 1 + (cx - d) + 2\sqrt{cx - d} \ . \tag{3}$$

From (2) and (3), we obtain that $d = \sqrt{cx - d}$, so that $x = (d^2 + d)/c$. From (2), we have that $x = (1 + b + d)/(a - c)$.

Select a, c, d so that $d > 0$ and $ac(a - c) \neq 0$, and choose b to satisfy

$$\frac{d^2 + d}{c} = \frac{1 + b + d}{a - c} \ .$$

Let

$$x = \frac{d^2 + d}{c} = \frac{1 + b + d}{a - c} = \frac{(d^2 + d) + (1 + b + d)}{c + (a - c)} = \frac{(d + 1)^2 + b}{a} \ .$$

Then

$$\sqrt{ax - b} - \sqrt{cx - d} = \sqrt{(d + 1)^2} - \sqrt{d^2} = (d + 1) - d = 1$$

and

$$ax - b - cx - d = (d + 1)^2 + b - b - d^2 - d - d = 1 \ .$$

Comments. In Solution 2, if we take $c = d = 1$, we get the family of parameters $(a, b, c, d) = (a, 2a - 4, 1, 1)$. R. Barrington Leigh found the set of parameters given in Solution 1. A. Feizmohammadi and Y. Wang provided the parameters $(a, b, c, d) = (2c, 0, c, 1)$. J. Schneider took $(a, b, c, d) = (n, n - 4, 2, 1)$, with both equations satisfied by $x = 1$. A. Tavakoli had the paramatrization $(a, b, c, d) = (2c, b, c, \sqrt{1+b}) = (2c, d^2 - 1, c, d)$, with $c \neq 0$ and $b > -1$. A. Remorov offered $(a, b, c, d) = (a, 1, 2a/5, 1)$ with $a \neq 0$ and $x = 5/a$, while D. Shi offered $(a, b, c, d) = (d + 2, d + 1, d, d)$ with $d > 0$. But C. Sun had the simplest family of all with $(a, b, c, d) = (a, b, 0, 0)$ with $a > 0$ and $b > 0$.

If $a = c$, then $ax - b - cx - d = 1$ is satisfied by any value of x as long as $1 + b + d = 0$. The surd equation becomes

$$\sqrt{ax - b} - \sqrt{ax + (b + 1)} = 1$$
$$\Longrightarrow ax - b = ax + (b + 1) + 1 + 2\sqrt{ax + (b + 1)}$$
$$\Longrightarrow b + 1 = -\sqrt{ax + (b + 1)}$$
$$\Longrightarrow (b + 1)^2 = ax + (b + 1) \Longrightarrow x = \frac{(b + 1)b}{a} \ .$$

Since $ax - b = b^2$ and $ax + (b + 1) = (b + 1)^2$, we should take $b \leq -1$ in order to satisfy the equation. This is a singular case in which the linear equation has infinitely many solutions and the surd equation either zero or one solution.

448. A criminal, having escaped from prison, travelled for 10 hours before his escape was detected. He was then pursued and gained upon at 3 miles per hour. When his pursuers had been 8 hours on the way, they met an express (train) going in the opposite direction at the same rate as themselves, which had met the criminal 2 hours and 24 minutes earlier. In what time from the beginning of the pursuit will the criminal be overtaken? [from *The high school algebra* by Robertson and Birchard, approved for Ontario schools in 1886]

Solution 1. It will take 20 hours to catch the criminal, so that he is at large for 30 hours. Let t be the time in hours from the time the pursuit begins until the time of capture, and let x be the speed of the criminal in miles per hour. Then

$$(10 + t)x = t(x + 3) \Longrightarrow 10x = 3t \ .$$

Consider the situation 2.4 hours before the pursuers met the express. The distance between the pursuers and criminal is the distance the pursuers can travel in 4.8 hours, namely $4.8(x + 3)$ miles (note that the train and pursuers travel equal distances during the 2.4 to a common meeting point). Since the relative speed of the pursuers relative to the criminal is 3 miles per hour, it will take the pursuers an additional $(1/3)(4.8)(x + 3) = 1.6(x + 3)$ hours to close in. Hence

$$t = (8 - 2.4) + 1.6(x + 3) \Longrightarrow t = 10.4 + 1.6x \ .$$

Since $x = 0.3t$, it follows that $104 = 0.52t$ and $t = 20$.

Solution 2. We note that the information that the pursuers travel 3 miles per hour faster than the criminal turns out to be redundant. Let u be the speed of the criminal and v be the speed of the pursuers and of the freight train. The distance from the prison to the place where the freight train encountered the criminal is

$$8v + (2.4)v = (18 - 2.4)u = 15.6u \Longrightarrow 10.4v = 15.6u \Longrightarrow 2v = 3u \ .$$

After 8 hours of pursuit, the distance between the criminal and his pursuers is

$$18u - 8v = t(v - u)$$

for some value of t. This equation reduces to

$$(18 + t)u = (t + 8)v$$

which with the earlier equation yields $t = 12$. Thus after 8 hours, the distance between the criminal and his pursuers is $18u - 8v = 12(v - u)$. Since the pursuers are travelling at speed $v - u$ relative to the criminal, it will take them 12 hours to close the gap. Hence it will take 20 hours after the pursuers begin to catch the criminal.

Solution 3. [J. Schneider] Let t be the elapsed time in hours since the escape of the criminal, u the speed of the criminal in miles per hour, and z the distance from the prison at time t in miles. For convenience, project the train's motion so that it reaches the prison at time s; since the train takes the same length of time (8 hours) to reach the prison from its encounter with the pursuers as the pursuers take to reach the train, $s = 10 + 8 + 8 = 26$. (This can also be obtained from equating $(u + 3)8 = -(u + 3)(18 - s)$.)

When $t \geq 10$, the criminal is distant from the prison vt miles, the pursuers $(u + 3)(t - 10)$ miles and the train $-(u + 3)(t - 26)$ miles. Since the train and criminal meet at $t = 15.6$,

$$15.6u = -(u + 3)(15.6 - 26) = 10.4(u + 3) \ ,$$

so that $3u = 2(u + 3)$ and $u = 6$. When the pursuers catch the criminal, we have that $6t = 9(t - 10)$, so that $t = 30$. Thus, it takes the pursuers 20 hours to catch the criminal.

Solution 4. [F. Ban] Let the speed of the criminal be u miles per hour and of the pursuers $u + 3$ miles per hour. When the criminal met the express, both were $15.6u$ miles from the prisoner and the pursuers were $5.6(u + 3)$ miles from the prison. Since the express and the pursuers went at the same speed, they met at a distance $8(u + 3)$ from the prison, which is exactly halfway between the former position of the pursuers at $5.6(u + 3)$ and of the criminal at $15.6u$. Thus

$$15.6u - 8(u + 3) = 8(u + 3) - 5.6(u + 3) \Longrightarrow 15.6u = 10.4(u + 3) \Longrightarrow u = 6, u + 3 = 9 \ .$$

If the time taken to catch the criminal after the pursuers start out is t, then the distance from the prison at time t is

$$9t = (10 + t)6 \Longrightarrow t = 20 \ ,$$

so that it takes the pursuers 20 hours after setting out to catch the criminal.

Comment. P. Chen set up the equation

$$8(u + 3) + 2.4(u + 3) = 15.6u \ .$$

449. Let $S = \{x : x > -1\}$. Determine all functions from S to S which both

(a) satisfies the equation $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all $x, y \in S$, and

(b) $f(x)/x$ is strictly increasing or strictly decreasing on each of the two intervals $\{x : -1 < x < 0\}$ and $\{x : x > 0\}$.

Solution. We first check that the function is viably defined. Suppose such a function $f(x)$ exists. Then for all $x, y \in S$,

$$1 + x + f(y) + xf(y) = (1 + x)(1 + f(y)) > 0 \ ,$$

so that $x + f(y) + xf(y) \in S$. Similarly, $y + f(x) + yf(x) \in S$.

If we set $x = y$, then we have, for all $x \in S$.

$$f(x + f(x) + xf(x)) = x + f(x) + xf(x) \ .$$

Thus, there is at least one number $a \in S$ for which $f(a) = a$. Let $b = a + f(a) + af(a) = 2a + a^2 = a(a + 2) = (a + 1)^2 - 1$. Then $b \in S$ and $f(b) = b$.

Suppose, if possible, that $a > 0$; then $b > a > 0$. However, $f(a)/a = f(b)/b = 1$, which contradicts condition (b). Suppose, if possible, that $-1 < a < 0$; then $-1 < b < a < 0$. However, as before, we get a contradiction with condition (b). Hence, the only remaining possibility is that $a = 0$, so that $x + f(x) + xf(x) = 0$ for all $x \in S$. Hence

$$f(x) = \frac{-x}{1+x} = \frac{1}{1+x} - 1$$

for $x > -1$.

We verify that this function works. We have that

$$x + f(y) + xf(y) = (1+x)(1+f(y)) - 1 = \frac{1+x}{1+y} - 1$$

and

$$y + f(x) + yf(x) = \frac{1+y}{1+x} - 1 \ .$$

Hence

$$f(x + f(y) + xf(y)) = \frac{(1+y) - (1+x)}{1+x} = y + f(x) + yf(x) \ .$$

Note that $f(x)/x = -1/(1+x)$ increases on $S \backslash \{0\}$.

Comment. We can establish more directly that $f(0) = 0$. First, note that $f(x)$ is one-one. Suppose that $f(a) = f(b)$. Then, for all $x \in S$, $x + f(a) + xf(a) = x + f(b) + xf(b)$. Aplying f yields that $a + f(x) + af(x) = b + f(x) + bf(x)$, so that $(a - b)(1 + f(x)) = 0$. Hence $a = b$. For all $x \in S$, $f(x + f(0) + xf(0)) = f(x)$, so that $f(0)(1 + x) = 0$ and $f(0) = 0$. One immediate consequence of this is that $f(f(x)) = x$ for all $x \in S$.

K. Huynh began by letting $f(0) = k$. Setting $(x, y) = (0, 0)$ led to $f(k) = k$, and $(x, y) = (0, k)$ to $0 = k(k + 1)$, whence $k = 0$. Thus $f(0) = 0$, from which $f(f(y)) = y$ for all $y \in S$.

450. The 4-sectors of an angle are the three lines through its vertex that partition the angle into four equal parts; adjacent 4-sectors of two angles that share a side consist of the 4-sector through each vertex that is closest to the other vertex.

Prove that adjacent 4-sectors of the angles of a parallelogram meet in the vertices of a square if and only if the parallelogram has four equal sides.

Solution 1. Let the parallelogram be $ABCD$ and let its diagonals intersect at P. Suppose that $ABCD$ has equal sides, so that $ABCD$ is a rhombus and its diagonals right bisect each other. The intersection of the adjacent 4-sectors are the respective incentres I, J, K, L of the triangle ABP, BCP, CDP, DAP. Since IP, JP, KP, LP bisect the respective angles APB, BPC, CPD, DPA, it follows that each of the angles IPJ, JPK, KPL, LPI is equal to $90°$. The four triangles APB, BPC, CPD, DPA are congruent with corresponding angles at P. Hence $IP = JP = KP = LP$ and $IJKL$ is square.

Suppose that $IJKL$ is square. Observe that $\angle BAI + \angle ABI$, being one quarter of the sum of adjacent angles of a parallelogram, is equal to $45°$, whence $\angle AIB = 135°$. Similarly, $\angle BJC = 135°$. Wolog, assume that $AB \geq AD = BC$. Then, by similar triangles, BAI and BCI, we have that $BJ \leq BI$, so that $\angle BIJ \leq \angle BJI$. Similarly, $AI \geq AL$, so that $\angle CJK = \angle ALI \leq \angle AIL$. Hence

$$\begin{aligned} 90° = \angle LIJ &= 360° - (\angle AIL + \angle AIB + \angle BIJ) \\ &\geq 360° - (\angle CJK + \angle BJC + \angle BJI) = \angle IJK = 90° \ . \end{aligned}$$

We must have equality throughout, so that

$$\angle BIJ = \angle BJI \Longrightarrow BI = BJ \Longrightarrow AB = BC$$

and $ABCD$ is a rhombus.

Solution 2. [A. Tavakoli] Use the same notation as in Solution 1. Suppose that the angles at A and C are 4α and at B and D are 4β. Since the angle sum of the parallelogram is $360°$, $\alpha + \beta = 45°$. Hence

$$\angle AIB = \angle BJC = \angle CKD = \angle DLA = 135° \ .$$

From the Sine Law, we have that

$$|BI| = \sqrt{2}(\sin\alpha)a \qquad |BJ| = \sqrt{2}(\sin\alpha)b$$

$$|AI| = \sqrt{2}(\sin\beta)a \qquad |AL| = \sqrt{2}(\sin\beta)b \ .$$

Assume that $IJKL$ is a square. Then, since $|IJ| = |IL|$, the Cosine Law reveals that

$$|BI|^2 + |BJ|^2 - 2|BI||BJ|\cos 2\beta = |AI|^2 + |AL|^2 - 2|AI||AL|\cos 2\alpha \ ,$$

whence

$$2a^2\sin^2\alpha + 2b^2\sin^2\alpha - 4ab\sin^2\alpha\cos 2\beta = 2a^2\sin^2\beta + 2b^2\sin^2\beta - 4ab\sin^2\beta\cos 2\alpha$$

$$\Longrightarrow (a^2+b^2)(\sin^2\alpha - \sin^2\beta) = 2ab[\sin^2\alpha(1-2\sin^2\beta) - \sin^2\beta(1-2\sin^2\alpha)] = 2ab(\sin^2\alpha - \sin^2\beta) \ .$$

If $\alpha = \beta$, the parallelogram is a rectangle. Suppose that $AD \geq AB$ and P is the centre of the rectangle. Compairing similar triangles AIB and ALD, we see that $AL \geq AI$, with equality if and only if $AD = AB$. Hence, since $\angle IAL = 45°$,

$$\angle AIL \geq 67\frac{1}{2}^{\circ} \geq \angle ALI \Rightarrow \angle LIP \geq 45° \leq \angle LIP$$

with equality if and only if $AD = AB$. Hence, $IJKL$ is a square if and only if $ABCD$ is a square.

Otherwise, $(a-b)^2 = a^2 + b^2 - 2ab = 0$ and $a = b$.

Assume that $ABCD$ is a rhombus, so that $b = a$. The triangles ABI, CBJ, ADL are congruent (ASA), so that $AI = AL$ and $BI = BJ$. Observe that, by the angle sum of triangles, $2\alpha + 2\angle AIL = 2\beta + 2\angle BIJ = 180°$. Hence

$$\angle LIJ = 360° - (\angle AIL + \angle AIB + \angle BIJ) = 360° - (90° - \alpha + 135° + 90° - \beta) = 90° \ .$$

By the Sine Law applied to triangle AIL, we have that

$$|IL| = \sin 2\alpha \cdot \frac{|AI|}{\sin(90° - \alpha)} = \frac{2\sin\alpha\cos\alpha\sqrt{2}(\sin\beta)a}{\cos\alpha} = 2\sin\alpha\sin\beta\sqrt{2}a \ .$$

Similarly, $|IJ| = |JK| = |KL| = 2\sin\alpha\sin\beta\sqrt{2}a = |IL|$. Hence $IJKL$ is a square.

Solution 3. [K. Huynh (first part)] Suppose that $ABCD$ is a rhombus. Then its diagonals bisect its opposite angles and $ABCD$ has reflective symmetry about each diagonal. The four triangles AIB, CJB, CKD, ALD are congruent (ASA); in particular, $BI = BJ = DK = DL$. The reflection about BD takes $A \leftrightarrow C$, $I \leftrightarrow J$ and $K \leftrightarrow L$, so that AC, IJ, LK are perpendicular to BD and $IL = JK$. Similarly BD, LI and JK are perpendicular to AC, and $IJ = KL$. Hence $IJKL$ is a rectangle.

Since BI bisects $\angle ABD$, I us equidistant from AB and BD. Similarly, I is equidistant from AB and AC. Hence, I is equidistant from AC and BD. Since $|IJ|$ is twice the distance from I to BD and $|IL|$ twice the distance from I to AC, $IJ = JK$ and so $IJKL$ is a square.

The reverse implication can be proved by a contradiction argument. Suppose that $AD > AB$. Determine point D' on segment AD and point C' on segment BC so that $ABC'D'$ is a rhombus. Let $IJKL$ be the internal figure determined by the 4-sectors for $ABCD$ and $I'J'K'L'$ for $ABC'D'$. Observe that $I' = I$, that $IJ'K'L'$ is a square and that L and J both lie within the angle $L'I'J'$ (since $D'L' \| DL$ and $C'J' \| CJ$). Hence $\angle LIJ < \angle L'IJ' = 90°$, so that $IJKL$ is not a square. Thus, if $IJKL$ is a square, then $AD = AB$ and the result follows.

451. Let a and b be positive integers and let $u = a + b$ and $v = lcm\ (a, b)$. Prove that

$$gcd\ (u, v) = gcd\ (a, b)\ .$$

Solution 1. Suppose that $d|a$ and $d|b$. Then d divides any multiple of these two numbers and so divides $lcm(a, b) = v$. Also, $d|a + b$. Hence $d|gcd(u, v)$.

On the other hand, suppose that $d|u$ and $d|v$. Let $g = gcd(d, a)$ and $d = gh$. We have that

$$v = lcm(a, b) = a \cdot \frac{b}{gcd(a, b)}\ .$$

Since d divides v, h divides d and $\gcd(h, a) = 1$, it follows that

$$h \Big| \frac{b}{gcd(a, b)}\ .$$

Now $g|a + b$ and $g|a$, so g divides $b = (a + b) - b$. Also $h|a + b$ and $h|b$, so h also divides a. But, as $gcd(h, a) = 1$, $h = 1$. Hence $d|a$. Similarly, $d|b$. Hence the pairs (a, b) and (u, v) have the same divisors and the result follows.

Solution 2. Let d be the greatest common divisor of a and b, and write $a = da_1$ and $b = db_1$. The pair (a_1, b_1) is coprime. We have that $u = d(a_1 + b_1)$ and $v = d(a_1 b_1)$. The greatest common divisor of u and v is equal to $d \cdot gcd(a_1 + b_1, a_1 b_1)$.

Suppose, if possible, that there is a prime p that divides both $a_1 + b_1$ and $a_1 b_1$. Then p must divide one of the factors a_1, b_1 of the product, say a_1. Then p must also divide $b_1 = (a_1 + b_1) - a_1$, which contradicts the coprimality of the pair (a_1, b_1). Hence $gcd(a_1 + b_1, a_1 b_1) = 1$, and the result follows.

Solution 3. Let $\gcd(a, b) = \prod p^k$, where the product is taken over all primes dividing the left side and p^k is the largest power of the prime dividing it. Then p^k divides a and b, and hence u and v, and so divides $gcd(u, v)$. Hence $gcd(a, b)|gcd(u, v)$.

Suppose that $gcd(u, v) = \prod p^r$. Then p^{r+1} divides neither a nor b and p^r divides at least one of a and b, say a. Then, as p^r divides $u = a + b$ and a, it follows that p^r divides b, and therefore divides $gcd(a, b)$. Hence $gcd(u, v)|gcd(a, b)$. The result follows.

452. (a) Let m be a positive integer. Show that there exists a positive integer k for which the set

$$\{k + 1, k + 2, \ldots, 2k\}$$

contains exactly m numbers whose binary representation has exactly three digits equal to 1.

(b) Determine all intgers m for which there is exactly one such integer k.

Solution 1. (a) For each positive integer k, let $f(k)$ be the number of integers in the set

$$\{k + 1, k + 2, \ldots, 2k\}$$

whose binary representation has exactly three digits equal to 1. When we move from $k - 1$ to k, the set corresponding to $k - 1$ drops the number k and adds the numbers $2k - 1$ and $2k$ to for the set correponding to k. Since k and $2k$ have exactly the same number of ones in their binary representations, we find that, for $k \geq 2$,

$$f(k) = f(k-1)$$

when $2k-1$ does not have three digits equal to one, and

$$f(k) = f(k-1) + 1$$

when $2k-1$ has exactly three digits equal to one (*i.e.*, has the form $2^a + 2^b + 2^c$ for distinct nonnegative integers a, b, c. There are infinitely many numbers of this form.

Hence $f(k)$ increases by 0 or 1 with every unit increase in k and takes arbitrarily large value. Since $f(1) = 0$, the function f assumes every nonnegative integer.

(b) Suppose that $f(k)$ assumes some value m exactly once. Then, there must be a positive integer r for which $f(r-1) = m-1$, $f(r) = m$ and $f(r+1) = m+1$, so that $2r - 1 = 2^t + 2^s + 1$ for some positive integers t and s with $t > s > 0$ (so $t \geq 2$) and the binary representation of $2r + 1 = 2^t + 2^s + 2 + 1$ has exactly three digits equal to 1. This can happen only of $s = 1$, so that $2r - 1 = 2^t + 3$, $2r + 1 = 2^t + 5$ and $r = 2^{t-1} + 2$.

We count the number of integers with three unit binary digits in

$$\{2^{t-1} + 2 + 1, 2^{t-1} + 2^2, \cdots, 2^t, 2^t + 1, 2^t + 2, 2^t + 2 + 1, 2^t + 2^2\} \ .$$

This set includes all the numbers with exactly t digits, except for 2^{t-1} and $2^{t-1} + 1$, neither of which has three unit digits, and exactly $\binom{t-1}{2}$ of them have three unit digits (corresponding to all possible choices of pairs of digit positions). There is one additional number $2^t + 2 + 1$ with three digits. Hence $f(k)$ assumes the value m exactly once if and only if m has the form $1 + \binom{n}{2}$ and $k = 2^n + 2$.

Solution 2. (a) [A. Remorov] Let $k = 2^a + 2^b + 1$, where $a > b \geq 1$. There are $\binom{a}{2}$ numbers with exactly three unit binary digits between 2^a and 2^{a+1} inclusive, since there are a positions in which to place the last two unit digits. There are $\binom{b}{2}$ numbers between 2^a and $2^a + 2^b$ inclusive, since there are b positions available for the last two unit digits. Thus there are $\binom{a}{2} - \binom{b}{2}$ numbers with three unit digits between $2^a + 2^b$ and $2^{a+1} - 1$ inclusive, and so

$$\binom{a}{2} - \binom{b}{2} - 1$$

numbers with three unit digits between $k+1 = 2^a + 2^b + 2$ and $2^{a+1} - 1$ inclusive (the number $k = 2^a + 2^b + 1$ is not included).

There are $\binom{b+1}{2} + 2$ numbers with three unit digits between 2^{a+1} and $2k = 2^{a+1} + 2^{b+1} + 2$ inclusive, since the last two ones can be chosen arbitrarily from the last $b+1$ digits and since $2k-1$ and $2k$ are also included. Hence the number of digits between $k+1$ and $2k$ inclusive is equal to $\binom{a}{2} + b + 1$. Since b can be any integer for which $1 \leq b \leq a-1$, the set of numbers m for which there are exactly m numbers with exactly three unit digits between $k+1$ and $2k$ inclusive contain all the numbers between $\binom{a}{2} + 2$ and $\binom{a}{2} + a = \binom{a+1}{2}$ for $a \geq 2$ (*i.e.*, $3, 5, 6, 8, 9, 10, \cdots$).

There is one such integer when $k = 4$ and two such integers when $k = 6$. When $a \geq 2$ and $k = 2^a + 3$, there are $\binom{a}{2} - 1$ such integers between $2^a + 4$ and $2^{a+1} - 1$ inclusive and also 2 more, $2^{a+1} + 3$ and $2^{a+1} + 6$ for a total of $\binom{a}{2} + 1$ between $k+1$ and $2k$ inclusive. Hence, all values of m can be assumed.

Solution 3. [D. Shi] Let x_m be the mth binary number that contains exactly two digits equal to 1 (so that $x_1 = 3$, $x_2 = 5$, $x_3 = 6$, $x_4 = 9$). We prove that $\{x_m + 1, x_m + 2, \cdots, 2x_m\}$ contains exactly $m-1$ numbers with exactly three unit binary digits.

First, note that there are exactly $n-1$ binary numbers with n digits with exactly two unit digits (the left digit and one other). Suppose that $1 + 2 + \cdots + (n-1) < m \leq 1 + 2 + \cdots + n$, so that $m = \binom{n}{2} + r$ for $1 \leq r \leq n$. Then x_m has $n+1$ binary digits and so $x^m = 2^n + 2^{r-1}$. In the set $\{x_m + 1, \cdots, 2x_m\}$, there are

$(r-1)+r+\cdots+(n-1)=\binom{n}{2}-\binom{r-1}{2}$ numbers of the form $2^n+2^a+2^b$ with $a\geq r-1, a>b\geq 0$ and $\binom{r}{2}$ numbers of the form $2^{n+1}+2^a+2^b$ with $r-1\geq a>b\geq 0$. Hence there are

$$\binom{n}{r}-\binom{r-1}{2}+\binom{r}{1}=\binom{n}{r}-(r-1)=m-1$$

numbers in $\{x_{m+1},\cdots,2x_m\}$ with three unit digits.

The number m of numbers being an increasing function of k, the number $m-1$ is unique if and only if $x_{m+1}=x_m+1$. This occurs if r is chosen so that $x_m=2^n+2^{r-1}+1$ has two digits equal to 1, which is equivalent to $r=1$. Hence, the numbers m which occur exactly once are of the form $\binom{n}{2}+1$ for $n\geq 2$.

453. Let A, B be two points on a circle, and let AP and BQ be two rays of equal length that are tangent to the circle that are directed counterclockwise from their tangency points. Prove that the line AB intersects the segment PQ at its midpoint.

Solution 1. [D. Dziabenko, Y. Wang] If A and B are at opposite ends of a diameter, then AP and BQ are mutual images with respect to a reflection in the centre of the circle and AB bisects PQ at the centre of the circle. Otherwise, wolog, we may suppose that the arc from A to B is less than a semicircle.

Let the lines AP and BQ meet at C and suppose that PA is produced to D so that $DP=2AP$. Since (in triangle CDQ), $DA:AC=AP:AC=BQ:CB$, $AB\|DQ$. Suppose that AB meets PQ at K. Then (in triangle PDQ), $AK\|DQ$, so that $PA:AD=PK:KQ$. Since $PA=AD$, $PK=KQ$ as desired.

Solution 2. [K. Huynh] The rotation with centre O, the centre of circle, that takes A to B also takes P to Q. Let $\beta=\angle AOP$. Consider the spiral similarity of a rotation about O with angle β followed by a dilation of factor $|OP|/|OA|$. This takes triangle OAB to triangle OPQ and takes the midpoint M of AB to the midpoint N of PQ. Our task is to show that A, B and N are collinear.

Since $OP:OA=ON:OM$ and $\angle AOP=\angle MON=\beta$, triangles OAP and OMN are similar. Hence $\angle OMN=\angle OAP=90°$. Since triangle OAB is isosceles, $OM\perp AB$, so that $\angle OMB=90°=\angle OMN$. Hence A,M,B,N are collinear and the lines AB meets the segment PQ at its midpoint.

Solution 3. [P. Chu] Suppose that AB and PQ intersect at M, and that OP and AM intersect at X. We have that $\Delta OAP\sim\Delta OBQ$ and $\Delta OAB\sim\Delta OPQ$. Since $\angle OAB=\angle OPQ$ and $\angle OXA=\angle MXP$, triangles OAX and MPX are similar, and so $AX:OX=PX:MX$. Since, also, $\angle AXP=\angle OXM$, triangles AXP and OXM are similar. Now,

$$\angle MOP+\angle MPO=\angle MOX+\angle QPO=\angle XAP+\angle BAO=90°$$

whence $\angle OMP=90°$. Since $OP=OQ$, triangle POQ is isosceles and its altitude OM bisects the base PQ. The result follows.

Solution 4. Let N be the midpoint of PQ. The half-turn (180° rotation) about N interchanges P and Q and takes A to A', so that N is the midpoint of AA'. We show that B lies on AA'.

Let O be the centre of the circle and let $\angle AOB=2\alpha$. The rotation with centre O that takes A to B also takes P to Q, so that the angle between AP and BQ is equal to 2α. Since AP is carried to $A'Q$ by the half-turn about N, the angle formed by BQ and QA' at Q is equal to 2α. This is an exterior angle to the triangle BQA'.

Since $BQ=PA=PA'$, triangle BQA' is isosceles and so $\angle BA'Q=\angle QBA'$. Hence

$$\angle NAP=\angle A'AP=\angle AA'Q=\angle BA'Q=\frac{1}{2}(\angle BA'Q+\angle QBA')=\alpha\ .$$

However, $\angle BAP$ is equal to the angle between chord and tangent and so equal to half the angle subtended by the chord at the centre O. Hence $\angle BAP=\alpha=\angle NAP$, so that A,B,N are collinear and the result follows.

Solution 5. [C. Sun] Let AB intersect PQ at M. Note that triangle OAB and OPQ are similar isosceles triangles.

$$\begin{aligned}\angle MBO &= 180^\circ - \angle ABO = 180^\circ - (90^\circ - \frac{1}{2}\angle AOB) \\ &= 180^\circ - (90^\circ - \frac{1}{2}\angle POQ) = 180^\circ - \angle PQO \\ &= 180^\circ - \angle MQO \ .\end{aligned}$$

Hence $\angle MBO + \angle MQO = 180^\circ$, so that the quadrilateral $OBMQ$ is concyclic. Therefore $\angle OMQ = \angle OBQ = 90^\circ$, from which $OM \perp PQ$. Because triangle OPQ is isosceles, M is the midpoint of PQ, as desired.

454. Let ABC be a non-isosceles triangle with circumcentre O, incentre I and orthocentre H. Prove that the angle OIH exceeds 90°.

Solution 1. Suppose that $\angle A > 90^\circ$. Then O and H are both external to the triangle on opposite sides of BC. The points O and H are opposite vertices of a rectangle, two of whose sides are the altitude from A to BC and the right bisector of BC. Since the angle bisector of angle BAC lies between these sides within triangle ABC [why?], I lies inside the rectangle and within the circle of diameter OH. Hence $\angle OIH > 90^\circ$. If $\angle A = 90^\circ$, then O is the midpoint of BC and $H = A$. he same argument can be used (noting that I is not on OH since the triangle is not isosceles).

Suppose that ABC is an acute triangle with $AB < AC < BC$. Let the altitudes be AP, BQ, CR and the medians AL, BM, CN. We have that $AR < AN$, $BP < BL$, $AQ < AM$. Hence H lies inside the quadrilateral $AMON$. Since $\angle RHP > 90^\circ$, $\angle PHC < 90^\circ$. The parallelogram with sides AP, OL, CR, ON has an acute angle at H and O and so is contained in the circle with diameter HO.

Since $AB < AC$, $\angle BAP < \angle CAP$ and $\angle BAL > \angle CAL$, so that the bisector AI of the angle A lies between AP and AL. Similarly, CI lies between CR and CN. Thus I lies within the parallelogram with sides AP, OL, CR, ON and so is contained within the circle of diameter OH. Hence $\angle OIH > 90^\circ$.

Solution 2. Recall some preliminary facts. The nine-point circle of a triangle ABC passes through the midpoints of the sides, the midpoints of the segments joining its vertices to the orthocentre H and the pedal points (*i.e.*, the feet of its altitudes to the sides). Its centre is the midpoint N of the segment joining the the circumcentre O and the orthocentre H of the triangle. Its radius $\frac{1}{2}R$ is equal to half the circumradius R of the triangle ABC and it touches internally the incircle with radius r (as well as all three excircles). (See the book, H.S.M. Coxeter & S.L. Greitzer, *Geometry revisited*, MAA, 1967, §1.8, 5.6). The square of the length of the segment OI is $|OI|^2 = R^2 - 2Rr = R(R - 2r)$ (*ibid*, §2.1)

[Y. Wang] Produce OI to M so that $OI = IM$, and let R and r be be the circumradius and inradius, respectively. Consider triangle OHM. Since N is the midpoint of OH and I is the midpoint of OM, $NI \| HM$ so that $|HM| = 2|NI| = R - 2r$. Since $|IM| = |OI| = \sqrt{R(R-2r)}$ and $\sqrt{R(R-2r)} > R - 2r$, $|IM| > |HM|$, so that $\angle IHM > \angle MIH$. Hence $\angle MIH < 90^\circ$ so that $\angle OIH > 90^\circ$.

Solution 3. [D. Dziabenko] See background information in Solution 2. The centre N of the nine-point circle is the midpoint of OH, so that $\overrightarrow{IH} = 2\overrightarrow{IN} - \overrightarrow{IO}$. Since

$$\overrightarrow{IN} \cdot \overrightarrow{IO} = |\overrightarrow{IN}||\overrightarrow{IO}| \cos \angle OIN = \frac{1}{2}(R - 2r)\sqrt{R^2 - 2Rr} \cos \angle OIN \ ,$$

it follows that

$$\begin{aligned}|\overrightarrow{IH}||\overrightarrow{IO}| \cos \angle OIH &= \overrightarrow{IH} \cdot \overrightarrow{IO} = (2\overrightarrow{IN} - \overrightarrow{IO} \cdot \overrightarrow{IO}) \\ &= 2(\overrightarrow{IN} \cdot \overrightarrow{IO}) - |IO|^2 \\ &= (R - 2r)(\sqrt{R^2 - 2Rr}) \cos \angle OIN - (R^2 - 2Rr) \\ &\leq (R - 2r)\sqrt{R^2 - 2Rr} - (R - 2r)R = (R - 2r)[\sqrt{R^2 - 2Rr} - R] < 0 \ .\end{aligned}$$

Hence $\cos \angle OIH < 0$ and so $\angle OIH > 90^\circ$.

455. Let $ABCDE$ be a pentagon for which the position of the base AB and the lengths of the five sides are fixed. Find the locus of the point D for all such pentagons for which the angles at C and E are equal.

Solution 1. [C. Bao] We use analytic geometry, with the assignment $A \sim (0,0)$, $B \sim (1,0)$, $C \sim (a,b)$, $D \sim (x,y)$ and $E \sim (c,d)$. The lengths of the sides are $|AB| = 1$, $|BC| = u$, $|CD| = v$, $|DE| = w$ and $|EA| = t$. We have that $u^2 = (a-1)^2 + b^2$, $v^2 = (x-a)^2 + (y-b)^2$, $w^2 = (x-c)^2 + (y-d)^2$ and $t^2 = c^2 + d^2$.

Now

$$\begin{aligned}\overrightarrow{CB} \cdot \overrightarrow{CD} &= (a-1,b) \cdot (a-x, b-y) = (a-1)(a-x) + b(b-y) \\ &= a^2 + b^2 - ax - by + x - a \\ &= \frac{1}{2}[(a-1)^2 + b^2 + (x-a)^2 + (b-y)^2 - (x-1)^2 - y^2] \\ &= \frac{1}{2}[u^2 + v^2 - (x-1)^2 - y^2] \ ,\end{aligned}$$

so that

$$\cos C = \frac{u^2 + v^2 - [(x-1)^2 + y^2]}{2uv} \ .$$

Similarly,

$$\cos E = \frac{w^2 + t^2 - (x^2 + y^2)}{2wt} \ .$$

Hence

$$(u^2 + v^2)wt - [(x-1)^2 + y^2]wt = (w^2 + t^2)uv - [x^2 + y^2]uv$$

so that

$$(uv - wt)[x^2 + y^2] + 2wtx + [(u^2 + v^2 - 1)wt - (w^2 + t^2)uv] = 0 \ .$$

Thus, the point $C \sim (x,y)$ lies on a circle when $uv - wt \neq 0$ and on a straight line perpendicular to AB when $uv = wt$.

456. Let $n+1$ cups, labelled in order with the numbers $0, 1, 2, \cdots, n$, be given. Suppose that $n+1$ tokens, one bearing each of the numbers $0, 1, 2, \cdots, n$ are distributed randomly into the cups, so that each cup contains exactly one token.

We perform a sequence of moves. At each move, determine the smallest number k for which the cup with label k has a token with label m not equal to k. Necessarily, $k < m$. Remove this token; move all the tokens in cups labelled $k+1, k+2, \cdots, m$ to the respective cups labelled $k, k+1, m-1$; drop the token with label m into the cup with label m. Repeat.

Prove that the process terminates with each token in its own cup (token k in cup k for each k) in not more that $2^n - 1$ moves. Determine when it takes exactly $2^n - 1$ moves.

Solution. Let $(x_0, x_1, x_2, \cdots, x_n)$ denote the arrangement of tokens in which token number x_i is placed in cup i. When $n = 0$, token 0 is in cup 0, and $0 = 2^0 - 1$ moves are required. When $n = 1$, there are two possible distributions of tokens, and at most $1 = 2^1 - 1$ moves is needed, with this number required in the case of $(1,0)$. We will establish the result by an induction argument.

First, observe that, for any arrangement $(x_0, x_1, \cdots, x_i, \cdots, x_n)$, any token either remains stationary or moves one cup to the left at each move until it reaches the leftmost cup to the right of tokens already in their cups. Also, note that the number of moves required to first take token x_i to the position from which it first moves to its own cup depends only on the tokens $x_0, \cdots, x_{i-1}$ to the left of it. This can be seen by induction on i. This is clear for $i = 1$, since either x_0 will move and x_1 goes to cup 0, or $x_0 = 0$ and x_1 will move to its own cup. Suppose that this is true for $i = j - 1 \geq 1$. Then, if $(x_0, x_1, \cdots, x_{j-1})$ is a permutation of $0, 1, \cdots, j-1)$, then x_j will remain in position until its left neighbours are sorted, and then will move.

Otherwise, x_j will move one position to the left on the first occasion when on of the tokens on the left is moved to the right of it. Since this token is now in cup $j-1$, we can apply the induction hypothesis.

Back to the given problem, we suppose as an induction hypothesis that, for $n = k$, at most $2^k - 1$ moves are required, and this this number of moves is necessary if and only if the initial arrangement is $(1, 2, 3, \cdots, k, 0)$.

Consider an initial arrangement $(x_0, x_1, \cdots, x_k, x_{k+1})$ in the case $n = k+1$. If $x_{k+1} = k+1$, then this token will never be moved and by the induction hypothesis, the remaining tokens will be put into their proper cups in at most $2^k - 1 < 2^{k+1} - 1$ moves. Suppose that $x_i = k+1$ for $0 \le i \le k$. Consider two initial arrangements:

$$A = (x_0, x_1, \cdots, x_{i-1}, x_i = k+1, x_{i+1}, \cdots, x_{k+1})$$

and

$$B = (x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k+1}) ,$$

where B has $k+1$ tokens numbered from 0 to k inclusive sorted into $k+1$ cups. The number of moves required to move x_i in arrangement A to a position from which it moves to its own cup is equal to the number of moves to move x_{i+1} in arrangement B to a similar position, namely, no more than $2^k - 1$. This number of moves is actually equal to $2^k - 1$ if and only if $B = (1, 2, \cdots, k, 0)$ and $A = (1, 2, \cdots, k, k+1, 0)$ (*i.e.*, $i = k$).

Thus, after at most $2^k - 1$ moves, we have an arrangement with token $k+1$ in cup 0. One additional move takes this token to cup $k+1$ and the rest all in the left cups. Finally, at most $2^k - 1$ moves are required to restore the remaining tokens to their proper cups. Thus, we make at most $(2^k - 1) + 1 + (2^k - 1) = 2^{k+1} - 1$ moves. This maximum is attained only if we begin with $(1, 2, \cdots, k, k+1, 0)$. The first $2^k - 1$ moves take us to $(k+1, 1, 2, \cdots, k, 0)$; the next move yields $(1, 2, \cdots, k, 0, k+1)$ and the final $2^k - 1$ moves takes us to $(0, 1, 2, \cdots, k, k+1)$.

457. Suppose that $u_1 > u_2 > u_3 > \cdots$ and that there are infinitely many indices n for which $u_n \ge 1/n$. Prove that there exists a positive integer N for which

$$u_1 + u_2 + u_3 + \cdots + u_N > 2006 .$$

Solution. Since there are infinitely many values of n for which $u_n \ge 1/n$, we can select positive integers n_i such that $n_{i+1} > 2n_i$ for $i = 1, 2, 3, \cdots$. Then

$$\sum_{n=n_i+1}^{n_{i+1}} u_n \ge \sum_{n=n_i+1}^{n_{i+1}} u_{n_{i+1}} \ge \frac{n_{i+1} - n_i}{n_{i+1}} > \frac{1}{2}$$

for $i \ge 1$. Let $N = n_{4013}$. Then

$$\sum_{n=1}^{N} u_n \ge \sum_{n=n_1+1}^{n_{4013}} u_n > (4012)(1/2) = 2006 .$$

458. Let ABC be a triangle. Let A_1 be the reflected image of A with axis BC, B_1 the reflected image of B with axis CA and C_1 the reflected image of C with axis AB. Determine the possible sets of angles of triangle ABC for which $A_1B_1C_1$ is equilateral.

Solution. We establish a preliminary result: *For any angle* θ,

$$\cos 3\theta = \cos\theta - 4\cos\theta \sin^2\theta . \tag{1}$$

This is true, since

$$\begin{aligned}\cos 3\theta &= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\ &= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta \\ &= \cos^3\theta - 3\sin^2\theta\cos\theta = \cos\theta(\cos^2\theta - 3\sin^2\theta) \\ &= \cos\theta(1 - 4\sin^2\theta)\ .\end{aligned}$$

Let a, b, c be the sides of the triangle and α, β, γ be the respective angles opposite these sides. Since the triangles A_1BC, AB_1C and ABC_1 are all congruent to the triangle ABC, we have that $\angle C_1AB_1 = 3\alpha$ or $|2\pi - 3\alpha|$, $\angle A_1BC_1 = 3\beta$ or $|2\pi - 3\beta|$, and $\angle B_1CA_1 = 3\gamma$ or $|2\pi - 3\gamma|$.

Applying the Cosine Law to triangle B_1CA_1 yields that

$$A_1B_1^2 = a^2 + b^2 - 2ab\cos 3\gamma \qquad (2)\ .$$

Where R is the circumradius of triangle ABC, the Cosine and the Sine Laws applied to that triangle yields that

$$2ab\cos\gamma = a^2 + b^2 - c^2 \qquad (3)$$

and

$$\sin\gamma = \frac{c}{2R} \qquad (4)\ .$$

Applying (1), (3), (4) to (1) yields that

$$\begin{aligned}A_1B_1^2 &= a^2 + b^2 - 2ab\cos 3\gamma = a^2 + b^2 - 2ab\cos\gamma(1 - 4\sin^2\gamma \\ &= a^2 + b^2 - (a^2 + b^2 - c^2)(1 - 4\sin^2\gamma) \\ &= (a^2 + b^2)(1 - 1 + 4\sin^2\gamma) + c^2(1 - 4\sin^2\gamma) \\ &= (a^2 + b^2 - c^2)(4\sin^2\gamma) + c^2 = \frac{(a^2 + b^2 - c^2)c^2}{R^2} + c^2 = \frac{c^2}{R^2}(a^2 + b^2 - c^2 + R^2)\ .\end{aligned}$$

Similarly,

$$B_1C_1^2 = \frac{a^2}{R^2}(b^2 + c^2 - a^2 + R^2)$$

and

$$C_1A_1^2 = \frac{b^2}{R^2}(c^2 + a^2 - b^2 + R^2)\ .$$

It follows that $A_1B_1 = B_1C_1$ if and only if

$$c^2(a^2 + b^2 - c^2 + R^2) - a^2(b^2 + c^2 - a^2 + R^2) = 0\ .$$

Factoring the left side yields that

$$(c^2 - a^2)(R^2 + b^2 - a^2 - c^2) = 0\ .$$

The equality of other pairs of sides can be similarly handled. Thus, triangle $A_1B_1C_1$ is equilateral if and only if the following system of three equations is valid:

$$(c^2 - a^2)(R^2 + b^2 - a^2 - c^2) = 0\ ;$$

$$(a^2 - b^2)(R^2 + c^2 - b^2 - a^2) = 0\ ;$$

$$(b^2 - c^2)(R^2 + a^2 - b^2 - c^2 = 0\ .$$

If a is unequal to both b and c, then

$$R = a^2 + c^2 - b^2 = b^2 + a^2 - c^2$$

so that $b = c$. Hence, the triangle is isosceles in any case.

Wolog, assume that $b = c$. Then from the middle equation, we obtain that $(a^2 - b^2)(R^2 - a^2) = 0$. Therefore, either $a = b$, in which case the triangle is equilateral, or $R = a$ and $\sin\alpha = a/2R = 1/2$. Therefore, $\alpha = 30°$ or $\alpha = 150°$. Thus, there are three possible sets of angles fo the triangle ABC: $(60°, 60°, 60°)$, $(30°, 75°, 75°)$ and $(150°, 15°, 15°)$.

459. At an International Conference, there were exactly 2006 participants. The organizers observed that: (1) among any three participants, there were two who spoke the same language; and (2) every participant spoke at most 5 languages. Prove that there is a group of at least 202 participants who speak the same language.

Solution 1. Consider an arbitrary participant, a. Suppose, first, that a can communicate with all other participants. Then, as a speaks at most five languages, there will be at least $\lceil 2006/5 \rceil = 402 > 202$ who speak one of the five languages.

On the other hand, if there is a participant, b, with whom a cannot communicate, then out of the remaining 2004 people, everybody should be able to communicate with either a or b. Thus, one of the pair, say a, can communicate with at least 1002 people. Since a speaks at most five languages, one of these five must be spoken by at least $lceil 1002/5 \rceil = 201$ of these 1002 people. Including a among the speakers of this language yields the result.

Solution 2. [D. Dziabenko] Suppose, if possible, that the result is false. So, no language is spoken by 202 people. Let p be any person in the group. This person p spoke at least five languages, and can share each language with at most 200 people. Let P be the set of people with whom p shares at least one language; the set P has at most 1000 individuals.

Suppose that $q \not\in P$. By a similar argument, the set Q of individuals with whom q shares at least one language contains at most 1000 people. Thus the set $R = P \bigcup Q \bigcup \{p\} \bigcup \{q\}$ contains at most 2002 people. Let $r \not\in R$. Then, since $r \not\in P$, r and p do not share a language. Since $r \not\in Q$, r and q do not share a language. Since $q \not\in P$, p and q do not share a language. Then $\{p, q, r\}$ is a triplet that violates condition (1), and we obtain a contradiction. Hence, there must be 202 people with a common language.

460. Given two natural numbers x and y for which

$$3x^2 + x = 4y^2 + y \ ,$$

prove that their positive difference is a perfect square. Determine a nontrivial solution of this equation.

Solution 1. [D. Dziabenko] Since $3(x - y)(x + y + 1) = y^2$, $x > y$. Let the greatest common divisor of x and y be t, so that $x = at$, $y = bt$ and (a, b) is a coprime pair with $a > b$. Since $3a^2t^2 + at = 4b^2t^2 + bt$, $a - b = t(4b^2 - 3a^2)$. Therefore, $a - b$ is divisible by t, so that $a - b = st$ for some natural number s.

Since $s = 4b^2 - 3a^2 = 4(b^2 - a^2) + a^2 = a^2 + 4st(a + b)$, a^2 is divisible by s. Therefore, $b^2 = a^2 - (a^2 - b^2)$ is divisible by s. However, (a^2, b^2) is a coprime pair, whence $s = 14$ and so $t = a - b$, $x = a(a - b)$ and $y = b(a - b)$. Therefore,

$$x - y = a(a - b) - b(a - b) = (a - b)^2 \ ,$$

a perfect square, as desired.

To find a nontrivial solution, it is enough to find a solution (a, b) of the pellian equation $(2b)^2 - 3a^2 = 1$. One such solution, which can be found by trial and error, is $(a, b) = (15, 13)$, which leads to $(x, y) = (30, 26)$.

Solution 2. We have that

$$\begin{aligned} 0 &= (3x^2 + x) - (4y^2 + y) \\ &= 3(x^2 - y^2) + (x - y) - y^2 \\ &= 4(x^2 - y^2) + (x - y) - x^2 \ , \end{aligned}$$

whence

$$x^2 = (x-y)[4(x+y)+1]$$

and

$$y^2 = (x-y)[3(x+y)+1] \ .$$

Multiplying these two equations yields that

$$(xy)^2 = (x-y)^2[4(x+y)+1][3(x+y)+1] \ .$$

The greatest common divisor of $4(x+y)+1$ and $3(x+y)+1$ must divide their difference $x+y$, and hence must divide 1. Therefore $4(x+y)+1$ and $3(x+y)+1$ are coprime. Since their product times a square is square, each must be a square. But then, since $(x-y)[4(x+y)+1]$ is square, so must also be $x-y$. The result follows.

To find a solution, we observe that $3x^2+x = 4y^2+y$ is equivalent to

$$(8y+1)^2 = 48x^2+16x+1 \ .$$

Since $48x^2+16x+1$ is a square less than $(7x)^2$, let us consider the possibility that it might be equal to $(7x-1)^2$. The condition

$$48x^2+16x+1 = (7x-1)^2 = 49x^2-14x+1$$

leads to $x=30$. Plugging this into the original equation leads to

$$0 = 4y^2+y-2730 = (y-26)(4y+25)$$

and so $y=26$. Thus $(x,y) = (30,26)$ satisfies the equation.

Comment. We can use the theory of Pell's equation to generate a whole raft of solutions. Manipulating the given equation leads to

$$(8y+1)^2 = 16(3x^2+x)+1 \Longrightarrow 12(8y+1)^2 = 16(3x^2+12x)+12 = 16(6x+1)^2-4 \ .$$

Thus, we have to solve for positive integers the system of equations

$$u^2-3v^2 = 4 \ ;$$

$$u = 4(6x+1) \ ;$$

$$v = 2(8y+1) \ .$$

It can be shown, working modulo 4, that there are no solutions for which u and v are odd. Accordingly, $(u,v) = (2r,2s)$ where (r,s) satisfies the pellian equation $r^2-3s^2=1$. The solutions of this equation are given by $(r,s) = (r_k,s_k)$ where $r_k+\sqrt{3}s_k = (2+\sqrt{3})^k$. The first few solutions are given by $(r,s) = (2,1),(7,4),(26,15),(97,56),(362,209),\cdots$. Not all of these will yield integers x and y. However, $(r,s) = (362,209)$ leads to $(u,v) = (724,418)$ and $(x,y) = (30,26)$. Show how you can extend this to determine an infinite family of solutions (x,y).

461. Suppose that x and y are integers for which $x^2+y^2 \neq 0$. Determine the minimum value of the function

$$f(x,y) \equiv |5x^2+11xy-5y^2| \ .$$

Solution. Say that $f(x,y)$ *represents* n if it assumes the value n for some pair (x,y) of integers not both zero. We make a number of observations. Note that $f(x,y) = f(-y,x)$, so that it is enough to look at nonnegative values of x and y. Since

$$5 = f(1,0) = f(5,-2) = f(13,-5) = \cdots \ ,$$

the number 5 is representable. Since the discriminant of the quadratic $5x^2 + 11xy - 5y^2$, namely 221, is not square, 0 is not representable by $f(x, y)$.

Suppose $p = 2$ or $p = 3$. If both x and y are multiples of p, then $f(x, y)$ is divisible by p^2, and so does not represent p. If both x and y are not multiples of p, then $x^2 \equiv y^2 \equiv 1 \pmod{p}$ and so $f(x, y) \equiv |11xy| \not\equiv 0 \pmod{p}$. Finally, if exactly one of x and y is a multiple of p, then $f(x, y)$ does not represent p. Hence, $f(x, y)$ never represents either 2 or 3.

If either or both of x and y is odd, then $f(x, y)$ must be odd. Therefore, if $f(x, y) = 4$, then $(x, y) = (2u, 2v)$ for some integer pair (u, v) and $f(u, v) = 1$. It can be deduced that 4 is representable if and only if 1 is representable. Thus, the minimum representable value of f is either 1 or 5. Thus, we need to check whether the equation $5x^2 + 11xy - 5y^2 = \pm 1$ is solvable in integers.

If $f(x, y) = 1$, then $5y^2 + 11xy - 5y^2 = \pm 1$, whence $(10x + 11y)^2 - 221y^2 = \pm 20$. Let $z = 10x + 11y$. Since 221 is divisible by 13, $z^2 \equiv \pm 7 \pmod{13}$. Raising each side to the sixth power and taking account of the little Fermat theorem (that $a^{p-1} \equiv 1 \pmod{p}$ for any prime p and a not divisible by p), we find that $1 \equiv 7^6 \equiv (-3)^3 = -27 \equiv -1 \pmod{13}$, a contradiction. Hence $f(x, y)$ cannot assume the value 1.

462. For any positive real numbers a, b, c, d, establish the inequality

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+d}} + \sqrt{\frac{c}{d+a}} + \sqrt{\frac{d}{a+b}} > 2 \ .$$

Solution 1. From the arithemetic-geometric means inequality, we have that

$$\sqrt{\frac{b+c}{a}} \le \frac{1 + (b+c)/a}{2} = \frac{a+b+c}{2a} \ ,$$

which implies that

$$\sqrt{\frac{a}{b+c}} \ge \frac{2a}{a+b+c} > \frac{2a}{a+b+c+d} \ .$$

Applying an analogous inequality to the other terms of the left side, we obtain the desired result.

Solution 2. By the arithmetic-geometric means inequality, we have that

$$\begin{aligned}\sqrt{\frac{a}{b+c}} + \sqrt{\frac{c}{d+a}} &= \frac{\sqrt{a^2+ad} + \sqrt{bc+c^2}}{\sqrt{(b+c)(a+d)}} \\ &\ge \frac{2(\sqrt{a^2+ad} + \sqrt{bc+c^2})}{a+b+c+d}\end{aligned}$$

and, similarly, that

$$\sqrt{\frac{b}{c+d}} + \sqrt{\frac{d}{a+b}} \ge \frac{2(\sqrt{ab+b^2} + \sqrt{cd+d^2})}{a+b+c+d} \ .$$

Hence the left side of the inequality is not less than $2t/s$ where $t = \sqrt{a^2+ad} + \sqrt{b^2+ab} + \sqrt{c^2+bc} + \sqrt{d^2+cd}$ and $s = a + b + c = d$.

We observe that

$$\begin{aligned}t^2 &= (a^2+b^2+c^2+d^2) + (ad+ab+bc+cd) \\ &\quad + 2(\sqrt{a^2+ad}\sqrt{b^2+ab} + \sqrt{a^2+ad}\sqrt{c^2+bc} + \sqrt{a^2+ad}\sqrt{d^2+cd} \\ &\quad + \sqrt{b^2+ab}\sqrt{c^2+bc} + \sqrt{b^2+ab}\sqrt{d^2+cd} + \sqrt{c^2+bc}\sqrt{d^2+cd}) \\ &> (a^2+b^2+c^2+d^2) + (ad+ab+bc+cd) + 2(ab+ac+ad+bc+bd+cd) \\ &> (a+b+c+d)^2 = s^2 \ .\end{aligned}$$

Hence $t > s$, so that $2t/s > 2$ and the result follows.

Comment. The special cases are of interest. If $b = d = 0$, the inequality becomes $\sqrt{a/c} + \sqrt{c/a} > 2$, and the left side can be made arbitrarily close to 2 by making a close to c. If we take $d = 0$, we can parameterize

$$(a, b, c) = (x^2, r^2 \sin^2 \theta, r^2 \cos^2 \theta)$$

and find that

$$\begin{aligned}\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{a}} &= \frac{x}{2} + \tan\theta + \frac{r\cos\theta}{x} \\ &\geq \tan\theta + 2\sqrt{\cos\theta}\ .\end{aligned}$$

Set $f(\theta) = \tan\theta + 2(\cos\theta)^{1/2}$; then $f'(\theta) = (\cos\theta)^{-1/2}(\sec^{3/2}\theta - \sin\theta) > 0$, so $f(\theta)$ assumes its minimumn value of 2 when $\theta = 0$.

Some other special cases have interesting denouements. If $a = c$ and $b = d$, we get

$$\sqrt{\frac{a}{a+b}} + \sqrt{\frac{b}{a+b}} + \sqrt{\frac{a}{a+b}} + \sqrt{\frac{b}{a+b}} = 2\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}}\right) > 2\ .$$

If $a = b$ and $c = d$, we get

$$\begin{aligned}&\sqrt{\frac{a}{a+c}} + \sqrt{\frac{a}{2c}} + \sqrt{\frac{c}{a+c}} + \sqrt{\frac{c}{2a}} \\ &= \frac{\sqrt{a}+\sqrt{c}}{\sqrt{a+c}} + \frac{1}{\sqrt{2}}\left(\sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}\right) > 1 + \frac{2}{\sqrt{2}} = 1 + \sqrt{2} > 2\ .\end{aligned}$$

If you get stuck, it is often not a bad idea to try some special cases to see whether any light might be shed on how to approach the general case. In this case, the special cases were not particularly helpful, but on a contest, might help to garner a couple of marks that might not otherwise be had.

463. In *Squareland*, a newly-created country in the shape of a square with side length of 1000 km, there are 51 cities. The country can afford to build at most 11000 km of roads. Is it always possible, within this limit, to design a road map that provides a connection between any two cities in the country?

Solution. [Yifan Wang] It is possible to design a road map that provides a connection between any two cities so that the total length of the roads does not exceed 11000 km. We will provide an example.

Place the square map of the country on the Cartesian plane so that the corners are at the origin $(0, 0)$ and the points $(1000, 0)$, $(1000, 1000)$ and $(0, 1000)$. Build six main roads as follows:

R_1: A vertical road, a segment between the points $(100, 0)$ and $(100, 1000)$;

R_2: A horizontal road, a segment between the points $(100, 100)$ and $(1000, 100)$;

R_3: A horizontal road, a segment between the points $(100, 300)$ and $(1000, 300)$;

R_4: A horizontal road, a segment between the points $(100, 500)$ and $(1000, 500)$;

R_5: A horizontal road, a segment between the points $(100, 700)$ and $(1000, 700)$;

R_6: A horizontal road, a segment between the points $(100, 900)$ and $(1000, 900)$.

The six main roads are connected and have a total length of $1000 + 900 \times 5 = 5500$ km. Any of the 51 cities is at most 100 km away from one of the main roads, so the local roads that connect the cities to the main road can be built with a length of at most $51 \times 100 = 5100$ km. Thus, the length of all the roads will not exceed $5500 + 5100 = 10600 < 11000$ km.

464. A square is partitioned into non-overlapping rectangles. Consider the circumcircles of all the rectangles. Prove that, if the sum of the areas of all these circles is equal to the area of the circumcircle of the square, then all the rectangles must be squares, too.

Solution. Let s be the side length of the square and (a_i, b_i) be the dimensions of the ith rectangle. Then $s^2 = \sum a_i b_i$.

The circumcircle of the square has area $(\pi s^2)/2$ and the circumcircle of the ith rectangle has area $(\pi(a^2 + b^2))/4$. Hence, we have, using the arithmetic-geometric means inequality and the condition that the sum of the rectangular circumcircle areas is equal to the square circumcircle area,

$$\frac{\pi}{2}s^2 = \frac{\pi}{2}\sum a_i b_i \leq \frac{\pi}{4}\sum(a_i^2 + b_i^2) = \frac{\pi}{2}s^2 \ .$$

Since the extreme members of this inequality are equal, we must have equality everywhere. In particular, $a_i = b_i$ for each i and all of the partitioning rectangles are square.

465. For what positive real numbers a is

$$\sqrt[3]{2+\sqrt{a}} + \sqrt[3]{2-\sqrt{a}}$$

an integer?

Solution 1. Let $x = \sqrt[3]{2+\sqrt{a}}$, $y = \sqrt[3]{2-\sqrt{a}}$ and $z = x + y$. Then

$$z^3 = (x+y)^3 = x^3 + y^3 + 3(4-a)^{1/3}z = 4 + 3(4-a)^{1/3}z \ .$$

Hence $27(4-a)z^3 = (z^3 - 4)^3$, whence

$$a = 4 - \frac{(z^3-4)^3}{27z^3} = \frac{108z^3 - (z^3-4)^3}{27z^3} \ .$$

Since $a \geq 0$, z must be either (1) a positive integer for which $108z^3 \geq (z^3 - 4)^3$, or (2) a negative integer for which $108z^3 \leq (z^3 - 4)^3$.

Condition (1) forces $108 \geq (z^2 - (4/z))^3 \geq (z^2 - 4)^3$, so that $z = 1, 2$. Condition (2) forces $108 \geq (z^2 - (4/z))^3 \geq z^6$, which is satisfied by no negative integer value of z. Hence, we must have that $(z, a) = (1, 5), (2, 100/27)$. Since $z = x + y$ is equivalent to $z^3 = 4 + 3(4-a)^{1/3}z$, it is straightforward to check that both these answers are correct. Hence $a = 5$ or $a = 100/27$.

Solution 2. [Yifan Wang] With x and y defined as in the first solution, note that $x > y$ and that $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$. Since $x^2 + y^2 > (x+y)^2/2$ and $-xy > -(x+y)^2/4$, we have that $4 > (x+y)^3/4$, whence $x + y \leq 2$. Since $x^3 > -y^3$, $x > -y$, so that $x + y > 0$. Hence $x + y = 1$ or $x + y = 2$.

When $x + y = 1$, $x^2 - xy + y^2 = 4$ and so $xy = -1$, and $x = \frac{1}{2}(1 + \sqrt{5})$, $y = \frac{1}{2}(1 - \sqrt{5})$. Therefore $4 - a = x^3y^3 = -1$ so that $a = 5$.

When $x + y = 2$, then $x^2 - xy + y^2 = 2$, so that $xy = 2/3$. Therefore $x = \frac{1}{3}(3 + \sqrt{3})$, $y = \frac{1}{3}(3 - \sqrt{3})$ and $4 - a = 8/27$. Thus, $a = 100/27$. These solutions check out.

Solution 3. [A. Tavakoli] Denote the left side of the equation by $f(a)$. When $a \geq 4$,

$$0 \leq f(a) = (\sqrt{a}+2)^{1/3} - (\sqrt{a}-2)^{1/3} = \frac{4}{(\sqrt{a}+2)^{2/3} + (a-4)^{1/3} + (\sqrt{a}-2)^{2/3}} \leq 4^{1/3} < 3 \ .$$

Let $0 \leq a \leq 4$; again $f(a) > 0$. Observe that

$$\left(\frac{1}{2}(u+v)\right)^{\frac{1}{3}} \geq \frac{1}{2}u^{\frac{1}{3}} + \frac{1}{2}v^{\frac{1}{3}}$$

for all nonnegative values of u and v. (This can be seen by using the concavity of the function $t^{1/3}$, or from the power-mean inequality $(1/2)(s+t) \leq [(1/2)(s^3 + t^3)]^{1/3}$.) Setting $u = \sqrt[3]{2+\sqrt{a}}$ and $v = \sqrt[3]{2-\sqrt{a}}$, we find that $3 > 2 \times 2^{1/3} \geq f(a) > 0$ with equality if and only if $a = 0$. Hence the only possible integer values of $f(a)$ are 0 and 1.

Let $x = \sqrt[3]{2 - \sqrt{a}}$, so that $2 + \sqrt{a} = 4 - x^3$. Then

$$\begin{aligned} f(a) = 1 &\Longleftrightarrow x + (4 - x^3)^{1/3} = 1 \\ &\Longleftrightarrow 4 - x^3 = 1 - 3x + 3x^2 - x^3 \\ &\Longleftrightarrow x^2 - x - 1 = 0 \Longleftrightarrow x = (1 \pm \sqrt{5})/2 \\ &\Longleftrightarrow x^3 = 2 \pm \sqrt{5}\ . \end{aligned}$$

The larger root of the quadratic leads to $x^3 > 2$ and so is extraneous. Hence $x^3 = 2 - \sqrt{5}$, and so $\sqrt{a} = \sqrt{5}$, $a = 5$.

$$\begin{aligned} f(a) = 2 &\Longleftrightarrow x + (4 - x^3)^{1/3} = 2 \\ &\Longleftrightarrow 4 - x^3 = (2 - x)^3 = 8 - 12x + 6x^2 - x^3 \\ &\Longleftrightarrow 3x^2 - 6x + 2 = 0 \Longleftrightarrow x = \frac{3 \pm \sqrt{3}}{3}\ . \ . \end{aligned}$$

Now,

$$\left(\frac{3 \pm \sqrt{3}}{3}\right)^3 = 2 \pm \frac{10\sqrt{3}}{9}\ .$$

The larger value of x leads to $x^3 > 2$, and so is inadmissible. The smaller value of x leads to $x^3 = 2 - (10\sqrt{3}/9)$ and $\sqrt{a} = (10\sqrt{3}/9)$, $a = 100/27$. Both values of a check out.

466. For a positive integer m, let $\overline{m}$ denote the sum of the digits of m. Find all pairs of positive integers (m, n) with $m < n$ for which $(\overline{m})^2 = n$ and $(\overline{n})^2 = m$.

Solution. Let $m = m_k \cdots m_1 m_0$ where $0 \leq m_i \leq 9$ are the digits of m. Then

$$10^k \leq m < n = (m_k + \cdots + m_0)^2 \leq [(k+1)10]^2\ ,$$

whence $10^{k-2} \leq (k+1)^2$ and $0 \leq k \leq 3$.

Hence $m < n = (m_3 + m_2 + m_1 + m_0)^2 \leq (4 \times 9)^2 = 36^2$. Since m and n are both perfect squares, we need only consider $m = r^2$, where $1 \leq r \leq 36$.

In the case that $k = 3$, $\overline{m} < 1 + 9 + 9 + 9 = 28$. Since $28^2 < 1000 < m < n$, there are no examples. In the case that $k = 2$, $\overline{m} < 6 + 9 + 9 = 24$ and so $n^2 \leq 24^2$. The only possibility is $(m, n) = (169, 256)$. There are no possibilities when $k = 0$ or $k = 1$.

Hence, the only number pair is $(m, n) = (169, 256)$.

Comment. This is problem **621** from *The College Mathematics Journal*.

467. For which positive integers n does there exist a set of n distinct positive integers such that

(a) each member of the set divides the sum of all members of the set, *and*

(b) none of its proper subsets with two or more elements satisfies the condition in (a)?

Solution. When $n = 1$, condition (b) is satisfied vacuously, and any singleton will do. When $n = 2$, such a set cannot be found. If a and b are any two positive integers, then condition (b) entails that both a and b divide $a + b$, and so must divide each other. This cannot happen when a and b are distinct.

When $n \geq 3$, a set of the required type can be found. For example, let

$$S_n = \{1, 2, 2 \times 3, 2 \times 3^2, \cdots 2 \times 3^{n-3}, 3^{n-2} \ .$$

The sum of the elements in S_n is $2 \times 3^{n-2}$, which is divisible by each member of S_n.

Consider any proper subset R of S_n with at least three numbers. If 3^{n-2} belongs to R, then the sum of the elements of R must be strictly between 3^{n-2} and $2 \times 3^{n-2}$, and so not divisible by 3^{n-2}. If R does not contain 3^{n-2}, then its largest entry has the form 2×3^k with $1 \leq k \leq n-3$. Then the sum of R is greater than 2×3^k and does not exceed $1 + 2(1 + 3 + \cdots + 3^k) = 3^{k+1} < 2(2 \times 3^k)$. Hence this sum is not divisible by 2×3^k. As we have seen, no doubleton satisfies the condition. Hence (b) is satisfied for all subsets of S_n.

Comment. This is problem **1504** in the October, 1996 issue of *Mathematics Magazine*.

468. Let a and b be positive real numbers satisfying $a + b \geq (a-b)^2$. Prove that

$$x^a(1-x)^b + x^b(1-x)^a \leq \frac{1}{2^{a+b-1}}$$

for $0 \leq x \leq 1$, with equality if and only if $x = \frac{1}{2}$.

Comment. Denote the left side by $f(x)$. When $a = b$, $f(x) = 2x^a(1-x)^a$, which is maximized when $x = 1/2$, its maximum value being 2×4^{-a}. In the general case, the solution can be obtained by calculus. Since $f(0) = f(1) = 0$ and the function possesses a derivative everywhere, the maximum occurs when $f'(x) = 0$ and $0 < x < 1$. Wolog, assume that $a < b$. We have that

$$f'(x) = x^{a-1}(1-x)^{a-1}[(a - (a+b)x)(1-x)^{b-a} + (b - (a+b)x)x^{b-a}] \ .$$

This solution can be found in *Mathematics Magazine* 70:4 (October, 1997), 301-302 (Problem **1505**), and is fairly technical. It would be nice to have a more transparent argument. Is there a solution that avoids calculus, at least for rational a and b?

A second solution, employs the substitution $2x = 1 - y$ to get the equivalent inequality

$$(1-y)^a(1+y)^b + (1-y)^b(1+y)^a \leq 2$$

for $|y| \leq 1$. Wolog, we can let $a = b + c$ with $c \geq 0$. Then the condition becomes $2b \geq c^2 - c$. Then the inequality is equivalent to

$$(1-y^2)^b[(1-y)^c + (1+y)^c] \leq 2 \ ,$$

for $|y| \leq 1$.

Let $0 \leq c \leq 1$. Then, for $t > 0$, the function t^c is concave, so that, for $u, v > 0$,

$$\left(\frac{u+v}{2}\right)^c \geq \frac{u^c + v^c}{2} \ .$$

Setting $(u, v) = (1-y, 1+y)$, we find that $(1-y)^c + (1+y)^c \leq 2$ for $|y| \leq 1$. Hence the inequality holds, with equality occurring when $y = 0$ $(x = 1/2)$.

When $c > 1$, I do not have a clean solution. First, it suffices to consider the inequality when b is replaced by $\frac{1}{2}(c^2 - c)$. Thus, we need to establish that

$$(1-y^2)^{(1/2)(c^2-c)}[(1-y)^c + (1-y)^c] \leq 2 \qquad (*)$$

for $|y| \leq 1$. The derivative of the natural logarithm of the left side is a positive multiple of

$$g(y) = (1+y)^c(1-cy) - (1-y)^c(1+cy) \ .$$

If this can be shown to be nonpositive, then the result will follow. An equivalent inequality is

$$\left(1-\frac{2y}{1+y}\right)^2=\left(\frac{1-y}{1+y}\right)^c\geq\left(\frac{1-cy}{1+cy}\right)=\left(1-\frac{2cy}{1+cy}\right)\ ,$$

for $c>1$ and $|y|\leq 1$.

469. Solve for t in terms of a,b in the equation

$$\sqrt{\frac{t^3+a^3}{t+a}}+\sqrt{\frac{t^3+b^3}{t+b}}=\sqrt{\frac{a^3-b^3}{a-b}}$$

where $0<a<b$.

Solution 1. The equation is equivalent to

$$\sqrt{t^2-at+a^2}+\sqrt{t^2-bt+b^2}=\sqrt{a^2+ab+b^2}\ .$$

Square both sides of the equation, collect the nonradical terms on one side and the radical on the other and square again. Once the polynomials are expanded and like terms collected, we obtain the equation

$$0=t^2(a+b)^2-2ab(a+b)t+a^2b^2=[t(a+b)-ab]^2\ ,$$

whence $t=ab/(a+b)$. This can be checked by substituting it into the equation.

Solution 2. [Y. Wang] As in solution 1, we can find an equivalent equation, which can then be manipulated to

$$\sqrt{(t-(a/2))^2+(\sqrt{3}a/2)^2}+\sqrt{(t-(b/2))^2+(-\sqrt{3}b/2)^2}=\sqrt{(a/2-b/2)^2+(\sqrt{3}a/2+\sqrt{3}b/2)}\ .$$

If we consider the points $A\sim(a/2,\sqrt{3}a/2)$, $B\sim(b/2,-\sqrt{3}b/2)$ and $T\sim(t,0)$, then we can interpret this equation as stating that $AT+BT=AB$. By the triangle inequality, we see that T must lie on AB, so that the slopes of AT and BT are equal. Thus

$$\frac{\sqrt{3}a}{a-2t}=\frac{\sqrt{3}b}{2t-b}\ ,$$

whence $t=ab/(a+b)$.

470. Let ABC, ACP and BCQ be nonoverlapping triangles in the plane with angles CAP and CBQ right. Let M be the foot of the perpendicular from C to AB. Prove that lines AQ, BP and CM are concurrent if and only if $\angle BCQ=\angle ACP$.

Solution 1. [A. Tavakoli] Let BP and AQ intersect at K. Let $\angle BCQ=\alpha$, $\angle ACP=\beta$ and $\angle BCA=\gamma$. By the trigonometric form of Ceva's theorem, CM, AP and BQ are concurrent if and only if

$$\frac{\sin\angle BCM}{\sin\angle ACM}\cdot\frac{\sin\angle KAC}{\sin\angle KAB}\cdot\frac{\sin\angle KBA}{\sin\angle KBC}=1\ . \tag{1}$$

This holds whether K lies inside or outside of the triangle.

We have that $\sin\angle BCM=\cos\angle CBA$, $\sin\angle ACM=\cos\angle CAB$, and, by the Law of Sines applied to triangles ACQ and ABQ,

$$\sin\angle KAC=\sin\angle QAC=(\sin\angle ACQ)(|QC|)/(|AQ|)\ ,$$

and

$$\sin \angle KAB = \sin \angle QAB = (\sin \angle ABQ)(|QB|)/(|AQ|) \ .$$

Therefore

$$\frac{\sin \angle KAC}{\sin \angle KAB} = \left(\frac{\sin \angle ACQ}{\sin \angle ABQ} \right) \cdot \left(\frac{|QC|}{|QB|} \right) = \left(\frac{\sin(\gamma + \alpha)}{\sin(\angle ABC + 90^\circ)} \right) \cdot \left(\frac{1}{\sin \alpha} \right) = \frac{-\sin(\gamma + \alpha)}{(\cos \angle CBA) \sin \alpha} \ .$$

Similarly,

$$\sin \angle KBA = \sin \angle BAP(|AP|/|BP|)$$

$$\sin \angle KBC = \sin \angle BCP(|PC|/|BP|)$$

and so

$$\frac{\sin \angle KBA}{\sin \angle KBC} = \frac{\sin(\angle BAC + 90^\circ)}{\sin(\beta + \gamma)} \cdot \frac{|AP|}{|PC|} = \frac{-\cos(\angle BAC) \sin \beta}{\sin(\beta + \gamma)} \ .$$

Hence the condition for concurrency becomes

$$\frac{\sin(\gamma + \alpha)}{\sin \alpha} \cdot \frac{\sin \beta}{\sin(\gamma + \beta)} = 1$$

$$\iff \sin \gamma \cot \alpha + \cos \gamma = \sin \gamma \cot \beta + \cos \gamma$$

$$\iff \cot \alpha = \cot \beta \iff \angle BCQ = \alpha = \beta = \angle ACP \ .$$

This is the required result.

Solution 2. We do some preliminary work. Suppose that PB and AQ intersect at O, and that X and Y are the respective feet of the perpendiculars from C to PB and AQ. Since $\angle CXP = \angle CAP = 90^\circ$, $CAXP$ is concyclic and so $\angle ACP = \angle AXP$. Similarly $CQBY$ is concyclic and so $\angle BCQ = \angle BYQ$. Since $\angle CXO = \angle CYO = 90^\circ$, X and Y lie on the circle with diameter CO. Hence $\angle YCO = \angle YXO = \angle YXB$.

Now suppose that $\angle BCQ = \angle ACP$. Let CO produced meet AB at N. Since $\angle AXP = \angle ACP = \angle BCQ = \angle BYQ$, it follows that $\angle AXB = \angle AYB$ so that $BYXA$ is concyclic and so $\angle YXB = \angle YAB$. Therefore

$$\angle YCN = \angle YCO = \angle YXB = \angle YAB = \angle YAN$$

and $ANYC$ is concyclic/ Hence $\angle CNA = \angle CYA = 90^\circ$ and N must coincide with M.

On the other hand, let CM pass through O. Since $\angle CYA = \angle CMA = 90^\circ$, $AMYC$ is concyclic so that

$$\angle YAB = \angle YAM = \angle YCM = \angle YCO = \angle YXB \ .$$

Therefore $BAXY$ is concyclic and $\angle BXA = \angle BYA \Rightarrow \angle AXP = \angle BYQ$. Since $CAXP$ and $CYBQ$ are concyclic, $\angle ACP = \angle AXP = \angle BYQ = \angle BCQ$.

471. Let I and O denote the incentre and the circumcentre, respectively, of triangle ABC. Assume that triangle ABC is not equilateral. Prove that $\angle AIO \leq 90^\circ$ if and only if $2BC \leq AB + CA$, with equality holding only simultaneously.

Solution 1. Wolog, let $AB \geq AC$. Suppose that the circumcircle of triangle ABC intersects AI in D. Construct the circle Γ with centre D that passes through B and C. By the symmetry of AB and AC in the angle bisector AD, this circle intersects segment AB in a point F such that $AF = AC$. Let Γ intersect AD at P. Then chords CP and FP have the same length. If $AB > AC$, this implies that P is on the angle bisector of angle ABC. If $AB = AC$, then $\angle ABC = \angle ADC = \angle PDC = 2\angle PBC$. In either case, $P = I$.

Let E be on the ray BA produced such that $AE = AC$. Since $\angle DAC = \frac{1}{2}\angle BAC = \angle AEC$ and $\angle ADC = \angle ABC = \angle EBC$, triangles ADC and EBC are similar, and so

$$ID : AD = CD : AD = BC : BE = BC : (AB + AC) \ .$$

But $\angle AIO \leq 90°$ if and only if $ID/AD \leq 1/2$, and so is equivalent to $2BC \leq AB + AC$, with equality holding only simultaneously. (Solution due to Wu Wei Chao in China.)

Solution 2. We have that $\angle AIO \leq 90°$ if and only if $\cos\angle AIO \geq 0$, if and only if $|AO|^2 \leq |OI|^2 + |IA|^2$. Let a, b, c be the respective sidelengths of BC, CA, AB; let R be the circumradius and let r be the inradius of triangle ABC. Since, by Euler's formula, $|OI|^2 = R^2 - 2Rr$, and $r = |IA|\sin(A/2)$, the foregoing inequality is equivalent to

$$2R \leq \frac{r}{\sin^2(A/2)} = \frac{2r}{1-\cos A} \ .$$

Applying $R = a/(2\sin A)$, $r = bc\sin A/(a+b+c)$ and $2bc\cos A = b^2 + c^2 - a^2$, we find that

$$\begin{aligned} r - R(1-\cos A) &= \frac{bc\sin A}{a+b+c} - \frac{a(1-\cos A)}{2\sin A} \\ &= \sin A\left[\frac{bc}{a+b+c} - \frac{a(1-\cos A)}{2\sin^2 A}\right] \\ &\frac{\sin A}{2(1+\cos A)(a+b+c)}[2bc + 2bc\cos A - a(a+b+c)] \\ &\frac{\sin A}{2(1+\cos A)(a+b+c)}[2bc + b^2 + c^2 - a^2 - a(a+b+c)] \\ &\frac{\sin A}{2(1+\cos A)(a+b+c)}[(b+c)^2 - 2a^2 - a(b+c)] \\ &\frac{\sin A}{2(1+\cos A)(a+b+c)}[(b+c+a)(b+c-2a)] \ . \end{aligned}$$

Hence the inequality $R(1-\cos A) \leq r$ is equivalent to $2a \leq b + c$. The desired result follows. (Solution due to Can A. Minh, USA)

Solution 3. [Y. Wang] Let AI intersect the circumcircle of triangle ABC at D. Since AI bisects the angle BAC and the arc BC, we have that $BD = BC$. Also,

$$\angle DIC = \angle CAD + \angle ACI = \angle BCD + \angle BCI = \angle DCI \ ,$$

whence $DC = DI = DB$. Using Ptolemy's Theorem, we have that

$$AB \times CD + BD \times AC = AD \times BC \ ,$$

so that

$$AB \times DI + DI \times AC = (AI + ID) \times BC \ .$$

Hence

$$k \equiv \frac{AB + AC}{BC} = 1 + \frac{AI}{ID} \ .$$

If $AB = AC$, then A, O, I are collinear. Let $k < 2$; then $AI < ID$ and I lies between A and O and $\angle AIO = 180°$. Let $k > 2$; then $AI > ID$, O lies between A and I and $\angle AIO = 0°$. [If $k = 2$, then $AI = ID$, the incentre and circumcentre coincide and the triangle is equilateral – the excluded case.]

Wolog, suppose that $AB > AC$. Then the circumcentre O lies within the triangle ABD. Let P be the foot of the perpendicular from O to AD. Then P is the midpoint of AD and the angle AIO is greater than, equal to or less than $90°$ according as I is in the segment AP, coincides with P or is in the segment PD. These correspond to $k < 2$, $k = 2$ and $k > 2$, and the result follows.

472. Find all integers x for which

$$(4-x)^{4-x} + (5-x)^{5-x} + 10 = 4^x + 5^x \ .$$

Solution. If $x < 0$, then the left side is an integer, but the right side is positive and less than $\frac{1}{4} + \frac{1}{5} < 1$. If $x > 5$, then the left side is less than $\frac{1}{4}$, while the right side is a positive integer. Therefore, the only candidates for solution are the integers between 0 and 5 inclusive. Checking, we find that the only solution is $x = 2$.

473. Let $ABCD$ be a quadrilateral; let M and N be the respective midpoint of AB and BC; let P be the point of interesection of AN and BD, and Q be the point of intersection of DM amd AC. Suppose the $3BP = BD$ and $3AQ = AC$. Prove that $ABCD$ is a parallelogram.

Solution. Let $\overrightarrow{AB} = \mathbf{x}$, $\overrightarrow{BC} = \mathbf{y}$ and $\overrightarrow{CD} = a\mathbf{x} + b\mathbf{y}$, where a and b are real numbers. Then

$$\overrightarrow{AD} = (a+1)\mathbf{x} + (b+1)\mathbf{y}$$

and

$$\overrightarrow{AN} = \mathbf{x} + \frac{1}{2}\mathbf{y} \ .$$

But $\overrightarrow{BD} = 3\overrightarrow{BP}$, so that

$$\overrightarrow{AP} = \frac{2\overrightarrow{AB} + \overrightarrow{AD}}{3} = \frac{a+3}{3}\mathbf{x} + \frac{b+1}{3}\mathbf{y} \ .$$

Since the vectors $\overrightarrow{AP}$ and $\overrightarrow{AN}$ are collinear, $a + 3 : 1 = b + 1 : \frac{1}{2}$, whence $a - 2b + 1 = 0$. Also

$$\overrightarrow{DM} = \overrightarrow{AM} - \overrightarrow{AD} = \left(\frac{1}{2} - a - 1\right)\mathbf{x} - (b+1)\mathbf{y} = -\left(a + \frac{1}{2}\right)\mathbf{x} - (b+1)\mathbf{y}$$

and

$$\overrightarrow{DQ} = \overrightarrow{AQ} - \overrightarrow{AD} = \frac{1}{3}(\mathbf{x} + \mathbf{y}) - (a+1)\mathbf{x} - (b+1)\mathbf{y} = -\frac{1}{3}[(3a+2)\mathbf{x} + (3b+2)\mathbf{y}] \ .$$

Since the vectors $\overrightarrow{DQ}$ and $\overrightarrow{DM}$ are collinear, we must have $(3a + 2) : (a + \frac{1}{2}) = (3b + 2) : (b + 1)$, whence $2a + b + 2 = 0$. Therefore $(a, b) = (-1, 0)$, $\overrightarrow{CD} = -\mathbf{x} = \overrightarrow{BA}$ and $\overrightarrow{AD} = \mathbf{y} = \overrightarrow{BC}$. Hence $ABCD$ is a parallelogram.

474. Solve the equation for positive real x:

$$(2^{\log_5 x} + 3)^{\log_5 2} = x - 3 \ .$$

Solution. Recall the identity $u^{\log_b v} = v^{\log_b u}$ for positive u, v and positive base $b \neq 1$. (Take logarithms to base b.) Then, for all real t, $(2^t + 3)^{\log_5 2} = 2^{\log_5 (2^t + 3)}$. This is true in particular when $t = \log_5 x$.

Let $f(x) = 2^{\log_5 x} + 3$ for $x > 0$. Then $f(x) = x^{\log_5 2} + 3$ and the equation to be solved is $f(f(x)) = x$. The function $f(x)$ is an increasing function of the positive variable x. If $f(x) < x$, then $f(f(x)) < f(x)$; if $f(x) > x$, then $f(f(x)) > f(x)$. Hence, for $f(f(x)) = x$ to be true, we must have $f(x) = x$. With $t = \log_5 x$, the equation becomes $2^t + 3 = 5^t$, or equivalently, $(2/5)^t + 3(1/5)^t = 1$. The left side is a stricly decreasing function of t, and so equals the right side only when $t = 1$. Hence the unique solution of the equation is $x = 5$.

475. Let z_1, z_2, z_3, z_4 be distinct complex numbers for which $|z_1| = |z_2| = |z_3| = |z_4|$. Suppose that there is a real number $t \neq 1$ for which

$$|tz_1 + z_2 + z_3 + z_4| = |z_1 + tz_2 + z_3 + z_4| = |z_1 + z_2 + tz_3 + z_4| \ .$$

Show that, in the complex plane, z_1, z_2, z_3, z_4 lie at the vertices of a rectangle.

Solution. Let $s = z_1 + z_2 + z_3 + z_4$. Then

$$|s - (1-t)z_1| = |s - (1-t)z_2| = |s - (1-t)z_3| \ .$$

Therefore, s is equidistant from the three distinct points $(1-t)z_1$, $(1-t)z_2$ and $(1-t)z_3$; but these three points are on the circle with centre 0 and radius $(1-t)z_1$. Therefore $s = 0$.

Since $z_1 - (-z_2) = z_1 + z_2 = -z_3 - z_4 = (-z_4) - z_3$ and $z_2 - (-z_3) = z_2 + z_3 = -z_4 - z_1 = (-z_4) - z_1$, z_1, $-z_2$, z_3 and $-z_4$ are the vertices of a parallelogram inscribed in a circle centered at 0, and hence of a rectangle whose diagonals intersect at 0. Therefore, $-z_2$ is the opposite of one of z_1, z_3 and $-z_4$. Since z_2 is unequal to z_1 and z_3, we must have that $-z_2 = z_4$. Also $z_1 = -z_3$. Hence z_1, z_2, z_3 and z_4 are the vertices of a rectangle.

476. Let p be a positive real number and let $|x_0| \leq 2p$. For $n \geq 1$, define

$$x_n = 3x_{n-1} - \frac{1}{p^2}x_{n-1}^3 \ .$$

Determine x_n as a function of n and x_0.

Solution. Let $x_n = 2py_n$ for each nonnegative integer n. Then $|y_0| \leq 1$ and $y_n = 3y_{n-1} - 4y_{n-1}^3$. Recall that

$$\sin 3\theta = \sin 2\theta \cos\theta + \sin\theta\cos 2\theta = 2\sin\theta(1 - \sin^2\theta) + \sin\theta(1 - 2\sin^2\theta) = 3\sin\theta - 4\sin^3\theta \ .$$

Select $\theta \in [-\pi/2, \pi/2]$. Then, by induction, we determine that $y_n = \sin 3^n\theta$ and $x_n = 2p\sin 3^n\theta$, for each nonnegative integer n, where $\theta = \arcsin(x_0/2p)$.

477. Let S consist of all real numbers of the form $a + b\sqrt{2}$, where a and b are integers. Find all functions that map S into the set $\mathbf{R}$ of reals such that (1) f is increasing, and (2) $f(x+y) = f(x) + f(y)$ for all x, y in S.

Solution. Since $f(0) = f(0) + f(0)$, $f(0) = 0$ and $f(x) \geq 0$ for $x \geq 0$. Let $f(1) = u$ and $f(\sqrt{2}) = v$; u and v are both nonnegative. Since $f(0) = f(x) + f(-x)$, $f(-x) = -f(x)$ for all x. Since, by induction, it can be shown that $f(nx) = nf(x)$ for every positive integer n, it follows that

$$f(a + b\sqrt{2}) = au + bv \ ,$$

for every pair (a, b) of integers.

Since f is increasing, for every positive integer n, we have that

$$f(\lfloor n\sqrt{2}\rfloor) \leq f(n\sqrt{2}) \leq f(\lfloor n\sqrt{2}\rfloor + 1) \ ,$$

so that

$$\lfloor n\sqrt{2}\rfloor u \leq nv \leq (\lfloor n\sqrt{2}\rfloor + 1)u \ .$$

Therefore,

$$\left(\sqrt{2} - \frac{1}{n}\right)u \leq \left(\frac{\lfloor n\sqrt{2}\rfloor}{n}\right)u \leq v \leq \frac{1}{n}(\lfloor n\sqrt{2}\rfloor + 1)u \leq \left(\sqrt{2} + \frac{1}{n}\right)u \ ,$$

for every positive integer n. It follows that $v = u\sqrt{2}$, so that $f(x) = ux$ for every $x \in S$. It is readily checked that this equation satisfies the conditions for all nonegative u.

478. Solve the equation

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + x}}} + \sqrt{3}\sqrt{2 - \sqrt{2 + \sqrt{2 + x}}} = 2x$$

for $x \geq 0$

Solution. Since $2 - \sqrt{2 + \sqrt{2+x}} \geq 0$, we must have $0 \leq x \leq 2$. Therefore, there exists a number $t \in [0, \frac{1}{2}\pi]$ for which $\cos t = \frac{1}{2}x$. Now we have that,

$$\begin{aligned}
\sqrt{2+\sqrt{2+\sqrt{2+x}}} &= \sqrt{2+\sqrt{2+\sqrt{2+2\cos t}}} \\
&= \sqrt{2+\sqrt{2+\sqrt{4\cos^2(t/2)}}} = \sqrt{2+\sqrt{2+2\cos(t/2)}} \\
&= \sqrt{2+2\cos(t/4)} = 2\cos(t/8)\ .
\end{aligned}$$

Similarly, $\sqrt{2-\sqrt{2+\sqrt{2+x}}} = 2\sin(t/8)$. Hence the equation becomes

$$2\cos\frac{t}{8} + 2\sqrt{3}\sin\frac{t}{8} = 4\cos t$$

or

$$\frac{1}{2}\cos\frac{t}{8} + \frac{\sqrt{3}}{2}\sin\frac{t}{8} = \cot t\ .$$

Thus,

$$\cos\left(\frac{\pi}{3} - \frac{t}{8}\right) = \cos t\ .$$

Since the argument of the cosine on the left side lies between 0 and $\pi/3$, we must have that $(\pi/3) - (t/8) = t$, or $t = 8\pi/27$.