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Notes. A real-valued function f defined on an interval is *concave* iff $f((1-t)u+tv) \geq (1-t)f(u)+tf(v)$ whenever $0 < t < 1$ and u and v are in the domain of definition of $f(x)$. If $f(x)$ is a one-one function defined on a domain into a range, then the *inverse* function $g(x)$ defined on the set of values assumed by f is determined by $g(f(x)) = x$ and $f(g(y)) = y$; in other words, $f(x) = y$ if and only if $g(y) = x$.

A sequence $\{x_n\}$ *converges* if and only if there is a number c, called its *limit*, such that, as n increases, the number x_n gets closer and closer to c. If the sequences is *increasing* (*i.e.*, $x_{n+1} \geq x_n$ for each index n) and *bounded above* (*i.e.*, there is a number M for which $x_n \leq M$ for each n, then it must converge. [Do you see why this is so?] Similarly, a decreasing sequence that is bounded below converges. [Supply the definitions and justify the statement.] An infinite *series* is an expression of the form $\sum_{k=a}^{\infty} x_k = x_a + x_{a+1} + x_{a+2} + \cdots + x_k + \cdots$, where a is an integer, usually 0 or 1. The nth *partial sum* of the series is $s_n \equiv \sum_{k=a}^{n} x_k$. The series has sum s if and only if its sequence $\{s_n\}$ of partial sums converges and has limit s; when this happens, the series *converges*. If the sequence of partial sums fails to converge, the series *diverges*. If every term in the series is nonnegative and the sequence of partial sums is bounded above, then the series converges. If a series of nonnegative terms converges, then it is possible to rearrange the order of the terms without changing the value of the sum.

A *rectangular hyperbola* is an hyperbola whose asymmptotes are at right angles.

55. A textbook problem has the following form: *A man is standing in a line in front of a movie theatre. The fraction x of the line is in front of him, and the fraction y of the line is behind him, where x and y are rational numbers written in lowest terms. How many people are there in the line?* Prove that, if the problem has an answer, then that answer must be the least common multiple of the denominators of x and y.

56. Let n be a positive integer and let $x_1, x_2, \cdots, x_n$ be integers for which
$$x_1^2 + x_2^2 + \cdots + x_n^2 + n^3 \leq (2n-1)(x_1 + x_2 + \cdots + x_n) + n^2 \ .$$
Show that

(a) $x_1, x_2, \cdots, x_n$ are all nonnegative;

(b) $x_1 + x_2 + \cdots + x_n + n + 1$ is not a perfect square.

57. Let $ABCD$ be a rectangle and let E be a point in the diagonal BD with $\angle DAE = 15°$. Let F be a point in AB with $EF \perp AB$. It is known that $EF = \frac{1}{2}AB$ and $AD = a$. Find the measure of the angle $\angle EAC$ and the length of the segment EC.

58. Find integers a, b, c such that $a \neq 0$ and the quadratic function $f(x) = ax^2 + bx + c$ satisfies
$$f(f(1)) = f(f(2)) = f(f(3)) \ .$$

59. Let $ABCD$ be a concyclic quadrilateral. Prove that

$$|AC - BD| \leq |AB - CD| .$$

60. Let $n \geq 2$ be an integer and $M = \{1, 2, \cdots, n\}$. For every integer k with $1 \leq k \leq n-1$, let

$$x_k = \sum\{\min A + \max A : A \subseteq M, A \text{ has } k \text{ elements}\}$$

where $\min A$ is the smallest and $\max A$ is the largest number in A. Determine $\sum_{k=1}^{n}(-1)^{k-1}x_k$.

61. Let $S = 1!2!3!\cdots 99!100!$ (the product of the first 100 factorials). Prove that there exists an integer k for which $1 \leq k \leq 100$ and $S/k!$ is a perfect square. Is k unique? (*Optional:* Is it possible to find such a number k that exceeds 100?)

62. Let n be a positive integer. Show that, with three exceptions, $n! + 1$ has at least one prime divisor that exceeds $n + 1$.

63. Let n be a positive integer and k a nonnegative integer. Prove that

$$n! = (n+k)^n - \binom{n}{1}(n+k-1)^n + \binom{n}{2}(n+k-2)^n - \cdots \pm \binom{n}{n}k^n .$$

64. Let M be a point in the interior of triangle ABC, and suppose that D, E, F are points on the respective side BC, CA, AB. Suppose AD, BE and CF all pass through M. (In technical terms, they are *cevians.*) Suppose that the areas and the perimeters of the triangles BMD, CME, AMF are equal. Prove that triangle ABC must be equilateral.

65. Suppose that XTY is a straight line and that TU and TV are two rays emanating from T for which $\angle XTU = \angle UTV = \angle VTY = 60°$. Suppose that P, Q and R are respective points on the rays TY, TU and TV for which $PQ = PR$. Prove that $\angle QPR = 60°$.

66. (a) Let $ABCD$ be a square and let E be an arbitrary point on the side CD. Suppose that P is a point on the diagonal AC for which $EP \perp AC$ and that Q is a point on AE produced for which $CQ \perp AE$. Prove that B, P, Q are collinear.

    (b) Does the result hold if the hypothesis is weakened to require only that $ABCD$ is a rectangle?

67. (a) Consider the infinite integer lattice in the plane (*i.e.*, the set of points with integer coordinates) as a graph, with the edges being the lines of unit length connecting nearby points. What is the minimum number of colours that can be used to colour all the vertices and edges of this graph, so that

    (i) each pair of adjacent vertices gets two distinct colours; AND

    (ii) each pair of edges that meet at a vertex get two distinct colours; AND

    (iii) an edge is coloured differently that either of the two vertices at the ends?

    (b) Extend this result to lattices in real $n-$dimensional space.

68. Let $a, b, c > 0$, $a < bc$ and $1 + a^3 = b^3 + c^3$. Prove that $1 + a < b + c$.

69. Let n, a_1, a_2, $\cdots$, a_k be positive integers for which $n \geq a_1 > a_2 > a_3 > \cdots > a_k$ and the least common multiple of a_i and a_j does not exceed n for all i and j. Prove that $ia_i \leq n$ for $i = 1, 2, \cdots, k$.

70. Let $f(x)$ be a concave strictly increasing function defined for $0 \leq x \leq 1$ such that $f(0) = 0$ and $f(1) = 1$. Suppose that $g(x)$ is its inverse. Prove that $f(x)g(x) \leq x^2$ for $0 \leq x \leq 1$.

71. Suppose that lengths a, b and i are given. Construct a triangle ABC for which $|AC| = b$. $|AB| = c$ and the length of the bisector AD of angle A is i (D being the point where the bisector meets the side BC).

72. The centres of the circumscribed and the inscribed spheres of a given tetrahedron coincide. Prove that the four triangular faces of the tetrahedron are congruent.

73. Solve the equation:
$$\left(\sqrt{2+\sqrt{2}}\right)^x + \left(\sqrt{2-\sqrt{2}}\right)^x = 2^x \ .$$

74. Prove that among any group of $n + 2$ natural numbers, there can be found two numbers so that their sum or their difference is divisible by $2n$.

75. Three consecutive natural numbers, larger than 3, represent the lengths of the sides of a triangle. The area of the triangle is also a natural number.

    (a) Prove that one of the altitudes "cuts" the triangle into two triangles, whose side lengths are natural numbers.

    (b) The altitude identified in (a) divides the side which is perpendicular to it into two segments. Find the difference between the lengths of these segments.

76. Solve the system of equations:
$$\log x + \frac{\log(xy^8)}{\log^2 x + \log^2 y} = 2 \ ,$$
$$\log y + \frac{\log(x^8/y)}{\log^2 x + \log^2 y} = 0 \ .$$
    (The logarithms are taken to base 10.)

77. n points are chosen from the circumference or the interior of a regular hexagon with sides of unit length, so that the distance between any two of them is **not** less that $\sqrt{2}$. What is the largest natural number n for which this is possible?

78. A truck travelled from town A to town B over several days. During the first day, it covered $1/n$ of the total distance, where n is a natural number. During the second day, it travelled $1/m$ of the remaining distance, where m is a natural number. During the third day, it travelled $1/n$ of the distance remaining after the second day, and during the fourth day, $1/m$ of the distance remaining after the third day. Find the values of m and n if it is known that, by the end of the fourth day, the truck had travelled $3/4$ of the distance between A and B. (Without loss of generality, assume that $m < n$.)

79. Let x_0, x_1, x_2 be three positive real numbers. A sequence $\{x_n\}$ is defined, for $n \geq 0$ by
$$x_{n+3} = \frac{x_{n+2} + x_{n+1} + 1}{x_n} \ .$$
    Determine all such sequences whose entries consist solely of positive integers.

80. Prove that, for each positive integer n, the series
$$\sum_{k=1}^{\infty} \frac{k^n}{2^k}$$
    converges to twice an odd integer not less than $(n+1)!$.

81. Suppose that $x \geq 1$ and that $x = \lfloor x \rfloor + \{x\}$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x and the fractional part $\{x\}$ satisfies $0 \leq x < 1$. Define

$$f(x) = \frac{\sqrt{\lfloor x \rfloor} + \sqrt{\{x\}}}{\sqrt{x}} \quad .$$

(a) Determine the small number z such that $f(x) \leq z$ for each $x \geq 1$.

(b) Let $x_0 \geq 1$ be given, and for $n \geq 1$, define $x_n = f(x_{n-1})$. Prove that $\lim_{n\to\infty} x_n$ exists.

82. (a) A regular pentagon has side length a and diagonal length b. Prove that

$$\frac{b^2}{a^2} + \frac{a^2}{b^2} = 3 \ .$$

(b) A regular heptagon (polygon with seven equal sides and seven equal angles) has diagonals of two different lengths. Let a be the length of a side, b be the length of a shorter diagonal and c be the length of a longer diagonal of a regular heptagon (so that $a < b < c$). Prove that:

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = 6$$

and

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5 \ .$$

83. Let $\mathfrak{C}$ be a circle with centre O and radius 1, and let $\mathfrak{F}$ be a closed convex region inside $\mathfrak{C}$. Suppose from each point $\mathfrak{C}$, we can draw two rays tangent to $\mathfrak{F}$ meeting at an angle of $60°$. Describe $\mathfrak{F}$.

84. Let ABC be an acute-angled triangle, with a point H inside. Let U, V, W be respectively the reflected image of H with respect to axes BC, AC, AB. Prove that H is the orthocentre of ΔABC if and only if U, V, W lie on the circumcircle of ΔABC,

85. Find all pairs (a, b) of positive integers with $a \neq b$ for which the system

$$\cos ax + \cos bx = 0$$

$$a \sin ax + b \sin bx = 0$$

has a solution. If so, determine its solutions.

86. Let $ABCD$ be a convex quadrilateral with $AB = AD$ and $CB = CD$. Prove that

(a) it is possible to inscribe a circle in it;

(b) it is possible to circumscribe a circle about it if and only if $AB \perp BC$;

(c) if $AB \perp AC$ and R and r are the respective radii of the circumscribed and inscribed circles, then the distance between the centres of the two circles is equal to the square root of $R^2 + r^2 - r\sqrt{r^2 + 4R^2}$.

87. Prove that, if the real numbers a, b, c, satisfy the equation

$$\lfloor na \rfloor + \lfloor nb \rfloor = \lfloor nc \rfloor$$

for each positive integer n, then at least one of a and b is an integer.

88. Let I be a real interval of length $1/n$. Prove that I contains no more than $\frac{1}{2}(n+1)$ irreducible fractions of the form p/q with p and q positive integers, $1 \leq q \leq n$ and the greatest common divisor of p and q equal to 1.

89. Prove that there is only one triple of positive integers, each exceeding 1, for which the product of any two of the numbers plus one is divisible by the third.

90. Let m be a positive integer, and let $f(m)$ be the smallest value of n for which the following statement is true:

    given any set of n integers, it is always possible to find a subset of m integers whose sum is divisible by m

    Determine $f(m)$.

91. A square and a regular pentagon are inscribed in a circle. The nine vertices are all distinct and divide the circumference into nine arcs. Prove that at least one of them does not exceed $1/40$ of the circumference of the circle.

92. Consider the sequence 200125, 2000125, 20000125, $\cdots$, $200\cdots00125$, $\cdots$ (in which the nth number has $n+1$ digits equal to zero). Can any of these numbers be the square or the cube of an integer?

93. For any natural number n, prove the following inequalities:

$$2^{(n-1)/(2^n-2)} \leq \sqrt{2}\sqrt[4]{4}\sqrt[8]{8}\cdots\sqrt[2^n]{2^n} < 4 \ .$$

94. ABC is a right triangle with arms a and b and hypotenuse $c = |AB|$; the area of the triangle is s square units and its perimeter is $2p$ units. The numbers a, b and c are positive integers. Prove that s and p are also positive integers and that s is a multiple of p.

95. The triangle ABC is isosceles is isosceles with equal sides AC and BC. Two of its angles measure $40°$. The interior point M is such that $\angle MAB = 10°$ and $\angle MBA = 20°$. Determine the measure of $\angle CMB$.

96. Find all prime numbers p for which all three of the numbers $p^2 - 2$, $2p^2 - 1$ and $3p^2 + 4$ are also prime.

97. A triangle has its three vertices on a rectangular hyperbola. Prove that its orthocentre also lies on the hyperbola.

98. Let $a_1, a_2, \cdots, a_{n+1}, b_1, b_2, \cdots, b_n$ be nonnegative real numbers for which
    (i) $a_1 \geq a_2 \geq \cdots \geq a_{n+1} = 0$,
    (ii) $0 \leq b_k \leq 1$ for $k = 1, 2, \cdots, n$.
    Suppose that $m = \lfloor b_1 + b_2 + \cdots + b_n \rfloor + 1$. Prove that

$$\sum_{k=1}^{n} a_k b_k \leq \sum_{k=1}^{m} a_k \ .$$

99. Let E and F be respective points on sides AB and BC of a triangle ABC for which $AE = CF$. The circle passing through the points B, C, E and the circle passing through the points A, B, F intersect at B and D. Prove that BD is the bisector of angle ABC.

100. If 10 equally spaced points around a circle are joined consecutively, a convex regular inscribed decagon P is obtained; if every third point is joined, a self-intersecting regular decagon Q is formed. Prove that the difference between the length of a side of Q and the length of a side of P is equal to the radius of the circle. [*With thanks to Ross Honsberger.*]

101. Let a, b, u, v be nonnegative. Suppose that $a^5 + b^5 \leq 1$ and $u^5 + v^5 \leq 1$. Prove that

$$a^2u^3 + b^2v^3 \leq 1 \ .$$

    [*With thanks to Ross Honsberger.*]

102. Prove that there exists a tetrahedron $ABCD$, all of whose faces are similar right triangles, each face having acute angles at A and B. Determine which of the edges of the tetrahedron is largest and which is smallest, and find the ratio of their lengths.

103. Determine a value of the parameter θ so that

$$f(x) \equiv \cos^2 x + \cos^2(x+\theta) - \cos x \cos(x+\theta)$$

is a constant function of x.

104. Prove that there exists exactly one sequence $\{x_n\}$ of positive integers for which

$$x_1 = 1 \ , \qquad x_2 > 1 \ , \qquad\qquad x_{n+1}^3 + 1 = x_n x_{n+2}$$

for $n \geq 1$.

105. Prove that within a unit cube, one can place two regular unit tetrahedra that have no common point.

106. Find all pairs (x, y) of positive real numbers for which the least value of the function

$$f(x,y) = \frac{x^4}{y^4} + \frac{y^4}{x^4} - \frac{x^2}{y^2} - \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x}$$

is attained. Determine that minimum value.

107. Given positive numbers a_i with $a_1 < a_2 < \cdots < a_n$, for which permutation $(b_1, b_2, \cdots, b_n)$ of these numbers is the product

$$\prod_{i=1}^{n} \left(a_i + \frac{1}{b_i} \right)$$

maximized?

108. Determine all real-valued functions $f(x)$ of a real variable x for which

$$f(xy) = \frac{f(x) + f(y)}{x + y}$$

for all real x and y for which $x + y \neq 0$.

109. Suppose that

$$\frac{x^2 + y^2}{x^2 - y^2} + \frac{x^2 - y^2}{x^2 + y^2} = k \ .$$

Find, in terms of k, the value of the expression

$$\frac{x^8 + y^8}{x^8 - y^8} + \frac{x^8 - y^8}{x^8 + y^8} \ .$$

110. Given a triangle ABC with an area of 1. Let $n > 1$ be a natural number. Suppose that M is a point on the side AB with $AB = nAM$, N is a point on the side BC with $BC = nBN$, and Q is a point on the side CA with $CA = nCQ$. Suppose also that $\{T\} = AN \cap CM$, $\{R\} = BQ \cap AN$ and $\{S\} = CM \cap BQ$, where $\cap$ signifies that the singleton is the intersection of the indicated segments. Find the area of the triangle TRS in terms of n.

111. (a) Are there four different numbers, not exceeding 10, for which the sum of any three is a prime number?

(b) Are there five different natural numbers such that the sum of every three of them is a prime number?

112. Suppose that the measure of angle BAC in the triangle ABC is equal to α. A line passing through the vertex A is perpendicular to the angle bisector of $\angle BAC$ and intersects the line BC at the point M. Find the other two angles of the triangle ABC in terms of α, if it is known that $BM = BA + AC$.

113. Find a function that satisfies all of the following conditions:

(a) f is defined for every positive integer n;

(b) f takes only positive values;

(c) $f(4) = 4$;

(d)

$$\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + \cdots + \frac{1}{f(n)f(n+1)} = \frac{f(n)}{f(n+1)} \ .$$

114. A natural number is a multiple of 17. Its binary representation (*i.e.*, when written to base 2) contains exactly three digits equal to 1 and some zeros.

(a) Prove that there are at least six digits equal to 0 in its binary representation.

(b) Prove that, if there are exactly seven digits equal to 0 and three digits equal to 1, then the number must be even.

115. Let U be a set of n distinct real numbers and let V be the set of all sums of distinct pairs of them, *i.e.*,

$$V = \{x + y : x, y \in U, x \neq y\} \ .$$

What is the smallest possible number of distinct elements that V can contain?

116. Prove that the equation

$$x^4 + 5x^3 + 6x^2 - 4x - 16 = 0$$

has exactly two real solutions.

117. Let a be a real number. Solve the equation

$$(a-1)\left(\frac{1}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x \cos x}\right) = 2 \ .$$

118. Let a, b, c be nonnegative real numbers. Prove that

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc \ .$$

When does equality hold?

119. The medians of a triangle ABC intersect in G. Prove that

$$|AB|^2 + |BC|^2 + |CA|^2 = 3(|GA|^2 + |GB|^2 + |GC|^2) \ .$$

120. Determine all pairs of nonnull vectors $\mathbf{x}$, $\mathbf{y}$ for which the following sequence $\{a_n : n = 1, 2, \cdots\}$ is (a) increasing, (b) decreasing, where

$$a_n = |\mathbf{x} - n\mathbf{y}| \ .$$

121. Let n be an integer exceeding 1. Let $a_1, a_2, \cdots, a_n$ be posive real numbers and $b_1, b_2, \cdots, b_n$ be arbitrary real numbers for which

$$\sum_{i \neq j} a_i b_j = 0 \ .$$

Prove that $\sum_{i \neq j} b_i b_j < 0$.

122. Determine all functions f from the real numbers to the real numbers that satisfy

$$f(f(x)+y) = f(x^2-y) + 4f(x)y$$

for any real numbers x, y.

123. Let a and b be the lengths of two opposite edges of a tetrahedron which are mutually perpendicular and distant d apart. Determine the volume of the tetrahedron.

124. Prove that

$$\frac{(1^4+\frac{1}{4})(3^4+\frac{1}{4})(5^4+\frac{1}{4})\cdots(11^4+\frac{1}{4})}{(2^4+\frac{1}{4})(4^4+\frac{1}{4})(6^4+\frac{1}{4})\cdots(12^4+\frac{1}{4})} = \frac{1}{313} .$$

125. Determine the set of complex numbers z which satisfy

$$\text{Im }(z^4) = (\text{Re }(z^2))^2 ,$$

and sketch this set in the complex plane. (*Note:* Im and Re refer respectively to the imaginary and real parts.)

126. Let n be a positive integer exceeding 1, and let n circles (*i.e.*, circumferences) of radius 1 be given in the plane such that no two of them are tangent and the subset of the plane formed by the union of them is connected. Prove that the number of points that belong to at least two of these circles is at least n.

Solutions and Comments

55. A textbook problem has the following form: *A man is standing in a line in front of a movie theatre. The fraction x of the line is in front of him, and the fraction y of the line is behind him, where x and y are rational numbers written in lowest terms. How many people are there in the line?* Prove that, if the problem has an answer, then that answer must be the least common multiple of the denominators of x and y.

Solution. Let p and q be the denominators of x and y, when written in their lowest terms; each denominator has no positive divisor save 1 in common with the corresponding numerator. Let n be the number of people in line. Since xn and yn are integers, p and q must both divide n, so that n is a multiple of m, the least common multiple of p and q.

$1-x-y$ can be written as a fraction of the form u/m. Since $(u/m)n = u(n/m) = 1$, we must have that $u = n/m = 1$, so that in particular $m = n$.

Comment. This is quite a difficult question to write up, because you have to put your finger quite precisely on the main issues involved. Many solvers relied on statements that were equally if not more difficult to see than the result of the problem. The point of a solution is to show how a result can be obtained from simpler or more wellknown propositions.

56. Let n be a positive integer and let $x_1, x_2, \cdots, x_n$ be integers for which

$$x_1^2 + x_2^2 + \cdots + x_n^2 + n^3 \leq (2n-1)(x_1+x_2+\cdots+x_n) + n^2 .$$

Show that

(a) $x_1, x_2, \cdots, x_n$ are all nonnegative;

(b) $x_1 + x_2 + \cdots + x_n + n + 1$ is not a perfect square.

Solution 1. The inequality can be rewritten as

$$\sum_{i=1}^{n}(x_i - n)(x_i - \overline{n-1}) \leq 0 \ .$$

(The line over $n-1$ is a form of bracket.) Since each summand is positive for integers other than n and $n-1$, the inequality can hold only if each x_i is equal to either n or $n-1$. Therefore,

$$n^2 + 1 = n(n-1) + n + 1 \leq x_1 + x_2 + \cdots + x_n + n + 1 \leq n^2 + n + 1 < (n+1)^2 \ .$$

Parts (a) and (b) follow easily from this.

Solution 2. [R. Furmaniak] Taking the difference between the right and left sides leads to the equivalent inequality

$$\sum_{k=1}^{n}[2x_k - (2n-1)]^2 \leq n \ .$$

Since each term in the sum is odd, it can only be 1. Therefore $2x_k - (2n-1) = \pm 1$, whence x_k is equal to n or $n-1$ for each k. The solution can be concluded as before.

Solution 3. [O. Bormashenko, M. Holmes] The given inequality is equivalent to

$$\sum_{i=1}^{n} x_i(x_i - \overline{2n-1}) \leq n^2 - n^3 \ .$$

Observe that, for each i,

$$x_i(\overline{2n-1} - x_i) \leq [\frac{1}{2}(2n-1)]^2 = n^2 - n + \frac{1}{4} \ .$$

This is a consequence of the arithmetic-geometric means inequality when the left side is positive, and obvious when the left side is negative. Since x_i is supposed to be an integer, we must have

$$x_i(\overline{2n-1} - x_i) \leq n^2 - n$$

and equality occurs if and only if $x_i = n$ or $x_i = n-1$. (One way to see this is to note that the left side is a quadratic and can take each of its values no more than twice.) Multiplying by -1 yields that

$$x_i(x_i - \overline{2n-1}) \geq n - n^2$$

whence

$$\sum_{i=1}^{n} x_i(x_i - \overline{2n-1}) \geq n^2 - n^3 \ ,.$$

with equality if and only if each x_i is equal to n or $n-1$. But because of the given inequality, equality *does* occur. The solution can now be completed as before.

Solution 4. The given inequality is equivalent to

$$\sum_{i=1}^{n}(n - x_i)^2 \leq \sum_{i=1}^{n}(n - x_i) \ . \qquad (1)$$

However, since x_i is an integer for each i, $(n - x_i)^2 \geq (n - x_i)$ with equality if and only if $n - x_i$ is equal to 0 or 1. Thus

$$\sum_{i=1}^{n}(n - x_i)^2 \geq \sum_{i=1}^{n}(n - x_i) \qquad (2)$$

with equality if and only if $n - x_i = 0$ or 1 for each i. But (1) and (2) together imply that equality *does* occur for each i, whence x_i is equal to either n or $n-1$. The solution is completed as before.

57. Let $ABCD$ be a rectangle and let E be a point in the diagonal BD with $\angle DAE = 15^\circ$. Let F be a point in AB with $EF \perp AB$. It is known that $EF = \frac{1}{2}AB$ and $AD = a$. Find the measure of the angle $\angle EAC$ and the length of the segment EC.

Solution. We begin with the observation that $\tan 15^\circ = 2 - \sqrt{3}$. (*Exercise:* Use either $\tan\theta = (\sin 2\theta)/(1+\cos 2\theta)^{-1}$ or $\tan\theta = \tan(3\theta - 2\theta)$ with $\theta = 15^\circ$.) Let a and b be the respective lengths of AD and FE, so that $|AF| = (2-\sqrt{3})b$, $|AB| = 2b$ and $|FB| = \sqrt{3}b$. Then $\tan\angle FEB = \sqrt{3}$, so that $\angle FEB = \angle ADB = 60^\circ$. Hence $|BE| = 2b$, $|BD| = 2a$ and $|ED| = 2(a-b)$. Since $a/2b = \cot\angle ADB = 1/\sqrt{3}$, $3a^2 = 4b^2$.

Since $\angle DAC = \angle ADB = 60^\circ$ and $\angle DAE = 15^\circ$, it follows that $\angle EAC = 45^\circ$.

We can determine $|EC|$ using the law of cosines in ΔDEC. Thus,

$$\begin{aligned}
|EC|^2 &= 4b^2 + 4(a-b)^2 - 8(a-b)b\sqrt{3}/2 \\
&= 8b^2 - 8ab + 4a^2 - 4ab\sqrt{3} + 4b^2\sqrt{3} \\
&= 4b^2(2+\sqrt{3}) - 4ab(2+\sqrt{3}) + 4a^2 \\
&= 3a^2(2+\sqrt{3}) - 2a^2\sqrt{3}(2+\sqrt{3}) + 4a^2 \\
&= a^2(4-\sqrt{3})\ ,
\end{aligned}$$

whence $|EC| = a\sqrt{4-\sqrt{3}}$.

Comment. The length of EC can be found more straightforwardly by introducing an additional point. Produce the line FE to meet CD at G. Then EGC is a right triangle for which $|EG| = a-b$ and $|GC| = \sqrt{3}b$. Applying pythagoras theorem yields

$$|EC|^2 = a^2 - 2ab + 4b^2 = a^2 - \sqrt{3}a^2 + 3a^2 = (4-\sqrt{3})a^2\ .$$

58. Find integers a, b, c such that $a \neq 0$ and the quadratic function $f(x) = ax^2 + bx + c$ satisfies

$$f(f(1)) = f(f(2)) = f(f(3))\ .$$

Solution 1. Suppose that $f(p) = f(q)$ with $p \neq q$. Then $a(p^2 - q^2) + b(p-q) = 0$, from which $p + q = -b/a$. It follows from this (Explain!) that $f(x)$ can take the same value at at most two distinct points. In particular, it is not possible that $f(1) = f(2) = f(3)$. Nor can $f(1)$, $f(2)$, $f(3)$ take three different values. Therefore, two of these take one value and the third a second value.

We make another preliminary observation. Suppose $f(p) = f(q) = u$, $f(r) = v$ and $f(u) = f(v)$, where p, q, r are different, and also u and v are different. Then, from the first paragraph, we have that $p + q = u + v = -b/a$.

Suppose, now, that $(p,q,r) = (1,2,3)$ and that $f(1) = u$. Then

$$\begin{aligned}
a + b + c &= u \\
4a + 2b + c &= u \\
9a + 3b + c &= 3 - u
\end{aligned}$$

so that $2(5a + 2b + c) = f(1) + f(3) = 3$, which is not possible, since a, b and c are integers.

Suppose that $(p, q, r) = (2, 3, 1)$. Then we are led to $2(5a + 2b + c) = 5$, again an impossibility.

Suppose that $(p, q, r) = (1, 3, 2)$ and that $f(1) = u$. Then

$$\begin{aligned} a + b + c &= u \\ 9a + 3b + c &= u \\ 4a + 2b + c &= 4 - u \end{aligned}$$

so that $4a + b = 0$ and $5a + b = 2u - 4$. This leads to the assignment

$$(a, b, c) = (2u - 4, -8u + 16, 7u - 12)$$

so that

$$\begin{aligned} f(x) &= (2u - 4)(x^2 - 4x) + (7u - 12) \\ &= (2u - 4)(x - 2)^2 - (u - 4) \\ &= (2u - 4)(x - 1)(x - 3) + u \ . \end{aligned}$$

It can be checked that $f(u)$ and $f(4 - u)$ are equal to $2u^3 - 12u^2 + 23u - 12$.

We can get a particular example by taking $u = 0$ to obtain the polynomial $f(x) = -4(x - 1)(x - 3)$, in which case $f(1) = f(3) = 0$, $f(2) = 4$ and $f(0) = f(4) = -12$.

Solution 2. Let $f(x) = ax^2 + bx + c$. Then $f(1) = a + b + c$, $f(2) = 4a + 2b + c$ and $f(3) = 9a + 3b + c$. Then

$$\begin{aligned} f(f(2)) - f(f(1)) &= a[(4a + 2b + c)^2 - (a + b + c)^2] + b[(4a + 2b + c) - (a + b + c)] \\ &= (3a + b)[a(5a + 3b + 2c) + b] \\ &= (3a + b)(5a^2 + 3ab + 2ac + b) \end{aligned}$$

$$\begin{aligned} f(f(3)) - f(f(1)) &= a[(9a + 3b + c)^2 - (a + b + c)^2] + b[(9a + 3b + c) - (a + b + c)] \\ &= (8a + 2b)[a(10a + 4b + 2c) + b] \\ &= 2(4a + b)(10a^2 + 4ab + 2ac + b) \end{aligned}$$

Suppose that $f(f(1)) = f(f(2)) = f(f(3))$. If $b = -3a$, then we must have $0 = 10a^2 + 4ab + 2ac + b = 10a^2 - 12a^2 + 2ac - 3a$, whence $c = a + (3/2)$. In this case, a and c cannot both be integers. However, if $b = -4a$, then $0 = 5a^2 + 3ab + 2ac + b = 5a^2 - 12a^2 + 2ac - 4a$, whence $c = (7a/2) + 2$. So we can get integer coefficients by taking $(a, b, c) = (2t, -8t, 7t + 2)$ for some integer t.

Comment. To get a single solution quickly, just try to make $f(1) = f(3) = 1$ and $f(2) = 3$. This leads immediately to $f(x) = -(x - 2)^2 + 3 = -2x^2 + 8x - 5$.

59. Let $ABCD$ be a concyclic quadrilateral. Prove that

$$|AC - BD| \leq |AB - CD| \ .$$

Solution 1. [O. Bormashenko, P. Cheng] Let the diagonals intersect in M and assign the lengths: $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AM = r$. $BM = p$, $CM = s$, $DM = q$. Since $\angle ABD = \angle ACD$ and $\angle BAC = \angle BDC$, triangles MDC and MAB are similar, so that, for some k, $q = kr$, $s = kp$, $c = ka$. Wolog, we may assume that $k \geq 1$. Note that in triangle MAB, we have that $p < r + a$ and $r < p + a$, whence $-a < p - r < a$ or $|p - r| < a$.

$$\begin{aligned} |AC - BD| &= |(r + s) - (p + q)| = (k - 1)|p - r| \\ &\leq (k - 1)a = |c - a| = |AB - CD| \ . \end{aligned}$$

Solution 2. Let the diagonals intersect in M and assign the lengths: $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AM = r$. $BM = p$, $CM = s$, $DM = q$. Let $\angle BAC = \angle BDC = \beta$, $\angle ABD = \angle ACD = \delta$ and $\angle AMB = \angle DMC = \phi$.

Applying the Law of Sines, we find that

$$p = \frac{a\sin\beta}{\sin\phi} \qquad q = \frac{c\sin\delta}{\sin\phi}$$

$$r = \frac{a\sin\delta}{\sin\phi} \qquad s = \frac{c\sin\beta}{\sin\phi} \ .$$

Hence

$$p - r = \frac{a(\sin\beta - \sin\delta)}{\sin\phi} \qquad q - s = \frac{c(\sin\delta - \sin\beta)}{\sin\phi}$$

whence

$$\begin{aligned}(p+q) - (r+s) &= \frac{(a-c)(\sin\beta - \sin\delta)}{\sin\phi} \\ &= \frac{(a-c)2\cos((\beta+\delta)/2)\sin((\beta-\delta)/2)}{\sin(\beta+\delta)} \\ &= \frac{(a-c)\sin((\beta-\delta)/2)}{\sin((\beta+\delta)/2)} \ .\end{aligned}$$

Since $(\beta+\delta)/2 \leq 90°$, the absolute value of the sine in the numerator is less than the sine in the denominator, and so we find that

$$|(p+q) - (r+s)| \leq |a - c| \ ,$$

as desired.

Solution 3. [R. Furmaniak] Let O be the circumcentre of $ABCD$ with R the circumradius. Let $\angle AOB = \alpha$, $\angle BOC = \beta$, $\angle COD = \gamma$ and $\angle DOA = \delta$. Wolog, we can let δ be the largest of these angles.

We have that

$$AB = 2R\sin\frac{\alpha}{2} \qquad CD = 2R\sin\frac{\gamma}{2}$$

$$AC = 2R\sin\frac{\alpha+\beta}{2} \qquad BD = 2R\sin\frac{\beta+\gamma}{2} \ .$$

Thus

$$\begin{aligned}AB - CD &= 2R\left(\sin\frac{\alpha}{2} - \sin\frac{\gamma}{2}\right) \\ &= 4R\sin\frac{\alpha-\gamma}{4}\cos\frac{\alpha+\gamma}{4}\end{aligned}$$

and

$$AC - BD = 4R\sin\frac{\alpha-\gamma}{4}\cos\frac{\alpha+\gamma+2\beta}{4} \ .$$

Since $(\alpha+2\beta+\gamma)/4$ and $(\alpha+\gamma)/4$ both do not exceed $(\alpha+\beta+\gamma+\delta)/4 = 90°$, we have that

$$\cos\frac{\alpha+\gamma+2\beta}{4} \leq \cos\frac{\alpha+\gamma}{4} \ ,$$

and this yields the desired result.

60. Let $n \geq 2$ be an integer and $M = \{1, 2, \cdots, n\}$. For every integer k with $1 \leq k \leq n$, let

$$x_k = \sum\{\min A + \max A : A \subseteq M, A \text{ has } k \text{ elements}\}$$

where $\min A$ is the smallest and $\max A$ is the largest number in A. Determine $\sum_{k=1}^{n}(-1)^{k-1}x_k$.

Solution 1. For any set $S = \{a_1, a_2, \cdots, a_k\}$ we define the related set $S' = \{(n+1) - a_k, (n+1) - a_{k-1}, \cdots, (n+1) - a_1\}$. As S runs through all the subsets of $\{1, 2, \cdots, n\}$ with k elements, S' also runs through the same class of subsets; the mapping $S \rightarrow S'$ is one-one. If, say, a_1 is the minimum element and a_k the maximum element of S, then $(n+1) - a_1$ is the maximum and $(n+1) - a_k$ the minimum element of S'. Hence the sum of the minima and maxima of the two sets is

$$(a_1 + a_k) + 2(n+1) - a_1 - a_k = 2(n+1) \ .$$

The sum of the maxima and minima of all the sets $S \cup S'$ is equal to $2x_k = 2(n+1)\binom{n}{k}$, so that $x_k = (n+1)\binom{n}{k}$. Hence

$$\sum_{k=1}^{n}(-1)^{k-1}x_k = -(n+1)\sum_{k=1}^{n}(-1)^k\binom{n}{k} = -(n+1)[(1-1)^n - 1] = n+1 \ .$$

Solution 2. [R. Furmaniak] For a given k and m with $1 \leq k, m \leq n$, the number of $k-$subsets of M with m as the smallest element is $\binom{n-m}{k-1}$, and the number of $k-$subsets of M with m as the largest element is $\binom{m-1}{k-1}$, We use the convention that $\binom{0}{0} = 1$ and $\binom{x}{y} = 0$ when $x < y$. From this, we see that

$$\begin{aligned} x_k &= \left[\sum_{m=1}^{n-k+1} m\binom{n-m}{k-1}\right] + \left[\sum_{m=k}^{n} m\binom{m-1}{k-1}\right] \\ &= \left[\sum_{m=1}^{n} m\left(\binom{n-m}{k-1} + \binom{m-1}{k-1}\right)\right] . \end{aligned}$$

Hence

$$\begin{aligned} \sum_{k=1}^{n}(-1)^{k-1}x_k &= \sum_{m=1}^{n}\left[m\sum_{k=1}^{n}(-1)^{k-1}\binom{n-m}{k-1}\right] + \sum_{m=1}^{n}\left[m\sum_{k=1}^{n}(-1)^{k-1}\binom{m-1}{k-1}\right] . \\ &= \left[\sum_{m=1}^{n-1} m\sum_{k=1}^{n}(-1)^{k-1}\binom{n-m}{k-1}\right] + n\binom{0}{0} + \left[\sum_{m=2}^{n-1} m\sum_{k=1}^{n}\binom{m-1}{k-1}\right] + 1\binom{0}{0} . \end{aligned}$$

Now, when $n - m \geq 1$,

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n-m}{k-1} = (1-1)^{n-m} = 0$$

and, when $m \geq 2$,

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{m-1}{k-1} = (1-1)^{m-1} = 0 \ .$$

It follows that the required sum is equal to $n+1$.

Solution 3. [O. Bormashenko] Denote by σ the quantity $\sum_{k=1}^{n}(-1)^{k-1}x_k$. Let m be a number between 1 and $n-1$, inclusive. m is the minimum of a $k-$set for exactly $\binom{n-m}{k-1}$ sets. For a particular k, m as minimum contributes $n\binom{n-m}{k-1}$ to the sum for x_k. Fixing m, m as minimum contributes to σ the value

$$m\sum_{k=1}^{n}(-1)^{k-1}\binom{n-m}{k-1} = 0 \ ,$$

where $\binom{i}{j} = 0$ when $j > i$. The only other possible minimum is n, and this is a minimum only for the singelton $\{n\}$, so that n as minimum contributes n to the sum σ.

By a similar argument, for any number $m = 2, 3, \cdots, n$, m as maximum contributes 0 to the sum σ. 1 is maximum only for the singleton $\{1\}$ and contributes the value 1 to the sum σ. Hence $\sigma = n+1$.

61. Let $S = 1!2!3!\cdots 99!100!$ (the product of the first 100 factorials). Prove that there exists an integer k for which $1 \leq k \leq 100$ and $S/k!$ is a perfect square. Is k unique? (*Optional:* Is it possible to find such a number k that exceeds 100?)

Solution 1. Note that, for each positive integer j, $(2j-1)!(2j)! = [(2j-1)!]^2 \cdot 2j$. Hence

$$S = \prod_{j=1}^{50} [(2j-1)!]^2 [2j] = 2^{50} 50! \left[\prod_{j=1}^{50} (2j-1)! \right]^2 ,$$

from which we see that $k = 50$ is the required number.

We show that $k = 50$ is the only possibility. First, k cannot exceed 100, for otherwise 101! would be a factor of $k!$ but not S, and so $S/k!$ would not even be an integer. Let $k \leq 100$. The prime 47 does not divide $k!$ for $k \leq 46$ and divides 50! to the first power. Since $S/50!$ is a square, it evidently divides S to an odd power. So $k \geq 47$ in order to get a quotient divisible by 47 to an even power. The prime 53 divides each $k!$ for $k \geq 53$ to the first power and divides $S/50!$, and so S to an even power. Hence, $k \leq 52$.

The prime 17 divides 50! and $S/50!$, and hence S to an even power, but it divides each of 51! and 52! to the third power. So we cannot have $k = 51$ or 52. Finally, look at the prime 2. Suppose that 2^{2u} is the highest power of 2 that divides $S/50!$ and that 2^v is the highest power of 2 that divides 50!; then 2^{2u+v} is the highest power of 2 that divides S. The highest power of 2 that divides 48! and 49! is 2^{v-1} and the highest power of 2 that divides 46! and 47! is 2^{v-5}. From this, we deduce that 2 divides $S/k!$ to an odd power when $47 \leq k \leq 49$. The desired uniqueness of k follows.

Solution 2. Let p be a prime exceeding 50. Then p divides each of $m!$ to the first power for $p \leq m \leq 100$, so that p divides S to the even power $100 - (p-1) = 101 - p$. From this, it follows that if $53 \geq k$, p must divide $S/k!$ to an odd power.

On the other hand, the prime 47 divides each $m!$ with $47 \leq m \leq 93$ to the first power, and each $m!$ with $94 \leq m \leq 100$ to the second power, so that it divides S to the power with exponent $54 + 7 = 61$. Hence, in order that it divide $S/k!$ to an even power, we must make k one of the numbers $47, \cdots, 52$.

By an argument, similar to that used in Solution 1, it can be seen that 2 divides any product of the form $1!2!\cdots(2m-1)!$ to an even power and 100! to the power with exponent

$$\lfloor 100/2 \rfloor + \lfloor 100/4 \rfloor + \lfloor 100/8 \rfloor + \lfloor 100/16 \rfloor + \lfloor 100/32 \rfloor + \lfloor 100/64 \rfloor = 50 + 25 + 12 + 6 + 3 + 1 = 97 .$$

Hence, 2 divides S to an odd power. So we need to divide S by $k!$ which 2 divides to an odd power to get a perfect square quotient. This reduces the possibilities for k to 50 or 51. Since

$$S = 2^{99} \cdot 3^{98} \cdot 4^{97} \cdots 99^2 \cdot 100 = (2 \cdot 4 \cdots 50)(2^{49} \cdot 3^{49} \cdot 4^{48} \cdots 99)^2 = 50! \cdot 2^{50}(\cdots)^2 ,$$

$S/50!$ is a square, and so $S/51! = (S/50!) \div (51)$ is not a square. The result follows.

Solution 3. As above, $S/(50!)$ is a square. Suppose that $53 \leq k \leq 100$. Then 53 divides $k!/50!$ to the first power, and so $k!/50!$ cannot be square. Hence $S/k! = (S/50!) \div (k!/50!)$ cannot be square. If $k = 51$ or 52, then $k!/50!$ is not square, so $S/k!$ cannot be square. Suppose that $k \leq 46$. Then 47 divides $50!/k!$ to the first power, so that $50!/k!$ is not square and $S/k! = (S/50!) \times (50!/k!)$ cannot be square. If $k = 47, 48$ or 49, then $50!/k!$ is not square and so $S/k!$ is not square. Hence $S/k!$ is square if and only if $k = 50$ when $k \leq 100$.

62. Let n be a positive integer. Show that, with three exceptions, $n! + 1$ has at least one prime divisor that exceeds $n + 1$.

Solution. Any prime divisor of $n! + 1$ must be larger than n, since all primes not exceeding n divide $n!$. Suppose, if possible, the result fails. Then, the only prime that can divide $n! + 1$ is $n + 1$, so that, for some positive integer r and nonnegative integer K,

$$n! + 1 = (n+1)^r = 1 + rn + Kn^2 .$$

This happens, for example, when $n = 1, 2, 4$: $1! + 1 = 2$, $2! + 1 = 3$, $4! + 1 = 5^2$. Note, however, that the desired result does hold for $n = 3$: $3! + 1 = 7$.

Henceforth, assume that n exceeds 4. If n is prime, then $n + 1$ is composite, so by our initial comment, all of its prime divisors exceed $n + 1$. If n is composite and square, then $n!$ is divisible by the four distinct integers $1, n, \sqrt{n}, 2\sqrt{n}$, while is n is composite and nonsquare with a nontrivial divisor d. then $n!$ is divisible by the four distinct integers $1, d, n/d, n$. Thus, $n!$ is divisible by n^2. Suppose, if possible, the result fails, so that $n! + 1 = 1 + rn + Kn^2$, and $1 \equiv 1 + rn \pmod{n^2}$. Thus, r must be divisible by n, and, since it is positive, must exceed n. Hence

$$(n + 1)^r \geq (n + 1)^n > (n + 1)n(n - 1) \cdots 1 > n! + 1 \ ,$$

a contradiction. The desired result follows.

63. Let n be a positive integer and k a nonnegative integer. Prove that

$$n! = (n + k)^n - \binom{n}{1}(n + k - 1)^n + \binom{n}{2}(n + k - 2)^n - \cdots \pm \binom{n}{n}k^n \ .$$

Solution 1. Recall the *Principle of Inclusion-Exclusion*: *Let S be a set of n objects, and let P_1, P_2, $\cdots$, P_m be m properties such that, for each object $x \in S$ and each property P_i, either x has the property P_i or x does not have the property P_i. Let $f(i, j, \cdots, k)$ denote the number of elements of S each of which has properties P_i, P_j, $\cdots$, P_k (and possibly others as well). Then the number of elements of S each having none of the properties P_1, P_2, $\cdots$, P_m is*

$$n - \sum_{1 \leq i \leq m} f(i) + \sum_{1 \leq i < j \leq m} f(i, j) - \sum_{1 \leq i < j < l \leq m} f(i, j, l) + \cdots + (-1)^m f(1, 2, \cdots, m) \ .$$

We apply this to the problem at hand. Note that an ordered selection of n numbers selected from among $1, 2, \cdots, n + k$ is a permutation of $\{1, 2, \cdots, n\}$ if and only if it is constrained to contain each of the numbers 1, 2, $\cdots$, n. Let S be the set of all ordered selections, and we say that a selection has property P_i iff its fails to include at least i of the numbers $1, 2, \cdots, n$ $(1 \leq i \leq n)$. The number of selections with property P_i is the product of $\binom{n}{i}$, the number of ways of choosing the i numbers not included and $(n + k - i)^n$, the number of ways of choosing entries for the n positions from the remaining $n + k - 1$ numbers. The result follows.

Solution 2. We begin with a lemma:

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^m = \begin{cases} 0 & (0 \leq m \leq n - 1) \\ (-1)^n n! & (m = n) \ . \end{cases}$$

We use the convention that $0^0 = 1$. To prove this, note first that $i(i-1) \cdots (i-m) = i^{m+1} + b_m i^m + \cdots + b_1 i + b_0$ for some integers b_i. We use an induction argument on m. The result holds for each positive n and for $m = 0$, as the sum is the expansion of $(1 - 1)^n$. It also holds for $n = 1, 2$ and all relevant m. Fix $n \geq 3$. Suppose that it holds when m is replaced by k for $0 \le k \le m \le n-2$. Then

$$\begin{aligned}
\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^{m+1} &= \sum_{i=1}^{n}(-1)^i\binom{n}{i}i(i-1)\cdots(i-m) - \sum_{k=0}^{m} b_k \sum_{i=0}^{n}(-1)^i\binom{n}{i}i^k \\
&= \sum_{i=m+1}^{n}(-1)^i\binom{n}{i}i(i-1)\cdots(i-m) - 0 \\
&= \sum_{i=m+1}^{n}(-1)^i\frac{n!i!}{i!(n-i)!(i-m-1)!} = \sum_{j=0}^{n-m-1}(-1)^{m+1+j}\frac{n!}{(n-m-1-j)!j!} \\
&= \sum_{j=0}^{n-m-1}(-1)^{m+1}(-1)^j\frac{n(n-1)\cdots(n-m)[(n-m-1)!]}{(n-m-1-j)!j!} \\
&= (-1)^{m+1}n(n-1)\cdots(n-m)\sum_{j=0}^{n-m-1}(-1)^j\binom{n-m-1}{j} = 0 \ .
\end{aligned}$$

(Note that the $j=0$ term is 1, which is consistent with the $0^0 = 1$ convention mentioned earlier.) So $\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^m = 0$ for $0 \le m \le n-1$. Now consider the case $m = n$:

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}i^n = \sum_{i=1}^{n}(-1)^i\binom{n}{i}i(i-1)\cdots(i-n+1) - \sum_{k=0}^{n-1} b_k \sum_{i=0}^{n}(-1)^i\binom{n}{i}i^k \ .$$

Every term in the first sum vanishes except the nth and each term of the second sum vanishes. Hence $\sum_{i=1}^{n}(-1)^i\binom{n}{i}i^n = (-1)^n n!$.

Returning to the problem at hand, we see that the right side of the desired equation is equal to

$$\begin{aligned}
(n+k)^n - &\binom{n}{1}(n+k-1)^n + \binom{n}{2}(n+k-2)^n - \cdots + (-1)^n\binom{n}{n}(n+k-n)^n \\
&= \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i+k)^n = \sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{j=0}^{n}\binom{n}{j}(n-i)^j k^{n-j} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^i\binom{n}{i}\binom{n}{j}(n-i)^j k^{n-j} = \sum_{j=0}^{n}\binom{n}{j}k^{n-j}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^j \\
&= \sum_{j=0}^{n}\binom{n}{j}k^{n-j}\sum_{i=0}^{n}(-1)^i\binom{n}{n-i}(n-i)^j \\
&= \sum_{j=0}^{n}\binom{n}{j}k^{n-j}\sum_{i=0}^{n}(-1)^n(-1)^i\binom{n}{i}i^j \ .
\end{aligned}$$

When $0 \le j \le n-1$, the sum $\sum_{i=0}^{n}(-1)^i\binom{n}{n-i}(n-i)^j = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}i^j$ vanishes, while, when $j = n$, it assunes the value $n!$. Thus, the right side of the given equation is equal to $\binom{n}{n}k^0 n! = n!$ as desired.

Solution 3. Let $m = n+k$, so that $m \ge n$, and let the right side of the equation be denoted by R. Then

$$\begin{aligned}
R &= m^n - \binom{n}{1}(m-1)^n + \binom{n}{2}(m-2)^n - \cdots + (-1)^i\binom{n}{i}(m-i)^n + \cdots + (-1)^n\binom{n}{n}(m-n)^n \\
&= m^m\left[\sum_{j=0}^{n}(-1)^i\binom{n}{i}\right] - \binom{n}{1}m^{n-1}\left[\sum_{i=1}^{n}(-1)^i i\binom{n}{i}\right] + \binom{n}{2}m^{n-2}\left[\sum_{i=1}^{n}(-1)^i i^2\binom{n}{i}\right] + \cdots \\
&\qquad + (-1)^n\binom{n}{n}\left[\sum_{i=1}^{n}(-1)^i i^n\binom{n}{i}\right] \ .
\end{aligned}$$

Let

$$f_0(x) = (1-x)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i$$

and let

$$f_k(x) = xDf_{k-1}(x)$$

for $k \geq 1$, where Df denotes the derivative of a function f. Observe that, from the closed expression for $f_0(x)$, we can establish by induction that

$$f_k(x) = \sum_{i=0}^{n} (-1)^i i^k \binom{n}{i} x^i$$

so that $R = \sum_{k=0}^{n} (-1)^k \binom{n}{k} m^{n-k} f_k(1)$.

By induction, we establish that

$$f_k(x) = (-1)^k n(n-1)\cdots(n-k+1)x^k(1-x)^{n-k} + (1-x)^{n-k+1}g_k(x)$$

for some polynomial $g_k(x)$. This is true for $k = 1$ with $g_1(x) = 0$. Suppose if holds for $k = j$. Then

$$\begin{aligned} f_j'(x) = (-1)^j n(n-1)\cdots(n-j+1)x^{j-1}(1-x)^{n-j} - (-1)^j n(n-1)\cdots(n-j+1)(n-j)x^j(1-x)^{n-j-1} \\ - (n-j+1)(1-x)^{n-j}g_j(x) + (1-x)^{n-j+1}g_j'(x) \ , \end{aligned}$$

whence

$$\begin{aligned} f_{j+1}(x) = (-1)^{j+1}n(n_1)\cdots(n-j)x^j(1-x)^{n-(j+1)} + (1-x)^{n-(j+1)+1}[(-1)^j n(n-1)\cdots(n-j+1)x^j \\ - (n-j+1)xg_k(x) + x(1-x)g_j'(x)] \end{aligned}$$

and we obtain the desired representation by induction. Then for $1 \leq k \leq n-1$, $f_k(1) = 0$ while $f_n(1) = (-1)^n n!$. Hence $R = (-1)^n f_n(1) = n!$.

64. Let M be a point in the interior of triangle ABC, and suppose that D, E, F are respective points on the side BC, CA, AB, which all pass through M. (In technical terms, they are *cevians*.) Suppose that the areas and the perimeters of the triangles BMD, CME, AMF are equal. Prove that triangle ABC must be equilateral.

Solution. [L. Lessard] Let the common area of the triangles BMD, CME and AMF be a and let their common perimeter be p. Let the area and perimeter of ΔAME be u and x respectively, of ΔMFB be v and y respectively, and of ΔCMD be w and z respectively.

By considering pairs of triangles with equal heights, we find that

$$\frac{AF}{FB} = \frac{a}{v} = \frac{2a+u}{v+a+w} = \frac{a+u}{a+w} \ ,$$

$$\frac{BD}{DC} = \frac{a}{w} = \frac{2a+v}{u+a+w} = \frac{a+v}{a+u} \ ,$$

$$\frac{CE}{EA} = \frac{a}{u} = \frac{2a+w}{u+a+v} = \frac{a+w}{a+v} \ .$$

From these three sets of equations, we deduce that

$$\frac{a^3}{uvw} = 1 \ ;$$

$$a^2 + (w-u)a - uv = 0 \ ,$$
$$a^2 + (u-w)a - vw = 0 \ ,$$
$$a^2 + (v-u)a - uw = 0 \ ;$$

whence

$$a^3 = uvw \qquad \text{and} \qquad 3a^2 = uv + vw + uw \ .$$

This means that uv, vw, uw are three positive numbers whose geometric and arithmetic means are both equal to a^2. Hence $a^2 = uv = vw = uw$, so that $u = v = w = a$. It follows that $AF = FB$, $BD = DC$, $CE = EA$, so that AD, BE and CF are medians and M is the centroid.

Wolog, suppose that $AB \geq BC \geq CA$. Since $AB \geq BC$, $\angle AEB \geq 90°$, and so $AM \geq MC$. Thus $x \geq p$. Similarly, $y \geq p$ and $p \geq z$.

Consider triangles BMD and AME. We have $BD \geq AE$, $BM \geq AM$, $ME = \frac{1}{2}BM$ and $MD = \frac{1}{2}AM$. Therefore

$$p - x = (BD + MD + BM) - (AE + ME + AM) = (BD - AE) + \frac{1}{2}(BM - AM) \geq 0$$

and so $p \geq x$. Since also $x \geq p$, we have that $p = x$. But this implies that $AM = MC$, so that $ME \perp AC$ and $AB = BC$. Since BE is now an axis of a reflection which interchanges A and C, as well as F and D, it follows that $p = z$ and $p = y$ as well. Thus, $AB = AC$ and $AC = BC$. Thus, the triangle is equilateral.

65. Suppose that XTY is a straight line and that TU and TV are two rays emanating from T for which $\angle XTU = \angle UTV = \angle VTY = 60°$. Suppose that P, Q and R are respective points on the rays TY, TU and TV for which $PQ = PR$. Prove that $\angle QPR = 60°$.

Solution 1. Let $\mathfrak{R}$ be a rotation of $60°$ about T that takes the ray TU to TV. Then, if $\mathfrak{R}$ transforms $Q \rightarrow Q'$ and $P \rightarrow P'$, then Q' lies on TV and the line $Q'P'$ makes an angle of $60°$ with QP. Because of the rotation, $\angle P'TP = 60°$ and $TP' = TP$, whence $TP'P$ is an equilateral triangle.

Since $\angle Q'TP = \angle TPP' = 60°$, $TV \| P'P$. Let $\mathfrak{T}$ be the translation that takes P' to P. It takes Q' to a point Q'' on the ray TV, and $PQ'' = P'Q' = PQ$. Hence Q'' can be none other than the point R [why?], and the result follows.

Solution 2. The reflection in the line XY takes $P \rightarrow P$, $Q \rightarrow Q'$ and $R \rightarrow R'$. Triangles PQR' and $PQ'R$ are congruent and isosceles, so that $\angle TQP = \angle TQ'P = \angle TRP$ (since $PQ' = PR$). Hence $TQRP$ is a concyclic quadrilateral, whence $\angle QPR = \angle QTR = 60°$.

Solution 3. [S. Niu] Let S be a point on TU for which $SR \| XY$; observe that ΔRST is equilateral. We first show that Q lies between S and T. For, if S were between Q and T, then $\angle PSQ$ would be obtuse and $PQ > PS > PR$ (since $\angle PRS > 60° > \angle PSR$ in ΔPRS), a contradiction.

The rotation of $60°$ with centre R that takes S onto T takes ray RQ onto a ray through R that intersects TY in M. Consider triangles RSQ and RTM. Since $\angle RST = \angle RTM = 60°$, $\angle SRQ = 60° - \angle QRT = \angle TRM$ and $SR = TR$, we have that $\Delta RSQ \equiv \Delta RTM$ and $RQ = RM$. (ASA) Since $\angle QRM = 60°$, ΔRQM is equilateral and $RM = RQ$. Hence M and P are both equidistant from Q and R, and so at the intersection of TY and the right bisector of QR. Thus, $M = P$ and the result follows.

Solution 4. [H. Pan] Let Q' and R' be the respective reflections of Q and R with respect to the axis XY. Since $\angle RTR' = 120°$ and $TR = TR'$, $\angle QR'R = \angle TR'R = 30°$. Since Q, R, Q', R', lie on a circle with centre P, $\angle QPR = 2\angle QR'R = 60°$, as desired.

Solution 5. [R. Barrington Leigh] Let W be a point on TV such that $\angle WPQ = 60° = \angle WTU$. [Why does such a point W exist?] Then $WQTP$ is a concyclic quadrilateral so that $\angle QWP = 180° - \angle QTP = 60°$ and ΔPWQ is equilateral. Hence $PW = PQ = PR$.

Suppose $W \neq R$. If R is farther away from T than W, then $\angle RPT > \angle WPT > \angle WPQ = 60° \Rightarrow 60° > \angle TRP = \angle RWP > 60°$, a contradiction. If W is farther away from T than R, then $\angle WPT > \angle WPQ = 60° \Rightarrow 60° > \angle RWP = \angle WRP > 60°$, again a contradiction. So $R = W$ and the result follows.

Solution 6. [M. Holmes] Let the circle through T, P, Q intersect TV in N. Then $\angle QNP = 180° - \angle QTP = 60°$. Since $\angle PQN = \angle PTN = 60°$, ΔPQN is equilateral so that $PN = PQ$. Suppose, if possible, that $R \neq N$. Then N and R are two points on TV equidistant from P. Since $\angle PNT < \angle PNQ = 60°$ and ΔPNR is isosceles, we have that $\angle PNR < 90°$, so N cannot lie between T and R, and $\angle PRN = \angle PNR = \angle PNT < 60°$. Since $\angle PTN = 60°$, we conclude that T must lie between R and N, which transgresses the condition of the problem. Hence R and N must coincide and the result follows.

Solution 7. [P. Cheng] Determine S on TU and Z on TY for which $SR \| XY$ and $\angle QRZ = 60°$. Observe that $\angle TSR = \angle SRT = 60°$ and $SR = RT$.

Consider triangles SRQ and TRZ. $\angle SRQ = \angle SRT - \angle QRT = \angle QRZ - \angle QRT = \angle TRZ$; $\angle QSR = 60° = \angle ZTR$, so that $\Delta SRQ = \Delta TRZ$ (ASA).

Hence $RZ = RQ \Rightarrow \Delta RQZ$ is equilateral $\Rightarrow RZ = ZQ$ and $\angle RZQ = 60°$. Now, both P and Z lie on the intersection of TY and the right bisector of QR, so they must coincide: $P = Z$. The result follows.

Solution 8. Let the perpendicular, produced, from Q to XY meet VT, produced, in S. Then $\angle XTS = \angle VTY = 60° = \angle XTU$, from which is can be deduced that TX right bisects QS. Hence $PS = PQ = PR$, so that Q, R, S are all on the same circle with centre P.

Since $\angle QTS = 120°$, we have that $\angle SQT = \angle QSR = 30°$, so that QR must subtend an angle of $60°$ at the centre P of the circle. The desired result follows.

Solution 9. [A.Siu] Let the right bisector of QR meet the circumcircle of TQR on the same side of QR at T in S. Since $\angle QSR = \angle QTR = 60°$ and $QS = QR$, $\angle SQR = \angle SRQ = 60°$. Hence $\angle STQ = 180° - \angle SRQ = 120°$. But $\angle YTQ = 120°$, so S must lie on TY. It follows that $S = P$.

Solution 10. Assign coordinates with the origin at T and the $x-$axis along XY. The the respective coordinates of Q and R have the form $(u, -\sqrt{3}u)$ and $(v, \sqrt{3}v)$ for some real u and v. Let the coordinates of P be $(w, 0)$. Then $PQ = PR$ yields that $w = 2(u + v)$. [*Exercise:* work it out.]

$$\begin{aligned} |PQ|^2 - |QR|^2 &= (u-w)^2 + 3u^2 - (u-v)^2 - 3(u+v)^2 \\ &= w^2 - 2uw - 4v(u+v) = w^2 - 2uw - 2vw \\ &= w^2 - 2(u+v)w = 0 \ . \end{aligned}$$

Hence $PQ = QR = PR$ and ΔPQR is equilateral. Therefore $\angle QPR = 60°$.

Solution 11. [J.Y. Jin] Let $\mathfrak{C}$ be the circumcircle of ΔPQR. If T lies strictly inside $\mathfrak{C}$, then $60° = \angle QTR > \angle QPR$ and $60° = \angle PTR > \angle PQR = \angle PRQ$. Thus, all three angle of ΔPQR would be less than $60°$, which is not possible. Similarly, if T lies strictly outside $\mathfrak{C}$, then $60° = \angle QTR < \angle QPR$ and $60° = \angle PTR < \angle PQR = \angle PRQ$, so that all three angles of ΔPQR would exceed $60°$, again not possible. Thus T must be on $\mathfrak{C}$, whence $\angle QPR = \angle QTR = 60°$.

Solution 12. [C. Lau] By the Sine Law,

$$\frac{\sin \angle TQP}{|TP|} = \frac{\sin 120°}{|PQ|} = \frac{\sin 60°}{|PR|} = \frac{\sin \angle TRP}{|TP|} \ ,$$

whence $\sin \angle TQP = \sin \angle TRP$. Since $\angle QTP$, in triangle QTP is obtuse, $\angle TQP$ is acute.

Suppose, if possible, that $\angle TRP$ is obtuse. Then, in triangle TPR, TP would be the longest side, so $PR < TP$. But in triangle TQP, PQ is the longest side, so $PQ > TP$, and so $PQ \neq PR$, contrary to hypothesis. Hence $\angle TRP$ is acute. Therefore, $\angle TQP = \angle TRP$. Let PQ and RT intersect in Z. Then, $60° = \angle QTZ = 180° - \angle TQP - \angle QZT = 180° - \angle TRP - \angle RZP = \angle QPR$, as desired.

66. (a) Let $ABCD$ be a square and let E be an arbitrary point on the side CD. Suppose that P is a point on the diagonal AC for which $EP \perp AC$ and that Q is a point on AE produced for which $CQ \perp AE$. Prove that B, P, Q are collinear.

(b) Does the result hold if the hypothesis is weakened to require only that $ABCD$ is a rectangle?

Solution 1. Let $ABCD$ be a rectangle, and let E, P, Q be determined as in the problem. Suppose that $\angle ACD = \angle BDC = \alpha$. Then $\angle PEC = 90° - \alpha$. Because $EPQC$ is concyclic, $\angle PQC = \angle PEC = 90° - \alpha$. Because $ABCQD$ is concyclic, $\angle BQC = \angle BDC = \alpha$. The points B, P, Q are collinear $\Longleftrightarrow \angle BQC = \angle PQC \Leftrightarrow \alpha = 90° - \alpha \Leftrightarrow \alpha = 45° \Leftrightarrow ABCD$ is a square.

Solution 2. (a) $EPQC$, with a pair of supplementary opposite angles, is concyclic, so that $\angle CQP = \angle CEP = 180° - \angle EPC - \angle ECP = 45°$. Since $CBAQ$ is concyclic, $\angle CQB = \angle CAB = 45°$. Thus, $\angle CQP = \angle CQB$ so that Q, P, B are collinear.

(b) Suppose that $ABCD$ is a nonquare rectangle. Then taking $E = D$ yields a counterexample.

Solution 3. (a) The circle with diameter AC that passes through the vertices of the square also passes through Q. Hence $\angle QBC = \angle QAC$. Consider triangles PBC and EAC. Since triangles ABC and EPC are both isosceles right triangles, $BC : AC = PC : EC$. Also $\angle BCA = \angle PCE = 45°$. Hence $\Delta PBC \sim \Delta EAC$ (SAS) so that $\angle PBC = \angle EAC = \angle QAC = \angle QBC$. It follows that Q, P, B are collinear.

Solution 4. [S. Niu] Let $ABCD$ be a rectangle and let E, P, Q be determined as in the problem. Let EP be produced to meet BC in F. Since $\angle ABF = \angle APF$, the quadrilateral $ABPF$ is concyclic, so that $\angle PBC = \angle PBF = \angle PAF$. Since $ABCQ$ is concyclic, $\angle QBC = \angle QAC = \angle PAE$. Now B, P, Q are collinear

$$\Leftrightarrow \angle PBC = \angle QBC \Leftrightarrow \angle PAF = \angle PAE \Leftrightarrow AC \text{ right bisects } EF$$

$$\Leftrightarrow \angle ECA = \angle ACB = 45° \Leftrightarrow ABCD \text{ is a square .}$$

Solution 5. [M. Holmes] (a) Suppose that BQ intersects AC in R. Since $ABCQD$ is concyclic, $\angle AQR = \angle AQB = \angle ACB = 45°$, so that $\angle BQC = 45°$. Since $\angle EQR = \angle AQB = \angle ECR = 45°$, $ERCQ$ is concyclic, so that $\angle ERC = 180° - \angle EQC = 90°$. Hence $ER \perp AC$, so that $R = P$ and the result follows.

Solution 6. [L. Hong] (a) Let QC intersect AB in F. We apply Menelaus' Theorem to triangle AFC: B, P, Q are collinear if and only if

$$\frac{AB}{BF} \cdot \frac{FQ}{QC} \cdot \frac{CP}{PA} = -1 \ .$$

Let the side length of the square be 1 and the length of DE be a. Then $|AB| = 1$. Since $\Delta ADE \sim \Delta FBC$, $AD : DE = BF : BC$, so that $|BF| = 1/a$ and $|FC| = \sqrt{1+a^2}/a$. Since $\Delta ADE \sim \Delta CQE$, $CQ : EC = AD : EA$, so that $|CQ| = (1-a)/\sqrt{1+a^2}$. Hence

$$\frac{|FQ|}{|CQ|} = 1 + \frac{|FC|}{|CQ|} = 1 + \frac{1+a^2}{a(1-a)} = \frac{1+a}{a(1-a)} \ .$$

Since ΔECP is right isosceles, $|CP| = (1-a)/\sqrt{2}$ and $|PA| = \sqrt{2} - |CP| = (1+a)/\sqrt{2}$. Hence $|CP|/|PA| = (1-a)/(1+a)$. Multiplying the three ratios together and taking account of the directed segments gives the product -1 and yields the result.

Solution 7. (a) Select coordinates so that $A \sim (0,1)$, $B \sim (0,0)$, $C \sim (1,0)$, $D \sim (1,1)$ and $E \sim (1,t)$ for some t with $0 \leq t \leq 1$. It is straightforward to verify that $P \sim (1 - \frac{t}{2}, \frac{t}{2})$.

Since the slope of AE is $t - 1$, the slope of AQ should be $(1-t)^{-1}$. Since the coordinates of Q have the form $(1 + s, s(1-t)^{-1})$ for some s, it is straightforward to verify that

$$Q \sim \left(\frac{2-t}{1+(1-t)^2}, \frac{t}{1+(1-t)^2)} \right) \ .$$

It can now be checked that the slope of each of BQ and BP is $t(2-t)^{-1}$, which yields the result.

(b) The result fails if $A \sim (0,2)$, $B \sim (0,0)$, $C \sim (1,0)$, $D \sim (1,2)$. If $E \sim (1,1)$, then $P \sim (\frac{3}{5}, \frac{4}{5})$ and $Q \sim (\frac{3}{2}, \frac{1}{2})$.

67. (a) Consider the infinite integer lattice in the plane (*i.e.*, the set of points with integer coordinates) as a graph, with the edges being the lines of unit length connecting nearby points. What is the minimum number of colours that can be used to colour all the vertices and edges of this graph, so that

(i) each pair of adjacent vertices gets two distinct colours; AND

(ii) each pair of edges that meet at a vertex get two distinct colours; AND

(iii) an edge is coloured differently than either of the two vertices at the ends?

(b) Extend this result to lattices in real $n-$dimensional space.

Solution 1. Since each vertex and the four edges emanating from it must have different colours, at least five colours are needed. Here is a colouring that will work: Let the colours be numbered 0, 1, 2, 3, 4. Colour the point $(x,0)$ with the colour $x \pmod 5$; colour the point $(0,y)$ with the colour $2y \pmod 5$; colour the points along each horizontal line parallel to the $x-$axis consecutively; colour the vertical edge whose lower vertex has colour $m \pmod 5$ with the colour $m+1 \pmod 5$; colour the horizontal edge whose left vertex has the colour $n \pmod 5$ with the colour $n+3 \pmod 5$.

This can be generalized to an $n-$dimensional lattice where $2n+1$ colours are needed by changing the strategy of colouring. The integer points on the line and the edges between them can be coloured $1--(3)--2--(1)--3--(2)--1$ and so on, where the edge colouring is in parenthesis. Form a plane by stacking these lines unit distance apart, making sure that each vertex has a different coloured vertex above and below it; use colours 4 and 5 judiciously to colour the vertical edges. Now go to three dimensions; stack up planar lattices and struts unit distance apart, colouring each with the colours 1, 2, 3, 4, 5, while making sure that vertically adjacent vertices have separate colours, and use the colours 6 and 7 for vertical struts. Continue on.

Solution 2. Consider the $n-$ dimensional lattice. Let the colours be numbered $0, 1, 2, \cdots, 2n$. Assign the vertex with coordinates $(x_1, x_2, \cdots, x_n)$ the colour $x_1 + 2x_2 + \cdots + nx_n$, modulo $2n+1$. Adjacent vertices have distinct colours. For each adjacent vertex has the same coordinates, except in one position where the coordinates differ by 1; if this is the ith coordinate, then the numbers of the two colours differ by $\pm i \pmod{2n+1}$.

Consider an edge joining a vertex with colour u to one of colour v; assign this edge the colour $(n+1)(u+v) \pmod{2n+1}$. Since $(n+1)(u+v) - v \equiv n(u-v) \pmod{2n+1}$, the greatest common divisor of n and $2n+1$ is 1 and $u \not\equiv v \pmod{2n+1}$, it follows that the colour of this edge differs from v; similarly, it differs from u.

Finally, consider a pair of adjacent edges, with colours $(n+1)(u+v)$ and $(n+1)(v+w) \pmod{2n+1}$. The difference between these colours, modulo $2n+1$, is equal to $(n+1)(u-w)$. If the edges are collinear, then this difference is $\pm 2(n+1)i$ for some i with $1 \leq i \leq n$, and this is not congruent to zero modulo $2n+1$. If the edges are perpendicular, then this difference is nonzero and of the form $(n+1)(\pm i \pm j)$. This value, lying between $-2n$ and $2n$ is not congruent to zero modulo $2n+1$. Thus, adjacent edges have distinct colours.

Therefore, we can achieve our goal with $2n+1$ colours, and, by looking at a vertex and its adjacent edges, we see that this is minimal.

68. Let $a, b, c > 0$, $a < bc$ and $1 + a^3 = b^3 + c^3$. Prove that $1 + a < b + c$.

Solution 1. Since $(1+a)(1-a+a^2) = (b+c)(b^2-bc+c^2)$, and since $1-a+a^2$ and b^2-bc+c^2 are positive, we have that

$$1 + a < b + c \Leftrightarrow 1 - a + a^2 > b^2 - bc + c^2 \ .$$

Suppose, if possible, that $1 + a \geq b + c$. Then

$$\begin{aligned} b^2 - bc + c^2 &\geq 1 - a + a^2 \\ &\Rightarrow (b+c)^2 - 3bc \geq (1+a)^2 - 3a > (1+a)^2 - 3bc \\ &\Rightarrow (b+c)^2 > (1+a)^2 \Rightarrow b + c > 1 + a \end{aligned}$$

which is a contradiction.

Solution 2. [J. Chui] Let $u = (1 + a) - (b + c)$. Then

$$\begin{aligned} (1+a)^3 - (b+c)^3 &= u[(1+a)^2 + (1+a)(b+c) + (b+c)^2] \\ &= u[(1+a)^2 + (1+a)(b+c) + b^2 + 2bc + c^2] \ . \end{aligned}$$

But also

$$\begin{aligned} (1+a)^3 - (b+c)^3 &= (1+a^3) - (b^3+c^3) + 3a(1+a) - 3bc(b+c) \\ &= 0 + 3[a(1+a) - bc(b+c)] < 3bcu \ . \end{aligned}$$

It follows from these that

$$0 > u[(1+a)^2 + (1+a)(b+c) + b^2 - bc + c^2] = u[(1+a)^2 + (1+a)(b+c) + \frac{1}{2}(b-c)^2 + \frac{1}{2}(b^2+c^2)] \ .$$

Since the quantity in square brackets is positive, we must have that $u < 0$, as desired.

Solution 3. [A. Momin, N. Martin] Suppose, if possible, that $(1 + a) \geq (b + c)$. Then

$$0 \leq (1+a)^2 - (b+c)^2 = (1+a^2) - (b^2+c^2) - 2(bc-a) < (1+a^2) - (b^2+c^2) \ .$$

Hence $1 + a^2 > b^2 + c^2$. It follows that

$$(1 - a + a^2) - (b^2 - bc + b^2) = (1 + a^2) - (b^2 + c^2) + (bc - a) > 0$$

so that

$$(1 - a + a^2) > (b^2 - bc + c^2) \ .$$

However

$$(1+a)(1-a+a^2) = 1 + a^3 = b^3 + c^3 + (b+c)(b^2 - bc + c^2) \ ,$$

from which it follows that $1 + a < b + c$, yielding a contradition. Hence, the desired result follows.

Solution 4. [H. Pan] First, observe that $a = c$ leads to $b = 1$ and a contradiction of the given conditions, while $a = b$ leads to $c = 1$ and a contradiction. Suppose, if possible, that that $b > a > c$. Then $b^3 + 1 > a^3 + 1 = b^3 + c^3 > c^3 + 1$, and $c < 1 < b$. Therefore,

$$bc > a \Rightarrow b^3c^3 > b^3 + c^3 - 1 \Rightarrow (b^3 - 1)(c^3 - 1) > 0 \ ,$$

which contradicts $b > 1 > c$. In a similar way, we see that $c > a > b$ cannot occur.

Thus, a must be either the largest or the smallest of the three numbers. Hence $(a - b)(a - c) > 0$, whence $a^2 + bc > a(b + c)$. Therefore

$$\begin{aligned} (b+c-a)^3 &= b^3 + c^3 - a^3 + 3b^2c + 3bc^2 - 3ab^2 - 3ac^2 + 3a^2b + 3a^2c - 6abc \\ &= 1 + 3b(a^2+bc) + 3c(a^2+bc) - 3ab(b+c) - 3ac(b+c) \\ &= 1 + 3(b+c)[(a^2+bc) - a(b+c)] > 1 \end{aligned}$$

and the desired result follows.

Solution 5. [X. Li] If $1 + a^2 < b^2 + c^2$, then

$$(1 + a)^2 = 1 + a^2 + 2a < b^2 + c^2 + 2bc = (b + c)^2 \ ,$$

whence $b + c > 1 + a$. On the other hand, if $1 + a^2 \geq b^2 + c^2$, then

$$1 + a^2 - a > b^2 + c^2 - bc = (b^3 + c^3)/(b + c) > 0 \ ,$$

whereupon,

$$\begin{aligned}(b + c)(b^2 + c^2 - bc) &= b^3 + c^3 = 1 + a^3 \\ &= (1 + a)(1 + a^2 - a) > (1 + a)(b^2 + c^2 - bc)\end{aligned}$$

so that $b + c > 1 + a$.

Solution 6. [P. Gyrya] Let $p(x) = x^3 - 3ax$. Checking the first derivative yields that $p(x)$ is strictly increasing for $x > \sqrt{a}$. Now $1 + a \geq 2\sqrt{a} > \sqrt{a}$ and $b + c \geq 2\sqrt{bc} > 2\sqrt{a} > \sqrt{a}$, so both $1 + a$ and $b + c$ lie in the part of the domain of $p(x)$ where it strictly increases. Now

$$p(1 + a) = (1 + a)^3 - 3a(1 + a) = 1 + a^3 = b^3 + c^3 = (b + c)^3 - 3bc(b + c) < (b + c)^3 - 3a(b + c) = p(b + c)$$

from which it follows that $1 + a < b + c$.

Solution 7. Consider the function $g(x) = x(1 + a^3 - x) = x(b^3 + c^3 - x)$. Then $g(1) = g(a^3) = a^3$ and $g(b^3) = g(c^3) = (bc)^3$. Since $a^3 < (bc)^3$ and the graph of $g(x)$ is a parabola opening down, it follows that b^3 and c^3 lie between 1 and a^3.

Now consider the function $h(x) = x^{1/3} + (b^3 + c^3 - x)^{1/3} = x^{1/3} + (1 + a^3 - x)^{1/3}$ for $0 \leq x \leq 1 + a^3$. Then $h(1) = h(a^3) = 1 + a$ and $h(b^3) = h(c^3) = b + c$. The graph of $h(x)$ resembles an inverted parabola, so since b^3 and c^3 lie between 1 and a^3, it follows that $1 + a < b + c$, as desired.

69. Let n, a_1, a_2, $\cdots$, a_k be positive integers for which $n \geq a_1 > a_2 > a_3 > \cdots > a_k$ and the least common multiple of a_i and a_j does not exceed n for all i and j. Prove that $ia_i \leq n$ for $i = 1, 2, \cdots, k$.

Solution 1. The result can be established by induction. It clearly holds when $i = 1$. Suppose that it holds for $1 \leq i \leq m$, so that, in particular $ma_m \leq n$. The least common multiple is equal to $ba_{m+1} = ca_m$ for some positive integers b and c with $c < b$. If $b \geq m + 1$, then $(m + 1)a_{m+1} \leq ba_{m+1} \leq n$ by hypothesis.

Assume that $b \leq m$. Then

$$\begin{aligned}(m + 1)a_{m+1} &= \frac{(m + 1)c}{b} \ a_m \leq \frac{(m + 1)cn}{bm} \\ &= \left(\frac{m + 1}{m}\right)\left(\frac{c}{b}\right)n \leq \left(\frac{m + 1}{m}\right)\left(\frac{b - 1}{b}\right)n \\ &= \left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{b}\right)n \leq \left(1 + \frac{1}{m}\right)\left(1 - \frac{1}{m}\right)n = \left(1 - \frac{1}{m^2}\right)n < n\end{aligned}$$

as desired. The result now follows.

Solution 2. We can obtain the result by induction, it being known when $i = 1$. Suppose that the result holds up to $i = m$. If $(m + 1)a_m \leq n$, then the desired result for $i = m + 1$ follows from $a_{m+1} > a_m$. On the other hand, suppose that $(m + 1)a_m > n$. With $ba_{m+1} = ca_m$ the least common multiple of a_m and a_{m+1}, we have that $(m + 1)a_m > ca_m$, so that

$$a_{m+1} = \frac{c}{b}a_m \leq \frac{c}{c + 1}a_m \leq \frac{m}{m + 1}a_m \leq \frac{n}{m + 1}$$

and the result follows.

70. Let $f(x)$ be a concave strictly increasing function defined for $0 \leq x \leq 1$ such that $f(0) = 0$ and $f(1) = 1$. Suppose that $g(x)$ is its inverse. Prove that $f(x)g(x) \leq x^2$ for $0 \leq x \leq 1$.

Comment. Begin with a sketch. The graph of the function is like a bow on top of a bowstring along the line $y = x$. As x increases, the slope of the chord from the origin to $(x, f(x))$ decreases. The solution begins with an analytic verification of this fact, using the definition of concavity.

Solution. Let $0 < v \leq u$. Then, taking $t = (u - v)/u$ in the definition of concavity, we have that

$$f(v) \geq \frac{vf(u) + (u - v)f(0)}{u} = \frac{vf(u)}{u} \ .$$

When $(u, v) = (1, x)$, this yields $f(x) \geq x$, so that $x = g(f(x)) \geq g(x)$ (since $g(x)$ is an increasing function). Let $(u, v) = (x, g(x))$ to obtain, for $x \neq 0$ that

$$x = f(g(x)) \geq \frac{g(x)f(x)}{x} \ .$$

It is straightforward to verify now that $f(x)g(x) \leq x^2$ for all $x \in [0, 1]$.

Comment. A special case is that $f(x) = x^k$ for $0 < k < 1$, so that $g(x) = x^{1/k}$. Then $f(x)g(x) = x^{k+(1/k)}$ and the result holds since $0 \leq x \leq 1$ and $k + (1/k) \geq 2$.

71. Suppose that lengths b, c and i are given. Construct a triangle ABC for which $|AC| = b$. $|AB| = c$ and the length of the bisector AD of angle A is i (D being the point where the bisector meets the side BC).

Solution 1. `Analysis.` Let AD meet the line through B parallel to AC in T. Then $\angle BTA = \angle TAC = \angle TAB$, so that $|BT| = |AB| = c$. By similar triangles, we have that $|DT| = ic/b$ so that $|AT| = i(b + c)/b$.

`Construction.` Construct an isosceles triangle ABT with the lengths of AB and AT both equal to c and the length of AT equal to $i(b + c)/b$. Cut AD off AT to have the length i, and let C be the intersection of AD and the line through A parallel to BT.

`Proof of construction.` Since $\angle BAT = \angle BTA = \angle CAT$, the segment AT bisects angle BAC. The length of AD is i and the length of AB is c, by construction. From the similar triangles, DBT and ADC, we find that the length of AC is i multiplied by $c = |BT|$ and divided by $[i(b + c)/b] - i = ic/b = |DT|$.

`Feasibility.` In order for the construction to work, we require that the sum of the lengths of AB and BT exceed that of AT. This requires $2c > i(b + c)/b$ or $i < 2bc/(b + c)$.

Solution 2. `Analysis.` Let θ be equal to angles BAD and CAD, where ABC is the required triangle with bisector AD. Since the area of ΔABC is the sum of the areas of ΔABD and ΔADC, we have that $bc \sin 2\theta = i(b + c) \sin \theta$, whence $\cos \theta = (b + c)i/2bc$.

`Sketch of construction and proof.` By Euclidean means it is possible to construct the lengths $b+c$, $(b + c)i$, $2bc$ and $(b + c)/2bc$ using proportionalities. Thus, we can obtain the cosine of the angle θ, and so find θ itself. Construct triangle ABC with the respective lengths of AB and AC equal to c and b and $\angle BAC = 2\theta$. The calculation in the analysis can be used to verify that the length of the bisector is equal to $2bc \cos \theta/(b + c)$, and so equal to i. Note that for θ to be found, it is necessary to have $(b + c)i \leq 2bc$.

72. The centres of the circumscribed and the inscribed spheres of a given tetrahedron coincide. Prove that the four triangular faces of the tetrahedron are congruent.

Solution 1. Let O be the common centre of the circumscribed and inscribed spheres of the tetrahedron $ABCD$. The plane BCO bisects the dihedral angle formed by the planes BCD and BCA, so that the circles determined by BCD and BCA in these planes must be congruent. Thus, $\angle BAC = \angle BDC = \alpha$, say. Similarly, we find that $\angle ABC = \angle ADC = \beta$, $\angle ABC = \angle ADC = \gamma$, $\angle ACB = \angle ADB = \phi$, $\angle BAD = \angle BCD = \psi$, and $\angle BAD = \angle BCD = \omega$. From the sum of angles of various triangles, we find that $\beta + \gamma = \psi + \omega$, $\alpha + \gamma = \phi + \omega$ and $\alpha + \beta = \psi + \phi$, whence $\alpha = \phi$, $\beta = \psi$, $\gamma = \omega$. From this, we see that all the triangles are similar, and congruence follows from the fact that each pair of triangles have corresponding sides in common.

Solution 2. Let R and r be respectively the circumradius and the inradius of $ABCD$, let the faces BCD, ACD, ABD and ABC touch the insphere in the respective points P, Q, T, S, and let O be the common centre of the insphere and the circumsphere. Since triangle OSA is right with $|OS| = r$ and $|OA| = R$, we have that $|SA| = \sqrt{R^2 - r^2}$. Similarly, $|SB| = |SC| = \sqrt{R^2 - r^2}$, so that S is the centre of a circle with radius $\sqrt{R^2 - r^2}$ passing through A, B, C. The same can be said about P, Q, T, and the faces that contain them.

It can be seen that $\Delta ABT \equiv \Delta ABS$, $\Delta ACQ \equiv \Delta ACS$, $\Delta ADQ \equiv \Delta ADT$, $\Delta BCP \equiv \Delta BCS$, $\Delta BDP \equiv \Delta BDR$ and $\Delta CDP \equiv \Delta CDQ$. Now

$$\angle ABS + \angle ACS = 90^\circ - \angle BCS = 90^\circ - \angle BCP = \angle BDP + \angle CDP$$

and

$$\angle ABT + \angle BDT = 90^\circ - \angle DAT = 90^\circ - \angle DAQ = \angle ACQ + \angle DCQ\ .$$

Since $\angle ABS = \angle ABT$, $\angle ACS = \angle ACQ$, $\angle BDP = \angle BDT$ and $\angle CDP = \angle DCQ$, it follows that $\angle ABS - \angle CDP = \pm(\angle ACS - \angle BDP) = 0^\circ$ so $\angle ABC = \angle BCD$.

Obtaining other similar angle equalities, we can determine that the faces are equiangular. Taking note of common sides, we can then deduce their congruence.

73. Solve the equation:

$$\left(\sqrt{2+\sqrt{2}}\right)^x + \left(\sqrt{2-\sqrt{2}}\right)^x = 2^x\ .$$

Solution 1. By inspection, we find that $x = 2$ satisfies the equation. We show that no other value of x does so. Observe that $\sqrt{2-\sqrt{2}} < 1$. When $x < 0$, the second term of the left side exceeds 1 while the right side is less than 1, so the equation is not satisfied. Henceforth, let $x \geq 0$ and let

$$f(x) \equiv 2^x - \left(\sqrt{2+\sqrt{2}}\right)^x$$

and $g(x) = (\sqrt{2-\sqrt{2}})^x$.

Note that, if $a > b > 1$, then $a^x - b^x = b^x((a/b)^x - 1)$ is an increasing function of x. Thus, $f(x)$ is increasing and $g(x)$ is decreasing as x increases. If $0 \leq x < 2$, then $f(x) < f(2) = g(2) < g(x)$, while if $x > 2$, then $f(x) > f(2) = g(2) > g(x)$. The desired result follows.

Solution 2. The equation can be rewritten in the form

$$f(y) \equiv a^y + (1-a)^y = 1$$

where $2y = x$ and $a = \frac{1}{4}(2+\sqrt{2})$. Note that $0 < a < 1$, so that each term is a strictly decreasing function of y. Thus, $f(y)$ assumes each of its values at most once, and since $f(1) = 1$, we find that $x = 2$ is the only solution.

Solution 3. Observe that

$$1 + \frac{\sqrt{2}}{2} = 1 + \cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8}$$

and

$$1 - \frac{\sqrt{2}}{2} = 1 - \sin\frac{\pi}{4} = 2\sin^2\frac{\pi}{8}\ .$$

The equation becomes
$$\left(\cos\frac{\pi}{8}\right)^x + \left(\sin\frac{\pi}{8}\right)^x = 1 \ .$$
This holds for $x = 2$. If $x > 2$, then $x - 2 > 0$ and so
$$\left(\cos\frac{\pi}{8}\right)^x = \left(\cos\frac{\pi}{8}\right)^{x-2}\left(\cos\frac{\pi}{8}\right)^2 < \left(\cos\frac{\pi}{8}\right)^2$$
with a similar inequality for the sine function. Thus, when $x > 2$, the left side is less than 1. Similarly, it can be shown that when $x < 2$, the left side exceeds 1. Hence the unique solution is $x = 2$.

Comment. Generally, the solutions involved a function similar to that used in Solution 2, and it was shown that it was impossible for there to be more than one solution. Some students came up with the use of trigonometry as in Solution 3.

74. Prove that among any group of $n+2$ natural numbers, there can be found two numbers so that their sum or their difference is divisible by $2n$.

Solution 1. For $0 \leq k \leq n$, let S_k be the subset of numbers x among the $n+2$ numbers for which x differs from either k or $2n-k$ by a multiple of $2n$. Since there are $n+2$ numbers and only $n+1$ subsets, the Pigeonhole Principle provides that some subset must contain at least two numbers u and v, say. Either u and v both leave the same remainder upon division by $2n$ and so differ by a multiple of $2n$, or else one of them differs from k by a multiple of $2n$ while the other differs from $2n-k$ by a multiple of $2n$. In the latter case, $u+v$ is a multiple of $2n$.

Solution 2. [A. Fink] Consider an arbitrary set of $n+2$ natural numbers. If any two are congruent modulo $2n$, then their difference is divisible by $2n$ and the result follows. Suppose otherwise, that all numbers have distinct residues modulo $2n$. Apportion these residues into the $n+1$ sets: $\{0\}$, $\{1, 2n-1\}$, $\{2, 2n-2\}$, $\cdots$, $\{n-1, n+1\}$, $\{n\}$. Since there are $n+2$ numbers, at least one of these sets must contain two residues, and so the two numbers involved must sum to a multiple of $2n$.

75. Three consecutive natural numbers, larger than 3, represent the lengths of the sides of a triangle. The area of the triangle is also a natural number.

(a) Prove that one of the altitudes "cuts" the triangle into two triangles, whose side lengths are natural numbers.

(b) The altitude identified in (a) divides the side which is perpendicular to it into two segments. Find the difference between the lengths of these segments.

Solution 1. Let the side lengths be $x-1$, x, $x+1$. By Heron's formula, the area A of the triangle is given by
$$A^2 = \frac{3}{16}x^2(x+2)(x-2) = \frac{(3(x^2-4))x^2}{16} \ .$$
Since A is an integer, x must be even and $3(x^2-4)$ must be the square of a multiple of $2 \times 3 = 6$. Hence for some integer y, we have that $3(x^2-4) = (6y)^2 = 36y^2$ or $x^2 - 12y^2 = 4$. (*Comment.* This is a Pell's equation and it has infinitely many solutions $(x, y) = (x_n, y_n)$ given by $(x_n, y_n) = 2(7 + 2\sqrt{12})^n$ and $(4 + \sqrt{12})(7 + 2\sqrt{12})^n$.

(a) The area A of the triangle is $\frac{1}{4}(6xy) = \frac{3xy}{2}$ where x is even and $x^2 - 12y^2 = 4$. The altitude to the side of length x is $2A/x = 3y$, an integer. This is the desired altitude.

(b) The triangle is subdivided by the altitude in (a) into two right triangles whose hypotenuses have lengths $x-1$ and $x+1$. Hence, the side of length x is split into two parts of lengths
$$\sqrt{(x-1)^2 - (3y)^2} = \sqrt{x^2 - 2x + 1 - 9y^2} = \sqrt{(x^2/4) - 2x + 4} = \frac{1}{2}(x-4)$$

and

$$\sqrt{(x+1)^2-(3y)^2}=\sqrt{x^2+2x+1-9y^2}=\sqrt{(x^2/4)+2x+4}=\frac{1}{2}(x+4)\ .$$

The difference between the lengths of these segments is 4. (Note that the sum is x. as expected.) (*Exercise.* Give some numerical examples.)

Solution 2. [L. Tchourakov] With the above notation, we find that $A = x\sqrt{3(x^2-4)}/4$, so that the length of the altitude to the side of length x is $\frac{1}{2}\sqrt{3(x^2-4)}$. If x were odd, then the numerator of the fraction for A would be odd and A not an integer. Hence x is even, and so is the altitude. Let u be one of the two parts of the side cut off by the altitude. By the pythagorean theorem,

$$u^2=(x+1)^2-\frac{3(x^2-4)}{4}=\frac{(x+4)^2}{4}\ ,$$

so that $u = 2 + x/2$. Since x is even, u is an integer. The altitude cuts the side into parts of length u and $x - u = u - 4$, and so (b) follows.

76. Solve the system of equations:

$$\log x+\frac{\log(xy^8)}{\log^2 x+\log^2 y}=2\ ,$$

$$\log y+\frac{\log(x^8/y)}{\log^2 x+\log^2 y}=0\ .$$

(The logarithms are taken to base 10.)

Solution 1. Let $u = \log x$, $v = \log y$ and $w = u^2 + v^2$. Note that w is nonzero. The equations become

$$u+(u+8v)/w=2 \qquad \text{and} \qquad v+(8u-v)/w=0\ .$$

Squaring and adding the equations $u + 8v = (2 - u)w$ and $8u - v = -vw$ yields $65(u^2 + v^2) = (4 - 4w + u^2 + v^2)w^2$, or $65 = (4 - 4u + w)w$. We can also write the system as

$$\begin{aligned}(w+1)u+8v&=2w\\ 8u+(w-1)v&=0\ ,\end{aligned}$$

which can be solved to yield

$$u=\frac{2w(w-1)}{w^2-65} \qquad v=\frac{16w}{65-w^2}\ .$$

Hence

$$w=u^2+v^2=\frac{4w^2(w-1)^2+256w^2}{(65-w^2)}$$

$$\Longrightarrow (65-w^2)^2=4w(w-1)^2+256w$$

$$\Longrightarrow$$

$$\begin{aligned}0&=w^4-4w^3-122w^2-260w+65^2\\ &=(w-13)(w-5)[(w+7)^2+4^2]\ .\end{aligned}$$

The two relevant solutions are $w = 13$ and $w = 5$.

When $w = 13$, $17 - 4u = 5$, which leads to $(u, v) = (3, -2)$. When $w = 5$, $9 - 4u = 13$, so that $(u, v) = (-1, 2)$. The desired solutions are $(x, y) = (10^3, 10^{-2}), (10^{-1}, 10^2)$.

Solution 2. With the same notation as (1), we can write the given equations in terms of u and v. Multiply the first equation by u and the second by v and add them to obtain the equation $2uv + 8 = 2v$, whereupon $u = 1 - (4/v)$. Eliminating u from the second equation yields $v^4 - 16 = 0$, whereupon $v = 2$ or $v = -2$. The remaining part of the solution is easily completed.

77. n points are chosen from the circumference or the interior of a regular hexagon with sides of unit length, so that the distance between any two of them is **not** less than $\sqrt{2}$. What is the largest natural number n for which this is possible?

Solution. [O. Ivrii] Let the hexagon be $ABCDEF$. Consider the points A, C, E. Since the lengths of AC, AE and CE are each $\sqrt{3} > \sqrt{2}$, the task is possible for $n = 3$. We show that it is impossible for $n \geq 4$.

Consider four points, M, N, P, Q, within the hexagon with each pair distant at least $\sqrt{2}$; let O be the centre of the hexagon. We have that

$$2 \leq |MN|^2 = |OM|^2 + |ON|^2 - 2|OM||ON| \cos \angle MON \leq 1 + 1 - 2|OM||ON| \cos \angle MON \ .$$

It follows that $\cos \angle MON \leq 0$, or $\angle MON \geq 90°$. Similarly, all four angles MON, NOP, POQ, QOM should be at least $90°$. Since the four sum to $360°$, each must be exactly $90°$, and so the lengths of each of OM, ON, OP, OQ equal 1. Thus, all four points must be vertices of the hexagon. But this is impossible, since no two diagonals of the hexagon intersect in a right angle.

Thus, we have proved by contradiction that n cannot be 4. It is straightforward to see that n cannot exceed 4 either. So the largest n with the desired property is 3.

78. A truck travelled from town A to town B over several days. During the first day, it covered $1/n$ of the total distance, where n is a natural number. During the second day, it travelled $1/m$ of the remaining distance, where m is a natural number. During the third day, it travelled $1/n$ of the distance remaining after the second day, and during the fourth day, $1/m$ of the distance remaining after the third day. Find the values of m and n if it is known that, by the end of the fourth day, the truck had travelled $3/4$ of the distance between A and B. (Without loss of generality, assume that $m < n$.)

Solution. [R. Furmaniak, J. Rin] Let d be the distance remaining at the beginning of a two-day period. The distance remaining at the end of the period is

$$d - \left[\frac{d}{n} + \frac{1}{m}\left(d - \frac{d}{n}\right)\right] = d\left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{n}\right) \ .$$

Thus, every two days the remaining distance is reduced by a factor of $r = (1 - (1/m))(1 - (1/n))$. (Note that this is symmetric in m and n.) After four days, the distance remaining is reduced by a factor of r^2; it is given in the problem that this is $1/4$. Hence $r = 1/2$.

Hence

$$\begin{aligned} \frac{1}{2} &= \left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{n}\right) \\ &\Longrightarrow mn = 2(m-1)(n-1) \\ &\Longrightarrow 0 = mn - 2(m+n) + 2 \\ &\Longrightarrow 2 = (m-2)(n-2) \ . \end{aligned}$$

Since m and n are positive integers with $m < n$, we must have $n - 2 = 2$ and $m - 2 = 1$, *i.e.*, $(m, n) = (3, 4)$.

79. Let x_0, x_1, x_2 be three positive real numbers. A sequence $\{x_n\}$ is defined, for $n \geq 0$ by

$$x_{n+3} = \frac{x_{n+2} + x_{n+1} + 1}{x_n} \ .$$

Determine all such sequences whose entries consist solely of positive integers.

Solution 1. Let the first three terms of the sequence be x, y, z. Then it can be readily checked that the sequence must have period 8, and that the entries cycle through the following:

$$x, y, z, \frac{y+z+1}{x}, \frac{x+y+z+1+xz}{xy},$$

$$\frac{(x+y+1)(y+z+1)}{xyz}, \frac{x+y+z+1+xz}{yz}, \frac{x+y+1}{z} \ .$$

For all of these entries to be positive integers, it is necessary that $y+z+1$ be divisible by x, $x+y+1$ be divisible by z and $(x+1)(z+1)$ be divisible by y. In particular, $x \le y+z+1$ and $z \le x+y+1$.

Without loss of generality, we can assume that the smallest entry in the sequence is y. Then,

$$\frac{y+z+1}{x} \ge y$$

and

$$\frac{x+y+1}{z} \ge y$$

whence

$$\begin{aligned} xy^2 &= y(xy) \le y(y+z+1) \\ &= y^2+yz+y \le y^2+x+y+1+y \\ &= x+(y+1)^2 \ . \end{aligned}$$

Hence

$$x(y^2-1) \le (y+1)^2$$

so that either $y = 1$ or

$$y \le x \le \frac{y+1}{y-1} = 1 + \frac{2}{y-1} \ .$$

The latter yields $(y-1)^2 \le 2$ so that $y \le 2$.

Suppose that $y = 1$. Then $x \le y+z+1 = z+2 \le y+x+3 = x+4$. Since x divides x and $y+z+1$, it must divide their difference, which cannot exceed 4. Hence $x = y+z+1$ or x must be one of 1, 2, 3, 4. Similarly, $z = x+y+1$ or z must be one of 1, 2, 3, 4.

If $x = y+z+1 = z+2$, then $x+y+1 = z+4$ and so z divides 4. We get the periods

$$(3,1,1,1,3,5,9,5)$$

$$(4,1,2,1,4,3,8,3)$$

$$(6,1,4,1,6,2,9,2) \ .$$

The case $z = x+y+1$ yields essentially the same periods. Otherwise, $1 \le x, z \le 4$ and we find the additional possible period

$$(2,1,2,2,5,4,5,2)$$

For each period, x_0 can start anywhere.

Suppose that $y = 2$. Then $x \le y+z+1 \le z+3 \le x+6$, so that x must divide some number not exceeding 6. Similarly, z cannot exceed 6. If $x = 2$, then $x+y+1 = 5$ and so $z = 5$; this yields the period $(2,2,5,4,5,2,2,1)$ already noted. If $x = 3$, then z must be 2, 3 or 6, and we obtain $(3,2,3,2,3,2,3,2)$; the possibilities $z = 2,6$ do not work. If $x = 4$, $z = 7$, which does not work. If $x = 5$, then $z = 2,4$ and we get a period already noted. If $x = 6$, then $z = 3$, which does not work.

Hence there are five possible cycles and the sequence can begin at any term in the cycle:

$$(1, 1, 1, 3, 5, 9, 5, 3)$$

$$(1, 2, 1, 4, 3, 8, 3, 4)$$

$$(1, 2, 2, 5, 4, 5, 2, 2)$$

$$(1, 4, 1, 6, 2, 9, 2, 6)$$

$$(2, 3, 2, 3, 2, 3, 2, 3)$$

Solution 2. [O. Ivrii] We show that the sequence contains a term that does not exceed 2. Suppose that none of x_0, x_1 and x_2 is less than 3. Let $k = \max(x_1, x_2)$. Then, noting that k is an integer and that $k \geq 3$, we deduce that

$$x_3 = \frac{x_1 + x_2 + 1}{x_0} \leq \frac{2k+1}{3} < k$$

so $x_3 \leq k - 1$ and

$$x_4 = \frac{x_2 + x_3 + 1}{x_1} \leq \frac{2k}{3} \leq k - 1$$

so that $\max(x_3, x_4) = k - 1$. If $k - 1 \geq 3$, we repeat the process to get a strictly lower bound on the next two terms. Eventually, we obtain two consecutive terms whose maximum is less than 3. In fact, we can deduce that there is an entry equal to either 1 or 2 arbitrarily far out in the sequence.

Suppose, from some point on in the sequence, there is no term equal to 1. Then there are three consecutive terms $a, 2, b$. The previous term is $(a+3)/b$ and the following term is $(b+3)/a$, so that a divides $b+3$ and b divides $a+3$. Since $a - 3 \leq b \leq a + 3$, b divides two numbers that differ by at most 6; similarly with a. Hence, neither a nor b exceeds 6. Testing out possibilities leads to $(a, b) = (6, 3), (5, 2), (3, 3)$.

Finally, we suppose that the sequence has three consecutive terms $a, 1, b$ and by similar arguments are led to the sequences obtained in the first solution.

Comment. C. Lau established that $x_n x_{n+4} = x_{n+2} x_{n+6}$ and thereby obtained the periodicity. R. Barrington Leigh showed that, if all terms of the sequence were at least equal to 2, then the sequence $\{y_n\}$ defined by $y_n = \max(x_n, x_{n+1})$ satisfies $y_{n+1} \leq y_n$. Since the same recursion defines the sequence "going backwards", we also have $y_{n-1} \leq y_n$, for all n. Hence $\{y_n\}$ is a constant sequence, and so $\{x_n\}$ is either constant or has period 2. It is straightforward to rule out the constant possibility. If the periodic segment of the sequence is (a, b), then $b = (a + b + 1)/a$ or $(a - 1)(b - 1) = 2$ and we are led to the segment $(2, 3)$. Otherwise, there is a 1 in the sequence and we can conclude as before.

80. Prove that, for each positive integer n, the series

$$\sum_{k=1}^{\infty} \frac{k^n}{2^k}$$

converges to twice an odd integer not less than $(n + 1)!$.

Solution 1. Since the series consists of nonnegative terms, we can establish its convergence by eventually showing that it is dominated term by term by a geometric series with common ratio less than 1. Noting that $n \leq k \log_k(3/2)$ for k sufficiently large, we find that for large k, $k^n < (3/2)^k$ and the kth term of the series is dominated by $(3/4)^k$. Thus the sum of the series is defined for each nonnegative integer n.

For nonnegative integers n, let

$$S_n = \sum_{k=1}^{\infty} \frac{k^n}{2^k} .$$

Then $S_0 = 1$ and

$$\begin{aligned}
S_n - \frac{1}{2}S_n &= \sum_{k=1}^{\infty} \frac{k^n}{2^k} - \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}} \\
&= \sum_{k=1}^{\infty} \frac{k^n}{2^k} - \sum_{k=1}^{\infty} \frac{(k-1)^n}{2^k} \\
&= \sum_{k=1}^{\infty} \frac{k^n - (k-1)^n}{2^k} \\
&= \sum_{k=1}^{\infty} \left[\binom{n}{1} \frac{k^{n-1}}{2^k} - \binom{n}{2} \frac{k^{n-2}}{2^k} + \binom{n}{3} \frac{k^{n-3}}{2^k} - \cdots + (-1)^{n-1} \frac{1}{2^k} \right]
\end{aligned}$$

whence

$$S_n = 2\left[\binom{n}{1} S_{n-1} - \binom{n}{2} S_{n-2} + \binom{n}{3} S_{n-3} - \cdots + (-1)^{n-1} \right] .$$

An induction argument establishes that S_n is twice an odd integer.

Observe that $S_0 = 1$, $S_1 = 2$, $S_2 = 6$ and $S_3 = 26$. We prove by induction that, for each $n \geq 0$,

$$S_{n+1} \geq (n+2)S_n$$

from which the desired result will follow. Suppose that we have established this for $n = m - 1$. Now

$$S_{m+1} = 2\left[\binom{m+1}{1} S_m - \binom{m+1}{2} S_{m-1} + \binom{m+1}{3} S_{m-2} - \binom{m+1}{4} S_{m-3} + \cdots \right] .$$

For each positive integer r,

$$\begin{aligned}
&\binom{m+1}{2r-1} S_{m-2r+2} - \binom{m+1}{2r} S_{m-2r+1} \\
&\qquad \geq \left[\binom{m+1}{2r-1} (m-2r+3) - \binom{m+1}{2r} \right] S_{m-2r+1} \\
&\qquad = \binom{m+1}{2r-1} \left[(m-2r+3) - \left(\frac{m-2r+2}{2r} \right) \right] S_{m-2r+1} \geq 0 .
\end{aligned}$$

When $r = 1$, we get inside the square brackets the quantity

$$(m+1) - \frac{m}{2} = \frac{m+2}{2}$$

while when $r > 1$, we get

$$(m-2r+3) - \left(\frac{m-2r+2}{2r} \right) > (m-2r+3) - (m-2r+2) = 1 .$$

Hence

$$\begin{aligned}
S_{m+1} &\geq 2\left[\binom{m+1}{1} S_m - \binom{m+1}{2} S_{m-1} \right] \\
&\geq 2\left[(m+1)S_m - \frac{m(m+1)}{2} \cdot \frac{1}{m+1} S_m \right] \\
&= 2\left[m + 1 - \frac{m}{2} \right] s_m = (m+2)S_m .
\end{aligned}$$

Solution 2. Define S_n as in the foregoing solution. Then, for $n \geq 1$,

$$\begin{aligned} S_n &= \frac{1}{2} + \sum_{k=2}^{\infty} \frac{k^n}{2_k} \\ &= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{(k+1)^n}{2^k} \\ &= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{k^n + \binom{n}{1}k^{n-1} + \cdots + \binom{n}{n-1}k + 1}{2^k} \\ &= \frac{1}{2} + \frac{1}{2}\left[S_n + \binom{n}{1}S_{n-1} + \cdots + \binom{n}{n-1}S_1 + 1\right] \end{aligned}$$

whence

$$S_n = \binom{n}{1}S_{n-1} + \binom{n}{2}S_{n-2} + \cdots + \binom{n}{n-1}S_1 + 2 \ .$$

It is easily checked that $S_k \equiv 2 \pmod 4$ for $k = 0, 1$. As an induction hypothesis, suppose this holds for $1 \leq k \leq n-1$. Then, modulo 4, the right side is congruent to

$$2[\sum_{k=0}^{n}\binom{n}{k} - 2] + 2 = 2(2^n - 2) + 2 = 2^{n+1} - 2 \ ,$$

and the desired result follows.

For $n \geq 1$,

$$\begin{aligned} \frac{S_{n+1}}{S_n} &= \frac{\binom{n+1}{1}S_n + \binom{n+1}{2}S_{n-1} + \cdots + \binom{n+1}{n}S_1 + 2}{S_n} \\ &= (n+1) + \frac{\binom{n+1}{2}S_{n-1} + \binom{n+1}{3}S_{n-2} + \cdots + (n+1)S_1 + 2}{\binom{n}{1}S_{n-1} + \binom{n}{2}S_{n-2} + \cdots + nS_1 + 2} \\ &\geq (n+1) + 1 = n + 2 \ , \end{aligned}$$

since each term in the numerator of the latter fraction exceeds each corresponding term in the denominator.

Solution 3. [of the first part using an idea of P. Gyrya] Let $f(x)$ be a differentiable function and let D be the differentiation operator. Define the operator L by

$$L(f)(x) = x \cdot D(f)(x) \ .$$

Suppose that $f(x) = (1-x)^{-1} = \sum_{k=0}^{\infty} x^k$. Then, it is standard that $L^n(f)(x)$ has a power series expansion obtained by term-by-term differentiation that converges absolutely for $|x| < 1$. By induction, it can be shown that the series given in the problem is, for each nonnegative integer n, $L^n(f)(1/2)$.

It is straightforward to verify that

$$L((1-x)^{-1}) = x(1-x)^{-2}$$

$$L^2((1-x)^{-1}) = x(1+x)(1-x)^{-3}$$

$$L^3((1-x)^{-1}) = x(1+4x+x^2)(1-x)^{-4}$$

$$L^4((1-x)^{-1}) = x(1+11x+11x^2+x^3)(1-x)^{-5} \ .$$

In general, a straightforward induction argument yields that for each positive integer n,

$$L^n(f)(x) = x(1 + a_{n,1}x + \cdots + a_{n,n-2}x^{n-2} + x^{n-1})(1-x)^{-(n+1)}$$

for some integers $a_{n,1}, \cdots, a_{n,n-2}$. Hence

$$L^n(f)(1/2) = 2(2^{n-1} + a_{n,1}2^{n-2} + \cdots + a_{n,n-2}2 + 1) ,$$

yielding the desired result.

81. Suppose that $x \geq 1$ and that $x = \lfloor x \rfloor + \{x\}$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x and the fractional part $\{x\}$ satisfies $0 \leq x < 1$. Define

$$f(x) = \frac{\sqrt{\lfloor x \rfloor} + \sqrt{\{x\}}}{\sqrt{x}} .$$

(a) Determine the smallest number z such that $f(x) \leq z$ for each $x \geq 1$.

(b) Let $x_0 \geq 1$ be given, and for $n \geq 1$, define $x_n = f(x_{n-1})$. Prove that $\lim_{n\to\infty} x_n$ exists.

81. *Solution.* (a) Let $x = y + z$, where $y = \lfloor x \rfloor$ and $z = \{x\}$. Then

$$f(x)^2 = 1 + \frac{2\sqrt{yz}}{y+z} ,$$

which is less than 2 because $\sqrt{yz} \leq \frac{1}{2}(y+z)$ by the arithmetic-geometric means inequality. Hence $0 \leq f(x) \leq \sqrt{2}$ for each value of x. Taking $y = 1$, we find that

$$\lim_{x\uparrow 2} f(x)^2 = \lim_{z\uparrow 1} \left(1 + \frac{2\sqrt{z}}{1+z}\right) = 2 ,$$

whence $\sup\{f(x) : x \geq 1\} = \sqrt{2}$.

(b) In determining the fate of $\{x_n\}$, note that after the first entry, the sequence lies in the interval $[1, 2)$. So, without loss of generality, we may assume that $1 \leq x_0 < 2$. If $x_n = 1$, then each $x_n = 1$ and the limit is 1. For the rest, note that $f(x)$ simplifies to $(1 + \sqrt{x-1})/\sqrt{x}$ on $(1, 2)$. The key point now is to observe that there is exactly one value v between 1 and 2 for which $f(v) = v$, $f(x) > x$ when $1 < x < v$ and $f(x) < x$ when $v < x < 2$. Assume these facts for a moment. A derivative check reveals that $f(x)$ is strictly increasing on $(1, 2)$, so that for $1 < x < v$, $x < f(x) < f(v) = v$, so that the iterates $\{x_n\}$ constitute a bounded, increasing sequence when $1 < x_0 < v$ which must have a limit. (In fact, this limit must be a fixed point of f and so must be v.) A similar argument shows that, if $v < x_0 < 2$, then the sequence of iterates constitute a decreasing convergent sequence (with limit v).

It remains to show that a unique fixed point v exists. Let $x = 1 + u$ with $u > 0$. Then it can be checked that $f(x) = x$ if and only if $1 + 2\sqrt{u} + u = 1 + 3u + 3u^2 + u^3$ or $u^5 + 6u^4 + 13u^3 + 12u^2 + 4u - 4 = 0$. Since the left side is strictly increasing in u, takes the value -4 when $u = 0$ and the value 32 when $u = 1$, the equation is satified for exactly one value of u in $(0, 1)$; now let $v = 1 + u$. The value of V turns out to be about 1.375, (Note that $f(x) > x$ if and only if $x < u$.)

82. (a) A regular pentagon has side length a and diagonal length b. Prove that

$$\frac{b^2}{a^2} + \frac{a^2}{b^2} = 3 .$$

(b) A regular heptagon (polygon with seven equal sides and seven equal angles) has diagonals of two different lengths. Let a be the length of a side, b be the length of a shorter diagonal and c be the length of a longer diagonal of a regular heptagon (so that $a < b < c$). Prove that:

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = 6$$

and

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5 \ .$$

82. *Solution 1.* (a) Let $ABCDE$ be the regular pentagon, and let triangle ABC be rotated about C so that B falls on D and A falls on E. Then ADE is a straight angle and triangle CAE is similar to triangle BAC. Therefore

$$\frac{a+b}{b} = \frac{b}{a} \Longrightarrow \frac{b}{a} - \frac{a}{b} = 1 \Longrightarrow \frac{b^2}{a^2} + \frac{a^2}{b^2} - 2 = 1$$

so that $b^2/a^2 + a^2/b^2 = 3$, as desired.

(b) Let A, B, C, D, E be consecutive vertices of the regular heptagon. Let AB, AC and AD have respective lengths a, b, c, and let $\angle BAC = \theta$. Then $\theta = \pi/7$, the length of BC, of CD and of DE is a, the length of AE is c, $\angle CAD = \angle DAE = \theta$, since the angles are subtended by equal chords of the circumcircle of the heptagon, $\angle ADC = 2\theta$, $\angle ADE = \angle AED = 3\theta$ and $\angle ACD = 4\theta$. Triangles ABC and ACD can be glued together along BC and DC (with C on C) to form a triangle similar to ΔABC, whence

$$\frac{a+c}{b} = \frac{b}{a} \ . \tag{1}$$

Triangles ACD and ADE can be glued together along CD and ED (with D on D) to form a triangle similar to ΔABC, whence

$$\frac{b+c}{c} = \frac{b}{a} \ . \tag{2}$$

Equation (2) can be rewritten as $\frac{1}{b} = \frac{1}{a} - \frac{1}{c}$. whence

$$b = \frac{ac}{c-a} \quad .$$

Substituting this into (1) yields

$$\frac{(c+a)(c-a)}{ac} = \frac{c}{c-a}$$

which simplifies to

$$a^3 - a^2c - 2ac^2 + c^3 = 0 \quad . \tag{3}$$

Note also from (1) that $b^2 = a^2 + ac$.

$$\begin{aligned}
\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} - 6 &= \frac{a^4c^2 + b^4a^2 + c^4b^2 - 6a^2b^2c^2}{a^2b^2c^2} \\
&= \frac{a^4c^2 + (a^4 + 2a^3c + a^2c^2)a^2 + c^4(a^2 + ac) - 6a^2c^2(a^2 + ac)}{a^2b^2c^2} \\
&= \frac{a^6 + 2a^5c - 4a^4c^2 - 6a^3c^3 + a^2c^4 + ac^5}{a^2b^2c^2} \\
&= \frac{a(a^2 + 3ac + c^2)(a^3 - a^2c - 2ac^2 + c^3)}{a^2b^2c^2} = 0 \text{ `}.
\end{aligned}$$

$$\begin{aligned}
\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} - 5 &= \frac{(a^4 + 2a^3c + a^2c^2)c^2 + a^2c^4 + a^4(a^2 + ac) - 5a^2c^2(a^2 + ac)}{a^2b^2c^2} \\
&= \frac{a^6 + a^5c - 4a^4c^2 - 3a^3c^3 + 2a^2c^4}{a^2b^2c^2} \\
&= \frac{a^2(a + 2c)(a^3 - a^2c - 2ac^2 + c^3)}{a^2b^2c^2} = 0 \ .
\end{aligned}$$

Solution 2. (b) [R. Barrington Leigh] Let the heptagon be $ABCDEFG$; let AD and BG intersect at P, and BF and CG intersect at Q. Observe that $|PD| = |GE| = b$, $|AP| = c - b$, $|GP| = |DE| = a$, $|BP| = b - a$, $|GQ| = |AB| = a$, $|CQ| = c - a$. From similarity of triangles, we obtain the following:

$$\frac{a}{c} = \frac{c-b}{a} \Longrightarrow \frac{a}{c} - \frac{c}{a} + \frac{b}{a} = 0 \qquad (\Delta APG \sim \Delta ADE)$$

$$\frac{c-a}{a} = \frac{c}{b} \Longrightarrow \frac{c}{a} - \frac{c}{b} = 1 \qquad (\Delta QBC \sim \Delta CEG)$$

$$\frac{c-b}{a} = \frac{b-a}{b} \Longrightarrow \frac{c}{a} - \frac{b}{a} + \frac{a}{b} = 1 \qquad (\Delta APG \sim \Delta DPB)$$

$$\frac{b-a}{a} = \frac{b}{c} \Longrightarrow \frac{b}{a} - \frac{b}{c} = 1 \qquad (\Delta ABP \sim \Delta ADB)\ .$$

Adding these equations in pairs yields

$$\frac{b}{a} + \frac{a}{c} - \frac{c}{b} = 1 \Longrightarrow \frac{b^2}{a^2} + \frac{a^2}{c^2} + \frac{c^2}{b^2} + 2\left(\frac{b}{c} - \frac{c}{a} - \frac{a}{b}\right) = 1$$

and

$$\frac{c}{a} + \frac{a}{b} - \frac{b}{c} = 2 \Longrightarrow \frac{c^2}{a^2} + \frac{a^2}{b^2} + \frac{b^2}{c^2} + 2\left(\frac{c}{b} - \frac{b}{a} - \frac{a}{c}\right) = 4\ .$$

The desired result follows from these equations.

Solution 3. (b) [of the second result by J. Chui] Let the heptagon be $ABCDEFG$ and $\theta = \pi/7$. Using the Law of Cosines in the indicated triangles ACD and ABC, we obtain the following:

$$\cos 2\theta = \frac{a^2 + c^2 - b^2}{2ac} = \frac{1}{2}\left(\frac{a}{c} + \frac{c}{a} - \frac{b^2}{ac}\right)$$

$$\cos 5\theta = \frac{2a^2 - b^2}{2a^2} = 1 - \frac{1}{2}\left(\frac{b}{a}\right)^2$$

from which, since $\cos 2\theta = -\cos 5\theta$,

$$-1 + \frac{1}{2}\left(\frac{b}{a}\right)^2 = \frac{1}{2}\left(\frac{a}{c} + \frac{c}{a} - \frac{b^2}{ac}\right)$$

or

$$\frac{b^2}{a^2} = 2 + \frac{a}{c} + \frac{c}{a} - \frac{b^2}{ac}\ . \tag{1}$$

Examining triangles ABC and ADE, we find that $\cos\theta = b/2a$ and $\cos\theta = (2c^2 - a^2)/(2c^2) = 1 - (a^2/2c^2)$, so that

$$\frac{a^2}{c^2} = 2 - \frac{b}{a}\ . \tag{2}$$

Examining triangles ADE and ACF, we find that $\cos 3\theta = a/2c$ and $\cos 3\theta = (2b^2 - c^2)/(2b^2)$, so that

$$\frac{c^2}{b^2} = 2 - \frac{a}{c}\ . \tag{3}$$

Adding equations (1), (2), (3) yields

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 6 + \frac{c^2 - bc - b^2}{ac}\ .$$

By Ptolemy's Theorem, the sum of the products of pairs of opposite sides of a concylic quadrilaterial is equal to the product of the diagonals. Applying this to the quadrilaterals $ABDE$ and $ABCD$, respectively, yields $c^2 = a^2 + bc$ and $b^2 = ac + a^2$, whence $c^2 - bc - b^2 = a^2 + bc - bc - ac - a^2 = -ac$ and we find that

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 6 - 1 = 5 \ .$$

Solution 4. [of the second result by X. Jin] By considering isosceles triangles with side-base pairs (a, b), (c, a) and (b, c), we find that $b^2 = 2a^2(1 - \cos 5\theta)$, $a^2 = 2c^2(1 - \cos\theta)$, $c^2 = 2b^2(1 - \cos 3\theta)$, where $\theta = \pi/7$. Then

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 2[3 - (\cos\theta + \cos 3\theta + \cos 5\theta)] \ .$$

Now,

$$\begin{aligned}\sin\theta(\cos theta + \cos 3\theta + \cos 5\theta) &= \frac{1}{2}[\sin 2\theta + (\sin 4\theta - \sin 2\theta) + (\sin 6\theta - \sin 4\theta)] \\ &= \frac{1}{2}\sin 6\theta = \frac{1}{2}\sin\theta \ ,\end{aligned}$$

so that $\cos\theta + \cos 3\theta + \cos 5\theta = 1/2$. Hence $b^2/a^2 + c^2/b^2 + a^2/c^2 = 2(5/2) = 5$.

Solution 5. (b) There is no loss of generality in assuming that the vertices of the heptagon are placed at the seventh roots of unity on the unit circle in the complex plane. Then $\zeta = \cos(2\pi/7) + i\sin(2\pi/7)$ be the fundamental seventh root of unity. Then $\zeta^7 = 1$, $1 + \zeta + \zeta^2 + \cdots + \zeta^6 = 0$ and (ζ, ζ^6), (ζ^2, ζ^5), (ζ^3, ζ^4) are pairs of complex conjugates. We have that

$$\begin{aligned}a &= |\zeta - 1| = |\zeta^6 - 1| \\ b &= |\zeta^2 - 1| = |\zeta^9 - 1| \\ c &= |\zeta^3 - 1| = |\zeta^4 - 1| \ .\end{aligned}$$

It follows from this that

$$\frac{b}{a} = |\zeta + 1| \qquad \frac{c}{b} = |\zeta^2 + 1| \qquad \frac{a}{c} = |\zeta^3 + 1| \ ,$$

whence

$$\begin{aligned}\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} &= (\zeta + 1)(\zeta^6 + 1) + (\zeta^2 + 1)(\zeta^5 + 1) + (\zeta^3 + 1)(\zeta^4 + 1) \\ &= 2 + \zeta + \zeta^6 + 2 + \zeta^2 + \zeta^5 + 2 + \zeta^3 + \zeta^4 \\ &= 6 + (\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6) = 6 - 1 = 5 \ .\end{aligned}$$

Also

$$\frac{a}{b} = |\zeta^4 + \zeta^2 + 1| \qquad \frac{b}{c} = |\zeta^6 + \zeta^3 + 1| \qquad \frac{c}{a} = |\zeta^2 + \zeta + 1| \ ,$$

whence

$$\begin{aligned}&\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = (\zeta^4 + \zeta^2 + 1)(\zeta^3 + \zeta^5 + 1) + (\zeta^6 + \zeta^3 + 1)(\zeta + \zeta^4 + 1) + (\zeta^2 + \zeta + 1)(\zeta^5 + \zeta^6 + 1) \\ &\quad = (3 + 2\zeta^2 + \zeta^3 + \zeta^4 + 2\zeta^5) + (3 + \zeta + 2\zeta^3 + 2\zeta^4 + \zeta^6) + (3 + 2\zeta + \zeta^2 + \zeta^5 + 2\zeta^6) \\ &\quad = 9 + 3(\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6) = 9 - 3 = 6 \ .\end{aligned}$$

Solution 6. (b) Suppose that the circumradius of the heptagon is 1. By considering isosceles triangles with base equal to the sides or diagonals of the heptagon and apex at the centre of the circumcircle, we see that

$$\begin{aligned}a &= 2\sin\theta = 2\sin 6\theta = -2\sin 8\theta \\ b &= 2\sin 2\theta = -2\sin 9\theta \\ c &= 2\sin 3\theta = 2\sin 4\theta\end{aligned}$$

where $\theta = \pi/7$ is half the angle subtended at the circumcentre by each side of the heptagon. Observe that

$$\cos 2\theta = \frac{1}{2}(\zeta + \zeta^6) \qquad \cos 4\theta = \frac{1}{2}(\zeta^2 + \zeta^5) \qquad \cos 6\theta = \frac{1}{2}(\zeta^3 + \zeta^4)$$

where ζ is the fundamental primitive root of unity. We have that

$$\frac{b}{a} = 2\cos\theta = 2\cos 6\theta \qquad \frac{c}{b} = 2\cos 2\theta \qquad \frac{a}{c} = -2\cos 4\theta$$

whence

$$\begin{aligned}\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} &= 4\cos^2 6\theta + 4\cos^2 2\theta + 4\cos^2 4\theta \\ &= (\zeta^3 + \zeta^4)^2 + (\zeta + \zeta^6)^2 + (\zeta^2 + \zeta^5)^2 \\ &= \zeta^6 + 2 + \zeta + \zeta^2 + 2 + \zeta^5 + \zeta^4 + 2 + \zeta = 6 - 1 = 5\ .\end{aligned}$$

Also

$$\begin{aligned}\frac{a}{b} &= \frac{\sin 6\theta}{\sin 2\theta} = 4\cos^2 2\theta - 1 = (\zeta + \zeta^6)^2 - 1 = 1 + \zeta^2 + \zeta^5 \\ -\frac{b}{c} &= \frac{\sin 9\theta}{\sin 3\theta} = 4\cos^2 3\theta - 1 = 4\cos^2 4\theta - 1 = (\zeta^2 + \zeta^5)^2 - 1 = 1 + \zeta^4 + \zeta^3 \\ \frac{c}{a} &= \frac{\sin 3\theta}{\sin\theta} = 4\cos^2 6\theta - 1 = (\zeta^3 + \zeta^4)^2 - 1 = 1 + \zeta^6 + \zeta\ ,\end{aligned}$$

whence

$$\begin{aligned}\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} &= (3 + 2\zeta^2 + \zeta^3 + \zeta^4 + 2\zeta^5) + (3 + \zeta + 2\zeta^3 + 2\zeta^4 + \zeta^6) + (3 + 2\zeta + \zeta^2 + \zeta^5 + 2\zeta^6) \\ &= 9 + 3(\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6) = 9 - 3 = 6\ .\end{aligned}$$

83. Let $\mathfrak{C}$ be a circle with centre O and radius 1, and let $\mathfrak{F}$ be a closed convex region inside $\mathfrak{C}$. Suppose from each point on $\mathfrak{C}$, we can draw two rays tangent to $\mathfrak{F}$ meeting at an angle of $60°$. Describe $\mathfrak{F}$.

Solution. Let A be an arbitrary point on the circumference of $\mathfrak{C}$. Draw rays AC, AB, BD tangent to $\mathfrak{F}$; let AC and BD intersect at P. Since $\angle CAB = \angle ABD = 60°$, $\angle APM = 60°$ and ΔAPB is equilateral and contains $\mathfrak{F}$. Suppose, if possible, that P lies strictly inside the circle. Let CE be the second ray from C tangent to $\mathfrak{F}$. Then $\angle ACE = 60°$, so CE is parallel to DB and lies strictly on the opposite side of BD to $\mathfrak{F}$; thus, it cannot be tangent to $\mathfrak{F}$ and we have a contradiction. Similarly, if P lies strictly outside the circle, the second ray from C, CE, tangent to $\mathfrak{F}$ is parallel to and distinct from BD. We have that $\mathfrak{F}$ is tangent to AC, BD and CE, an impossibility since DE, within the circle, lies between CE and DB.

Hence $C = D = P$ and ABC is an equilateral triangle containing $\mathfrak{F}$ with all its sides tangent to $\mathfrak{F}$. Since A is arbitrary, $\mathfrak{F}$ is contained within the intersection of all such triangles, namely the circle $\mathfrak{D}$ with centre O and radius $1/2$, and every chord of the given circle tangent to $\mathfrak{F}$ has length $\sqrt{3}$. If $\mathfrak{F}$ were a proper subset of $\mathfrak{D}$, there would be a point Q on the circumference of $\mathfrak{D}$ outside $\mathfrak{F}$. and a tangent of $\mathfrak{F}$ separating $\mathfrak{F}$ from Q. This tangent chord would intersect the interior of $\mathfrak{D}$ and so be longer than $\sqrt{3}$, yielding a contradiction. Hence $\mathfrak{F}$ must be the circle $\mathfrak{D}$.

84. Let ABC be an acute-angled triangle, with a point H inside. Let U, V, W be respectively the reflected image of H with respect to axes BC, AC, AB. Prove that H is the orthocentre of ΔABC if and only if U, V, W lie on the circumcircle of ΔABC,

Solution 1. Suppose that H is the orthocentre of ΔABC. Let P, Q, R be the respective feet of the altitudes from A, B, C. Since BC right bisects HU, $\Delta HBP \equiv \Delta UBP$ and so $\angle HBP = \angle UBP$. Thus

$$\begin{aligned}\angle ACB = \angle QCB &= 90° - \angle QBC = 90° - \angle HBP \\ &= 90° - \angle UBP = \angle PUB = \angle AUB\ ,\end{aligned}$$

so that $ABUC$ is concyclic and U lies on the circumcircle of ΔABC. Similarly V and W lie on the circumcircle.

Now suppose that U, V, W lie on the circumcircle. Let $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ be the respective reflections of the circumcircle about the axes BC, CA, AB. These three circles intersect in the point H. If H' is the orthocentre of the triangle, then by the first part of the solution, the reflective image of H' about the three axes lies on the circumcircle, so that H' belongs to $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ and $H = H'$ or else HH' is a common chord of the three circles. But the latter does not hold, as the common chords AH, BH and CH of pairs of the circles intersect only in H.

Solution 2. Let H be the orthocentre, and P, Q, R the pedal points as defined in the first solution. Since $ARHQ$ is concyclic

$$\angle BAC + \angle BUC = \angle BAC + \angle BHC = \angle RAQ + \angle RHQ = 180^\circ$$

and so $ABUC$ is concyclic. A similar argument holds for V and W.

[A. Lin] Suppose that U, V, W are on the circumcircle. From the reflection about BC, $\angle BCU = \angle BCH$. From the reflections about BA and BC, we see that $BW = BH = BU$, and so, since the equal chords BW and BU subtend equal angles at C, $\angle BCW = \angle BCU$. Hence $\angle BCW = \angle BCH$, with the result that C, H, W are collinear and CW is an altitude. Similarly, AU and BV are altitudes that contain H, and so their point H of intersection must be the orthocentre.

85. Find all pairs (a, b) of positive integers with $a \neq b$ for which the system

$$\cos ax + \cos bx = 0$$

$$a \sin ax + b \sin bx = 0 \ .$$

Solution 1. Suppose that the system is solvable; note that $x = 0$ is not a solution. Then $\cos ax = -\cos bx$ so that $\sin ax = \epsilon \sin bx$ where $\epsilon = \pm 1$. Hence $(a + \epsilon b) \sin bx = 0$. Since a and b are positive and unequal, $a + \epsilon b \neq 0$, so that $\sin bx = 0$. Hence $bx = n\pi$ for some integer n. Also $\sin ax = 0$, so that $ax = m\pi$ for some integer m. Hence, we must have $an = bm$ for some integers m and n. Since $\cos m\pi = -\cos n\pi$, m and n must have opposite parity.

Suppose that $a = 2^u p$ and $b = 2^v q$ with u and v unequal integers and p and q odd. Then $x = 2^{-w}\pi$. where w is the minimum of u and v satisfies the system of equations.

Solution 2. First, observe that $x \neq 0$ for any solution. If the system is satisfied, then

$$\begin{aligned}
0 &= \cos ax + \cos bx = 2 \cos \frac{1}{2}(a+b)x \cos \frac{1}{2}(a-b)x \\
&\Longrightarrow \cos \frac{1}{2}(a+b)x = 0 \qquad \text{or} \qquad \cos \frac{1}{2}(a-b)x = 0 \\
&\Longrightarrow \frac{1}{2}(a+b)x = (2k+1)\frac{\pi}{2} \qquad \text{or} \qquad \frac{1}{2}(a-b)x = (2k+1)\frac{\pi}{2} \\
&\Longrightarrow ax \pm bx = (2k+1)\pi \quad \text{ for some integer } \ k \\
&\Longrightarrow \sin ax = -\sin(\pm bx) = \mp \sin bx \\
&\Longrightarrow 0 = a \sin ax + b \sin bx = (a \mp b) \sin bx \ .
\end{aligned}$$

Since $a \neq b$ and $a + b > 0$, $0 = \sin bx = \sin ax$, so that $ax = m\pi$ and $bx = n\pi$ for some integers m and n. Since $0 = \cos ax + \cos bx = (-1)^m + (-1)^n$, the integers m and n must have different parity. Hence

$$x = \frac{m\pi}{a} = \frac{n\pi}{b}$$

where m and n are integers not both even or both odd. Since $x \neq 0$, $a/b = m/n$, so a/b in lowest terms must have numerator and denominator of different parities.

We now show that, for any pair a, b satisfying this condition, there is a solution. Wolog, let $a = 2uw$ and $v = (2v+1)w$, where the greatest common divisor of $2u$ and $2v+1$ is 1, and w is an arbitrary positive integer. Suppose that $x = \pi/w$. Then

$$\cos ax + \cos bx = \cos 2u\pi + \cos(2v+1)\pi = 1 - 1 = 0$$

and

$$a \sin ax + b \sin bx = a \sin 2u\pi + b \sin(2v+1)\pi = 0 + 0 = 0$$

as desired.

Solution 3. Since $\cos^2 ax = \cos^2 bx$ and $a^2 \sin^2 ax + b^2 \sin^2 bx$, then

$$a^2 \cos^2 bx + b^2 \sin^2 bx = a^2 \Longrightarrow (b^2 - a^2) \sin^2 bx = 0$$

so that $bx = n\pi$ for some integer. Similarly $ax = m\pi$. The solution can be completed as before.

Comment. Note that there are two parts to the solution of this problem, and your write-up should make sure that these are carefully delineated. First, assuming that there is a solution, you derive necessary conditions on a and b that the two equations are consistent. Then, you assume these conditions on a and b, and then display a solution to the two equations. A complete solution requires noting that suitable numbers a and b actually do lead to a solution.

86. Let $ABCD$ be a convex quadrilateral with $AB = AD$ and $CB = CD$. Prove that

(a) it is possible to inscribe a circle in it;

(b) it is possible to circumscribe a circle about it if and only if $AB \perp BC$;

(c) if $AB \perp BC$ and R and r are the respective radii of the circumscribed and inscribed circles, then the distance between the centres of the two circles is equal to the square root of $R^2 + r^2 - r\sqrt{r^2 + 4R^2}$.

Comment. Most students picked up the typo in part (c) in which AC was given instead of the intended BC. I am sorry for the mistake. However, this does happen from time to time even on competitions, and you should be alert. From the context of this problem, the intention was probably pretty clear (in fact, some of you might not have realized that there was an error). The rule in such a situation is that, if you feel that there is an error, make a reasonable *nontrivial* interpretation of the problem, state it clearly and solve it.

Solution 1. (a) Triangles ABC and ADC are congruent (SSS) with the congruence implemented by a reflection in AC. Hence AC bisects angles DAB and DCB. The angle bisectors of $\angle ADB$ and $\angle ABC$ are reflected images and intersect in I, a point on AC. Since I is equidistant from the four sides of the kite $ABCD$, it is the centre of its incircle.

(b) If $AB \perp BC$, then the circle with diameter AC passes through B. By symmetry about AC, it must pass through D as well. Conversely, let $\mathfrak{C}$ be the circumcircle of $ABCD$. The circle goes to itself under reflection in AC, so AC must be a diameter of $\mathfrak{C}$. Hence $\angle ABC = \angle ADC = 90°$.

(c) Let I be the incentre and O the circumcentre of $ABCD$; both lie on AC. Suppose that J and K are the respective feet of the perpendiculars to AB and BC from I, and P and Q the respective feet of the perpendiculars to AB and BC from O. Let $x = |AB|$ and $y = |BC|$. Then

$$\frac{x}{y} = \frac{x - r}{r} \Rightarrow xy = r(x + y) \ .$$

Since $|AO| = |OC| = R$, $4R^2 = x^2 + y^2$. Noting that x and y both exceed r, we have that

$$
\begin{aligned}
(x+y)^2 = x^2+y^2+2xy &= 4R^2+2r(x+y) \\
&\Rightarrow (x+y-r)^2 = r^2+4R^2 \\
&\Rightarrow x+y = r+\sqrt{r^2+4R^2} \ .
\end{aligned}
$$

Now

$$
\begin{aligned}
|IO|^2 &= |JP|^2+|KQ|^2 = (|JB|-\frac{1}{2}|AB|)^2+(|KB|-\frac{1}{2}|BC|)^2 \\
&= 2r^2-r(x+y)+\frac{1}{4}(x^2+y^2) = r^2-r\sqrt{r^2+4R^2}+R^2 \ ,
\end{aligned}
$$

yielding the desired result.

Solution 2. (a) Since triangles ADB and CDB are isosceles, the angle bisectors of A and C right bisect the base BD and so they coincide. The line AC is an axis of reflective symmetry that interchanges B and D, and also interchanges the angle bisectors of B and D. The point P where one of the bisectors intersects the axis AC is fixed by the reflection and so lies on the other bisector. Hence, P is common to all four angle bisectors, and so is equidistant from the four sides of the quadrilateral. Thus, we can inscribe a circle inside $ABCD$ with centre P.

(b) Since AC is a line of symmetry, $\angle B = \angle D$. Note that, $ABCD$ has a circumcircle $\Leftrightarrow$ pairs of opposite angles sum to $180^\circ \Leftrightarrow \angle B+\angle D = 180^\circ \Leftrightarrow \angle B = \angle D = 90^\circ$. This establishes the result.

(c) [R. Barrington Leigh] Let a, b and c be the respective lengths of the segments BC, AC and AB. Let O and I be, respectively, the circumcentre and the incentre for the quadrilateral. Note that both points lie on the diagonal AC. Wolog, we may take $a \geq c$.

We observe that $\angle ABI = 45^\circ$ and that BI is the hypotenuse of an isosceles right triangle with arms of length r. We have, by the Law of Cosines,

$$
\begin{aligned}
d^2 &= R^2+2r^2-2\sqrt{2}Rr\cos\angle IBO \\
&= R^2+2r^2-2\sqrt{2}Rr\left[\cos\left(\angle ABO-\frac{\pi}{4}\right)\right] \\
&= R^2+2r^2-2\sqrt{2}Rr[\cos\angle ABO(1/\sqrt{2})+\sin\angle ABO(1/\sqrt{2})] \\
&= R^2+2r^2-2Rr[\cos\angle ABO+\cos\angle CBO] \\
&= R^2+2r^2-2Rr[\cos\angle BAC+\cos\angle BCA] \\
&= R^2+2r^2-2Rr\left(\frac{a+c}{b}\right) \\
&= R^2+2r^2-r(a+c) \ ,
\end{aligned}
$$

since $b = 2R$ and both triangle AOB and COB are isosceles with arms of length R.

By looking at the area of ΔABC in two ways, we have that $ac = r(a+c)$. Now

$$
(a+c-r)^2 = a^2+c^2+r^2+2[ac-r(a+c)] = a^2+c^2+r^2 = 4R^2+r^2 \ ,
$$

so that $a+c = r+\sqrt{4R^2+r^2}$. (The positive root is selected as a and c both exceed r.) Hence

$$
\begin{aligned}
d^2 &= R^2+2r^2-r[r+\sqrt{4R^2+r^2}] \\
&= R^2+r^2-r\sqrt{r^2+4R^2} \ .
\end{aligned}
$$

87. Prove that, if the real numbers a, b, c, satisfy the equation

$$
\lfloor na \rfloor + \lfloor nb \rfloor = \lfloor nc \rfloor
$$

for each positive integer n, then at least one of a and b is an integer.

Solution. We first show that $a+b=c$. Suppose, if possible, that $c>a+b$. Let $n \geq (c-a-b)^{-1}$. Then

$$\lfloor nc \rfloor > nc - 1 > n(a+b) > \lfloor na \rfloor + \lfloor nb \rfloor$$

yielding a contradiction. Similarly, if $c < a+b$ and $n \geq 2(a+b-c)^{-1}$, then

$$\lfloor nc \rfloor \leq nc \leq na - 1 + nb - 1 < \lfloor na \rfloor + \lfloor nb \rfloor$$

again yielding a contradiction. Hence $a+b=c$.

Let $a = \lfloor a \rfloor + \alpha$, $b = \lfloor b \rfloor + \beta$ and $c = \lfloor c \rfloor + \gamma$. From the condition for $n=1$, we have $\lfloor a \rfloor + \lfloor b \rfloor = \lfloor c \rfloor$. Then, $\lfloor na \rfloor = \lfloor n\lfloor a \rfloor + n\alpha \rfloor = n\lfloor a \rfloor + \lfloor n\alpha \rfloor$ with similar equations for b and c. Putting this together gives that $\lfloor n\alpha \rfloor + \lfloor n\beta \rfloor = \lfloor n\gamma \rfloor$, for all $n \geq 1$. As in the first part of the solution, we have that $\alpha+\beta=\gamma$, from which $\{n\alpha\} + \{n\beta\} = \{n\gamma\}$ for all $n \geq 1$, where $\{x\}$ denotes the fractional part $x - \lfloor x \rfloor$ of x.

We first show that α, β and γ cannot all be positive and rational. For, if they were rational, then for some positive integers i, j, k with $k \geq 2$, we would have $\alpha = i/k$, $\beta = j/k$ and $\gamma = (i+j)/k$. Then $\lfloor k\alpha \rfloor = i = \lfloor (k-1)\alpha \rfloor + 1$, with similar relations for β and γ. Thus,

$$\lfloor k\alpha \rfloor + \lfloor k\beta \rfloor - \lfloor k\gamma \rfloor - [\lfloor (k-1)\alpha \rfloor + \lfloor (k-1)\beta \rfloor - \lfloor (k-1)\gamma \rfloor] = 1$$

so that $\lfloor n\alpha \rfloor + \lfloor n\beta \rfloor = \lfloor n\gamma \rfloor$ must fail for either $n=k$ or $n=k-1$.

Wolog, suppose that all of α, β, γ are positive with α irrational. Let p be a nonnegative integer for which $\alpha + p\beta < 1 \leq \alpha + (p+1)\beta$ and suppose that $\epsilon = 1 - (\alpha + p\beta)$. Since $\alpha + \beta = \gamma < 1$, it follows that $p \geq 1$.

We show that there is a positive integer $m \geq 2$ for which $\alpha + p\beta < \{m\alpha\}$. Let $t = \lceil 1/\epsilon \rceil$ and consider the intervals $[i/t, (i+1)/t)$ where $0 \leq i \leq t-1$. By the Pigeonhole Principle, one of these intervals must contain two of the numbers $\{2\alpha\}, \{4\alpha\}, \cdots, \{2(t+1)\alpha\}$, say $\{q\alpha\}$ and $\{r\alpha\}$ with $q \geq r+2$. Thus, $|\{q\alpha\} - \{r\alpha\}| < 1/t \leq \epsilon$. Since

$$\{q\alpha\} - \{r\alpha\} = (q-r)\alpha - \lfloor q\alpha \rfloor + \lfloor r\alpha \rfloor = (q-r)\alpha \pm I$$

for some integer I, either $\{(q-r)\alpha\} < \epsilon$ or $\{(q-r)\alpha\} > 1-\epsilon$.

In the first case, we can find a positive integer s for which $1 > s\{(q-r)\alpha\} > 1-\epsilon$. Since

$$s(q-r)\alpha = s\lfloor (q-r)\alpha \rfloor + s\{(q-r)\alpha\} \ ,$$

it follows that

$$\{s(q-r)\alpha\} = s\{(q-r)\alpha\} > 1-\epsilon \ .$$

Thus, in either case, we can find $m \geq 2$ for which $\alpha + p\beta = 1 - \epsilon < \{m\alpha\}$.

Now, $\{m\alpha\} > \alpha$,

$$\{m\gamma\} = \{m\alpha\} + \{m\beta\} \geq \{m\alpha\} > \alpha + p\beta \geq \alpha + \beta = \gamma$$

and

$$\{m\beta\} = \{m\gamma\} - \{m\alpha\} < 1 - (\alpha + p\beta) = 1 - [\alpha + (p+1)\beta - \beta] \leq 1 - (1-\beta) = \beta \ .$$

Hence,

$$\lfloor m\alpha \rfloor = m\alpha - \{m\alpha\} = (m-1)\alpha - (\{m\alpha\} - \alpha) < (m-1)\alpha$$

so that $\lfloor (m-1) \rfloor = \lfloor m\alpha \rfloor$;

$$\lfloor m\gamma \rfloor = m\gamma - \{m\gamma\} = (m-1)\gamma - (\{m\gamma\} - \gamma) < (m-1)\gamma$$

so that $\lfloor (m-1)\gamma \rfloor = \lfloor m\gamma \rfloor$; and $\lfloor m\beta \rfloor = m\beta - \{m\beta\} > (m-1)\beta \geq \lfloor (m-1)\beta \rfloor$, so that $\lfloor (m-1)\beta \rfloor + 1 = \lfloor m\beta \rfloor$. Hence $\lfloor n\alpha \rfloor + \lfloor n\beta \rfloor = \lfloor n\gamma \rfloor$ must fail for either $n = m$ or $n = m - 1$.

The only remaining possibility is that either α or β vanishes, *i.e.*, that a or b is an integer. This possibility is feasible when the other two variables have the same fractional part.

88. Let I be a real interval of length $1/n$. Prove that I contains no more than $\frac{1}{2}(n+1)$ irreducible fractions of the form p/q with p and q positive integers, $1 \leq q \leq n$ and the greatest common divisor of p and q equal to 1.

Comment. The statement of the problem needs a slight correction. The result does not apply for closed intervals of length $1/n$ whose endpoints are consecutive fractions with denominator n. The interval $[1/3, 2/3]$ is a counterexample. So we need to strengthen the hypothesis to exclude this case, say by requiring that the interval be open (*i.e.*, not include its endpoints), or by supposing that not both endpoints are rational. Alternatively, we could change the bound to $\frac{1}{2}(n+3)$ and ask under what circumstances this bound is achieved.

Solution 1. We first establish a lemma: *Let $1 \leq q \leq n$. Then there exists a positive integer m for which $\frac{1}{2}(n+1) \leq mq \leq n$.* For, let $m = \lfloor n/q \rfloor$. If $q > n/2$, then $m = 1$ and clearly $\frac{1}{2}(n+1) \leq mq = q \leq n$. If $q \leq n/2$, then $(n/q) - 1 < m \leq (n/q)$, so that

$$\frac{n}{2} \leq n - q < mq \leq n$$

and $\frac{1}{2}(n+1) \leq mq \leq n$.

Let p/q and p'/q' be two irreducible fractions in I with m and m' corresponding integers as determined by the lemma. Suppose, if possible, that $mq = m'q'$. Then

$$\left| \frac{p}{q} - \frac{p'}{q'} \right| = \left| \frac{mp}{mq} - \frac{m'p'}{mq} \right| \geq \frac{1}{mq} \geq \frac{1}{n} ,$$

contradicting the fact that no two fractions in I can be distant at least $\frac{1}{n}$.

It follows that the mapping $p/q \to mq$ from the set of irreducible fractions in I into the set of integers in the interval $[(n+1)/2, n]$ is one-one. But the latter set has at most $n - ((n+1)/2) + 1 = (n+1)/2$ elements, and the result follows.

Solution 2. [M. Zaharia] For $1 \leq i \leq \frac{1}{2}(n+1)$, define

$$S_i = \{2^j(2i-1) : j = 0, 1, 2, \cdots\} .$$

(Thus, $S_1 = \{1, 2, 4, 8, \cdots\}$, $S_3 = \{3, 6, 12, 24, \cdots\}$ and $S_5 = \{5, 10, 20, 40, \cdots\}$, for example.) We show that each S_i contains at most one denominator not exceeding n among the irreducible fractions in I. For suppose

$$\frac{a}{2^u(2i-1)} \quad \text{and} \quad \frac{b}{2^v(2i-1)}$$

are distinct irreducible fractions in I, with $u \leq v$. Then

$$\left| \frac{a}{2^u(2i-1)} - \frac{b}{2^v(2i-1)} \right| = \left| \frac{2^{v-u}a - b}{2^v(2i-1)} \right| \geq \frac{1}{2^v(2i-1)} \geq \frac{1}{n} .$$

But I cannot contain two fractions separated by a distance of $1/n$ or larger. Thus, we get a contradiction, and it follows that there cannot be more than one fraction with a denominator in each of the at most $(n+1)/2$ sets S_i. The result follows.

89. Prove that there is only one triple of positive integers, each exceeding 1, for which the product of any two of the numbers plus one is divisible by the third.

Solution 1. Let a, b, c be three numbers with the desired property; wolog, suppose that $2 \leq a \leq b \leq c$. Since $a|(bc+1)$, a has greatest common divisor 1 with each of b and c. Similarly, the greatest common divisor of b and c is 1. Since $ab+bc+ca+1$ is a multiple of each of a, b, c, it follows that $ab+bc+ca+1$ is a multiple of abc. Therefore, $abc \leq ab+bc+ca+1$.

Since a, b, c are distinct and so $2 \leq a < b < c$, we must have $a \geq 2$ and $b \geq 3$. Suppose, if possible that $b \geq 4$, so that $c \geq 5$. Then $abc \geq 40$ and

$$ab+bc+ca+1 \leq \frac{abc}{5} + \frac{abc}{2} + \frac{abc}{4} + 1 \leq \frac{19abc}{20} + \frac{abc}{40} < abc \ ,$$

a contradiction. Hence b must equal 3 and a must equal 2. Since $c|(ab+1)$, c must equal 7. The triple $(a,b,c) = (2,3,7)$ satisfies the desired condition and is the only triple that does so.

Solution 2. As in Solution 1, we show that $abc \leq ab+bc+ca+1$, so that

$$1 \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} \ .$$

However, if $a \geq 3$, the right ride of the inequality cannot exceed $(1/3)+(1/4)+(1/5)+(1/60) = 4/5 < 1$. Hence $a = 2$. If $a = 2$ and $b \geq 4$, then $b \geq 5$ (why?) and the right side cannot exceed $(1/2)+(1/5)+(1/7)+(1/70) = 6/7 < 1$. Hence $(a,b) = (2,3)$. If now c exceeds 7, then $c \geq 11$ and the right side of the inequality cannot exceed $(1/2)+(1/3)+(1/11)+(1/66) = 31/33 < 1$. Hence $c \leq 7$, and we are now led to the solution.

Solution 3. [P. Du] As in Solution 1, we show that a, b, c are pairwise coprime and that $ab+bc+ca+1$ is a multiple of abc. Assume $2 \leq a < b < c$. Then $abc > ac$ and $bc(a-1) \geq bc > ac \geq a(b+1) > ab+1$, whence, adding these, we get $2abc - bc > ac+ab+1$, so that $2abc > ab+bc+ca+1$. Hence,

$$abc = ab+bc+ca+1 = bc-(c-a)b+bc+bc-(b-a)c+1 = 3bc-(c-a)b-(c-b)a+1 < 3bc$$

so that $a < 3$. Hence $a = 2$. Plugging this into the equation yields

$$bc = 2b+2c+1 = 4c-2(c-b)+1 < 4c$$

so that $b < 4$. Hence $b = 3$, and we find that $6c = 6+5c+1$ or $c = 7$.

Solution 4. [O. Ivrii] As before, we show that a, b and c are pairwise coprime, and take $2 \leq a < b < c$. Then $bc|ab+ac+1$. Since $bc > ac+1$ and $bc > ab+1$, we have that $2bc > ab+ac+1$. Hence $bc = ab+ac+1$, so that $(bc+1)+ab = 2(ab+1)+ac$. Since a divides each of $bc+1$, ab and ac, but is coprime with $ab+1$, it follows that a divides 2. Hence $a = 2$ and

$$bc = 2(b+c)+1 \Longrightarrow (b-2)(c-2) = 5 \Longrightarrow (b,c) = (3,7) \ .$$

Solution 5. As above, we can take $2 \leq a < b < c$. Since

$$\frac{(ab+1)}{c} \cdot \frac{(ca+1)}{b} = a^2 + \frac{a}{c} + \frac{a}{b} + \frac{1}{bc} \ ,$$

we see that $(a/c)+(a/b)+(1/(bc))$ is a positive integer less than 3.

Suppose, if possible, that $(a/c)+(a/b)+(1/(bc)) = 2$. Then $ab+ac+1 = 2bc$, whence $b(c-a)+c(b-a) = 1$, an impossibility. Hence $a(b+c)+1 = bc$, so that

$$2 = (bc+1) - a(b+c) \ .$$

Since both terms on the right are divisible by a, 2 must be a multiple of a. Hence $a = 2$, and we obtain $(b-2)(c-2) = 5$, so that $(b, c) = (3, 7)$.

90. Let m be a positive integer, and let $f(m)$ be the smallest value of n for which the following statement is true:

    given any set of n integers, it is always possible to find a subset of m integers whose sum is divisible by m

    Determine $f(m)$.

Solution. [N. Sato] The value of $f(m)$ is $2m-1$. The set of $2m-2$ numbers consisting of $m-1$ zeros and $m-1$ ones does not satisfy the property; from this we can see that n cannot be less than $2m-1$.

We first establish that, if $f(u) = 2u-1$ and $f(v) = 2v-1$, then $f(uv) = 2uv-1$. Suppose that $2uv-1$ numbers are given. Select any $2u-1$ at random. By hypothesis, there exists a $u-$subset whose sum is divisible by u; remove these u elements. Continue removing $u-$subsets in this manner until there are fewer than u numbers remaining. Since $2uv-1 = (2v-1)u + (u-1)$, we will have $2v-1$ sets of u numbers summing to a multiple of u. For $1 \leq i \leq 2v-1$, let ua_i be the sum of the ith of these $2v-1$ sets. We can choose exactly v of the a_i whose sum is divisible by v. The v $u-$sets corresponding to these form the desired uv elements whose sum is divisible by uv. Thus, if we can show that $f(p) = 2p-1$ for each prime p, we can use the fact that each number is a product of primes to show that $f(m) = 2m-1$ for each positive integer m.

Let $x_1, x_2, \cdots, x_{2p-1}$ be $2p-1$ integers. Wolog, we can assume that the x_i have been reduced to their least non-negative residue modulo p and that they are in increasing order. For $1 \leq i \leq p-1$, let $y_i = x_{p+i} - x_i$; we have that $y_i \geq 0$. If $y_i = 0$ for some i, then $x_{i+1} = \cdots = x_{p+i}$, in which case $x_{i+1} + \cdots + x_{p+i}$ is a multiple of p and we have achieved our goal. Henceforth, assume that $y_i > 0$ for all i

Let $s = x_1 + x_2 + \cdots + x_p$. Replacing x_i by x_{p+i} in this sum is equivalent to adding y_i. We wish to show that there is a set of the y_i whose sum is congruent to $-s$ modulo p; this would indicate which of the first p x_i to replace to get a sum which is a multiple of p.

Suppose that $A_0 = \{0\}$, and, for $k \geq 1$, that A_k is the set of distinct numbers i with $0 \leq i \leq p-1$ which either lie in A_{k-1} or are congruent to $a + y_k$ for some a in A_{k-1}. Note that the elements of each A_k is equal to 0 or congruent (modulo p) to a sum of distinct y_i. We claim that the number of elements in A_k must increase by at least one for every k until A_k is equal to $\{0, 1, \cdots, p-1\}$.

Suppose that going from A_{j-1} to A_j yields no new elements. Since $0 \in A_{j-1}$, $y_j \in A_j$, which means that $y_j \in A_{j-1}$. Then $2y_j = y_j + y_j \in A_j = A_{j-1}$, $3y_j = 2y_j + y_j \in A_j = A_{j-1}$, and so on. Thus, all multiples of y_j (modulo p) are in A_{j-1}. As p is prime, we find that A_{j-1} must contain $\{0, 1, \cdots, p-1\}$. We deduce that some sum of the y_i is congruent to $-s$ modulo p and obtain the desired result.

Comment. This turned out to be a hard problem. Here is a partial solution that was originally obtained. Let m be any positive integer. Note that the any set of $2(m-1)$ integers with $m-1$ of them congruent to 0, modulo m, and the other $m-1$ of them congruent to 1, modulo m, does not have a subset of m integers whose sum is divisible by m. Therefore, $f(m) \geq 2m-1$.

Let a and b be two positive integers. We establish that

$$f(ab) \leq f(a) + (f(b)-1)a \quad . \tag{1}$$

Suppose that we have any set of at least $f(a) + (f(b)-1)a$ integers. Since the number of elements in the set exceeds $f(a)$, we can find a set T_1 of a integers whose sum is divisible by a; let this sum be as_1. Remove these a integers from the original set to leave $f(a) + (f(b)-2)a$ integers. We can find a new set T_2 of a integers whose sum is of the form as_2 for some integer s_2. After doing this $f(b)-1$ times, we are left with a residue of at least $f(a)$ integers and $f(b)-1$ sets T_i $(1 \leq i \leq f(b)-1)$ whose elements add up to as_i

respectively. Finally, we can find a set $T_{f(b}$ of a integers from the final $f(a)$ numbers for which the sum is $as_{f(b)}$ for some integer $s_{f(b}$.

Now consider the set

$$\{s_1, s_2, \cdots, s_{f(b)}\} \ .$$

By the definition of $f(b)$, we can find b of them whose sum is divisible by b, say s_j where j belongs to a set U. Then $\sum\{as_j : j \in U\}$ is divisible by ab. There are b summands and each of them is a sum of a elements of our original set, so we there are ab numbers in the original set whose sum is divisible by ab.

Let $W = \{m; f(m) = 2m-1\}$. The set W is nonvoid, as it contains the number 1. Also, the product of any two numbers in W also lies inside W. For suppose that $f(a) = 2a-1$ and $f(b) = 2b-1$. Then

$$f(ab) \leq f(a) + (f(b) - 1)a = (2a - 1) + (2b - 2)a = 2ab - 1 \ .$$

But, we know that $f(ab) \geq 2ab - 1$, from our initial observation. Hence $f(ab) = 2ab - 1$. Thus, to complete the problem and show that $f(m) = 2m - 1$ for all m, we just have to deal with the case that m is prime.

91. A square and a regular pentagon are inscribed in a circle. The nine vertices are all distinct and divide the circumference into nine arcs. Prove that at least one of them does not exceed 1/40 of the circumference of the circle.

Solution 1. Let the four points of the square be at A, B, C, D. We can partition the circumference into eight arcs as follows: (1) four closed arcs (containing endpoints), each of length 1/20 of the circumference centered at A, B, C and D; call these the *shorter arcs*: (2) four open arcs (not containing endpoints), each of length 1/5 of the circumference and interpolated between two of the arcs centred at the vertices of the square; call these the *longer arcs*.

Suppose a regular pentagon is inscribed in a circle. Since any pair of adjacent vertices terminate a closed arc whose length is 1/5 of the circumference, no two vertices of the pentagon can belong to the same longer arc. Since there are only four longer arcs, at least one vertex of the pentagon must lie inside a shorter arc and so be no more distant that $\frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$ from a vertex of the square.

Solution 2. By the Pigeonhole Principle, two vertices of the pentagon must be on the arc joining two adjacent vertices of the square. Let the two vertices of the square be A and D and let the two vertices of the pentagon lying between them be B and C, If the order is $ABCD$, then arc $AB+$ arc $CD = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$ of the circumference. It is not possible for both the arcs AB and CD to exceed 1/40 of the circumference and the result follows.

92. Consider the sequence 200125, 2000125, 20000125, $\cdots$, $200\cdots00125$, $\cdots$ (in which the nth number has $n+1$ digits equal to zero). Prove that none of these numbers is the square, cube or fifth power of an integer.

Solution 1. Each number has the form $2 \times 10^{n+1} + 125$ where $n \geq 4$. Since both 10^n and 120 are multiples of 8, each number in the sequence is congruent to 5, modulo 8, and so cannot be square.

Observe that $10^{n+1} = 10^{n-2} \times 8 \times 125$, so that

$$2 \times 10^{n+1} + 125 = 125(16 \times 10^{n-2} + 1) = 5^3(16 \times 10^{n-2} + 1) \ .$$

If such a number is a kth power, it must be divisible by 5^k. Since $16 \times 10^{n-2} + 1$ is not a multiple of 5, no number in the sequence can be a kth power for $k \geq 4$.

Let $u_n = 16 \times 10^{n-2} + 1$, for $n \geq 5$. Suppose that $(10x_n + 1)^3 = u_n$. Then

$$1000x_n^3 + 300x_n^2 + 30x_n = 16 \times 10^{n-2}$$

so that $x_n = 10^{n-3}y_n$ for some y_n and we find that

$$16 = 10^{3n-6}y_n^3 + 3 \times 10^{2n-4}y_n^2 + 3y_n \geq y_n^2 + 3y_n$$

so that $y_n = 1$ or 2. It is straightfoward to check that neither works, so that u_n can never be a cube. Hence no number in the given sequence can be a cube.

Solution 2. The nth number is equal to $5^3 \times 16 \times 10^{n-1} + 1$, from which it is seen that 5 divides it to an odd power. Hence it cannot be square.

Suppose the number is the cube of some natural number k. Then $k = 10b + 5$ for some natural number b. Hence

$$(10b + 5)^3 = 2 \times 10^{n+1} + 125 \ .$$

This reduces to $b(4b^2 + 6b + 3) = 2^i \cdot 5^j$ for some natural numbers i and j exceeding 4. Since the factor in brackets on the left is odd, 2^i divides b. Checking, modulo 5, we see that the factor in brackets is never divisible by 5, so 5^j must divide b. Hence $4b^2 + 6b + 3 = \pm 1$, which is impossible, since b is a natural number. The desired result follows.

As before, it is straightforward by checking divisibility by 5 that the numbers cannot be higher powers.

93. For any natural number n, prove the following inequalities:

$$2^{(n-1)/(2^n-2)} \leq \sqrt{2}\sqrt[4]{4}\sqrt[8]{8}\cdots \sqrt[2^n]{2^n} < 4 \ .$$

Solution. The middle member of the inequality is 2 raised to the power $s \equiv \frac{1}{2} + \frac{2}{4} + \cdots + \frac{n}{2^n}$. Note that

$$s - \frac{1}{2}s = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} - \frac{n}{2^{n+1}} = 1 - \frac{1}{2^n} - \frac{n}{2^{n+1}} = 1 - \frac{n+2}{2^{n+1}}$$

whence $s = 2 - (n+2)2^{-n}$. Clearly, $s < 2$.

On the other hand,

$$s - \frac{n-1}{2^{n-2}} = \frac{2^{n+1} - (n+2) - 4(n-1)}{2^n} = \frac{2^{n+1} - (5n-2)}{2^n} \ .$$

When $n = 1$, $2^{n+1} - (5n-2) = 1 > 0$; when $n = 2$, $2^{n+1} - (5n-2) = 0$, and when $n = 3$, $2^{n+1} - (5n-2) = 3 > 0$. Suppose, as an induction hypothesis, that $2^{k+1} > 5k - 2$ for some $k \geq 3$. Then

$$2^{k+2} > 10k - 4 = 5(k+1) - 2 + (5k - 7) > 5(k+1) - 2 \ ,$$

so that, for each positive integer n, $2^{n+1} \geq 5n - 2$, with equality if and only if $n = 2$. The desired result follows.

Comment. Solvers approached the proof of $2^{n+1} \geq 5n - 2$ in two ways. Some used induction. The inequality holds for $n = 1, 2, 3$. Suppose that it holds for $n = k > 2$, Then, using this fact, we find that

$$2^{k+1} = 2^{k+1} + 2^{k+1} \geq 5k - 2 + 5 = 5(k+1) - 2 \ ,$$

from which a proof by induction can be constructed.

Another approach is to note that both sides are equal for $n = 2$ and then argue that the left increases more rapidly than the right with increase of n. One might note, for example, that for each $n \geq 2$, $5(n+1) - 2 < 2(5n-2)$.

94. ABC is a right triangle with arms a and b and hypotenuse $c = |AB|$; the area of the triangle is s square units and its perimeter is $2p$ units. The numbers a, b and c are positive integers. Prove that s and p are also positive integers and that s is a multiple of p.

Solution. Since $a^2 + b^2 = c^2$ and squares are congruent to 0 or 1, modulo 4, it is not possible for a and b to be both odd. Since, at least one of them is even, $s = \frac{1}{2}ab$ is an integer. Also, since either two or none of a, b, c are odd, $p = \frac{1}{2}(a+b+c)$ is an integer. Let r be the inradius of the triangle, so that $s = rp$. It remains to find r and show that it is an integer. In any triangle with sides a, b, c, the length of the tangents from vector to incircle are $\frac{1}{2}(a+b-c)$, $\frac{1}{2}(b+c-a)$ and $\frac{1}{2}(b+c-a)$. For a right triangle with hypotenuse c, one of the lengths is r, namely $r = \frac{1}{2}(a+b-c)$. Since one of the legs of the triangle is even, the other leg and the hypotenuse must have the same parity, so that $a+b-c$ is even and r is an integer.

Comment. T. Chao noted that it suffices to show the result for primitive right triangles, for which $(a,b,c) = (m^2-n^2, 2mn, m^2+n^2)$ for some coprime pair (m,n) of integers. (Prove this.) Then, it turns out that $s = (m^2-n^2)mn$ and $p = m(m+n)$, from which the result follows. A lot of students used the representation in their proofs without proving it. Since this is not part of the regular curriculum, it is prudent to provide a proof. There were several solutions to this problem in which this "well-known" fact was improperly used and yielded a particular rather than a general solution. Some studenets thought that the above form gave a general representation of *all* pythagorean triples; but $(15, 36, 39)$ for example cannot be so represented. How should the form be modified to embrace such examples?

95. The triangle ABC is isosceles is isosceles with equal sides AC and BC. Two of its angles measure $40°$. The interior point M is such that $\angle MAB = 10°$ and $\angle MBA = 20°$. Determine the measure of $\angle CMB$.

Solution 1. Let BM be produced to meet AC at N. Since $\angle AMB = 150°$, we have that $\angle NMA = 30° = \angle NAM$ so that $NA = NM$. Let $a = |BC| = |AC|$, $c = |AB|$, $u = |CN|$ and $v = |NA| = |NM|$. Since BN bisects angle CBA, we have that $v/u = c/a$.

By the Law of Sines,

$$\frac{c}{a} = \frac{\sin 100°}{\sin 40°} = \frac{\sin 80°}{\sin 40°} = 2\cos 40°$$

and

$$\frac{v}{u} = \frac{\sin(\alpha - 60°)}{\sin\alpha}$$

where $\alpha = \angle CMB$. Hence

$$\sin(\alpha - 60°) = 2\sin\alpha\cos 40° = \sin(\alpha + 40°) + \sin(\alpha - 40°)$$

whence

$$2\cos(\alpha - 50°)\sin 10° = \sin(\alpha - 60°) - \sin(\alpha - 40°) = \sin(\alpha + 40^{^circ}) \ .$$

Since $\alpha + 40° = (\alpha - 50°) + 90°$, $\sin(\alpha + 40°) = \cos(\alpha - 50°)$. Therefore, we find that $2\cos(\alpha - 50°)\sin 10° = \cos(\alpha - 50°)$. Since $\sin 10° \neq \frac{1}{2}$, we have that $\cos(\alpha - 50°) = 0$ and $\alpha = 140°$.

Solution 2. [R. Mong] Let A' be on BC produced so that $A'B = AB$. The triangle $A'BM$ and ABM are congruent (SAS), so $A'M = AM$, and $\angle BA'M = \angle BAM = 10°$. Triangle $A'AB$ is isosceles, with vertex angle $B = 40°$, so $\angle A'AB = \angle AA'B = (180° - 40°)/2 = 70°$. Then $\angle MA'A = \angle BA'A - \angle BA'M = 60°$. Similarly, $\angle MAA' = 60°$. Therefore, the triangle $MA'A$ is equilateral.

$\angle CAM = \angle CAB - \angle MAB = 40° - 10° = 30°$, so CA is the angle bisector in the triangle $MA'A$. It follows that CA is the perpendicular bisector of MA', and $A'C = CM$. Applying the Exterior Angle Theorem for the triangle $C'AM$, $\angle MCB = 2\angle MA'B = 20°$, and $\angle CMB = 180° - \angle MBC - \angle MCB = 180° - 20° - 20° = 140°$.

96. Find all prime numbers p for which all three of the numbers $p^2 - 2$, $2p^2 - 1$ and $3p^2 + 4$ are also prime.

Solution. Modulo 7, we find that $p^2 - 2 \equiv 0$ when $p \equiv 3, 4$, $2p^2 - 1 \equiv 0$ when $p \equiv 2, 5$ and $3p^2 + 4 \equiv 0$ when $p \equiv 1, 6$. Thus, when $p > 7$, at least one of the four numbers is a proper multiple of 7. The only primes p for which all numbers are prime at 3 and 7, and we get the quadruples $(3, 7, 17, 31)$ and $(7, 47, 97, 151)$.

Notes. A *rectangular hyperbola* is an hyperbola whose asymptotes are at right angles.

97. A triangle has its three vertices on a rectangular hyperbola. Prove that its orthocentre also lies on the hyperbola.

Solution 1. A rectangular hyperbola can be represented as the locus of the equation $xy = 1$. Let the three vertices of the triangle be at $(a, 1/a)$, $(b, 1/b)$, $(c, 1/c)$. The altitude to the points $(c, 1/c)$ has slope $-(a-b)/(a^{-1}-b^{-1}) = ab$ and its equation is $y = abx + (1/c) - abc$. The altitude to the point $(a, 1/a)$ has equation $y = bcx + (1/a) - abc$. These two lines intersect in the point $(-1/abc, -abc)$ and the result follows.

Solution 2. [R. Barrington Leigh] Suppose that the equation of the rectangular hyperbola is $xy = 1$. Let the three vertices be at (x_i, y_i) $(i = 1, 2, 3)$, and let the orthocentre be at (x_0, y_0). Then

$$(x_1 - x_2)(x_0 - x_3) = -(y_1 - y_2)(y_0 - y_3)$$

and

$$(x_1 - x_3)(x_0 - x_2) = -(y_1 - y_3)(y_0 - y_2) \ .$$

Cross-multiplying these equations yields that

$$(x_1 - x_2)(y_1 - y_3)(x_0 - x_3)(y_0 - y_2) = (x_1 - x_3)(y_1 - y_2)(x_0 - x_2)(y_0 - y_3) \ ,$$

whence

$$(1 - x_1y_3 - x_2y_1 + x_2y_3)(x_0y_0 - x_0y_2 - x_3y_0 + x_3y_2) = (1 - x_1y_2 - x_3y_1 + x_3y_2)(x_0y_0 - x_0y_3 - x_2y_0 + x_2y_3) \ .$$

Collecting up the terms in x_0y_0, x_0, y_0, and the rest, and simplifying, yields that $x_0y_0 = 1$, as desired.

98. Let $a_1, a_2, \cdots, a_{n+1}, b_1, b_2, \cdots, b_n$ be nonnegative real numbers for which
(i) $a_1 \geq a_2 \geq \cdots \geq a_{n+1} = 0$,
(ii) $0 \leq b_k \leq 1$ for $k = 1, 2, \cdots, n$.
Suppose that $m = \lfloor b_1 + b_2 + \cdots + b_n \rfloor + 1$. Prove that

$$\sum_{k=1}^{n} a_k b_k \leq \sum_{k=1}^{m} a_k \ .$$

Solution. Note that $m - 1 \leq b_1 + b_2 + \cdots + b_m < m$. We have that

$$\begin{aligned}
a_1b_1 + a_2b_2 + \cdots + a_mb_m + a_{m+1}b_{m+1} + \cdots + a_nb_n & \\
&\leq a_1b_1 + a_2b_2 + \cdots + a_mb_m + a_m(b_{m+1} + b_{m+2} + \cdots + b_n) \\
&< a_1b_1 + a_2b_2 + \cdots + a_mb_m + a_m(m - b_1 - b_2 - \cdots - b_m) \\
&= a_1b_1 + a_2b_2 + \cdots + a_mb_m + a_m(1 - b_1) + a_m(1 - b_2) + \cdots + a_m(1 - b_m) \\
&\leq a_1b_1 + a_2b_2 + \cdots + a_mb_m + a_1(1 - b_1) + a_2(1 - b_2) + \cdots + a_m(1 - b_m) \\
&= a_1 + a_2 + \cdots + a_m \quad .
\end{aligned}$$

99. Let E and F be respective points on sides AB and BC of a triangle ABC for which $AE = CF$. The circle passing through the points B, C, E and the circle passing through the points A, B, F intersect at B and D. Prove that BD is the bisector of angle ABC.

Solution 1. Because of the concyclic quadrilaterals, $\angle DEA = 180^\circ - \angle BED = \angle DCF$ and $\angle DFC = 180^\circ - \angle DFB = \angle DAB$. Since, also, $AE = CF$, $\Delta DAE \equiv \Delta DFC$ (ASA) so that $AD = DF$. In the circle through $ABFD$, the equal chords AD and DF subtend equal angles ABD and FBD at the circumference. The result follows.

Solution 2. $\angle CDF = \angle CDE - \angle FDE = 180^\circ - \angle ABC - \angle FDE = \angle FDA - \angle FDE = \angle EDA$ and $\angle AED = 180^\circ - \angle BED = \angle BCD = \angle FCD$. Since $AE = CF$, $\Delta EAD \equiv \Delta CFD$ (ASA). The altitude from D to AE is equal to the altitude from D to FC, and so D must be on the bisector of $\angle ABC$.

Solution 3. Let B be the point $(0, -1)$ and D the point $(0, 1)$. The centres of both circles are on the right bisector of BD, namely the $x-$axis. Let the two circles have equations $(x-a)^2 + y^2 = a^2 + 1$ and $(x-b)^2 + y^2 = b^2 + 1$. Suppose that $y = mx - 1$ is a line through B; this line intersects the circle of equation $(x-a)^2 + y^2 = a^2 + 1$ in the point

$$\left(\frac{2(m+a)}{m^2+1}, \frac{m^2+2am-1}{m^2+1}\right)$$

and the circle of equation $(x-b)^2 + y^2 = b^2 + 1$ in the point

$$\left(\frac{2(m+b)}{m^2+1}, \frac{m^2+2bm-1}{m^2+1}\right)$$

The distance between these two points is the square root of

$$\left[\frac{2(a-b)}{m^2+1}\right]^2 + \left[\frac{2m(a-b)}{m^2+1}\right]^2 = \frac{4(a-b)^2(1+m^2)}{(m^2+1)^2} = \frac{4(a-b)^2}{m^2+1} \ .$$

Now suppose that the side AB of the triangle has equation $y = m_1 x - 1$ and the side BC the equation $y = m_2 x - 1$, so that (A, E) and (C, F) are the pairs of points where the lines intersect the circles. Then, from the foregoing paragraph, we must have $m_1^2 + 1 = m_2^2 + 1$ or $0 = (m_1 - m_2)(m_1 + m_2)$. Since the sides are distinct, it follows that $m_1 = -m_2$ and so BD bisects $\angle ABC$.

100. If 10 equally spaced points around a circle are joined consecutively, a convex regular inscribed decagon P is obtained; if every third point is joined, a self-intersecting regular decagon Q is formed. Prove that the difference between the length of a side of Q and the length of a side of P is equal to the radius of the circle. [*With thanks to Ross Honsberger.*]

Solution 1. Let the decagon be $ABCDEFGHIJ$. Let BE and DI intersect at K and let AF and DI intersect at L. Observe that $AB \| DI \| EH$ and $BE \| AF \| HI$, so that $ABKL$ and $KIHE$ are parallelograms. Now AB is a side of P and HE is a side of Q, and the length of the segment IL is the difference of the lengths of $EH = IK$ and $AB = KL$. Since L, being the intersection of the diameters AF and DI, is the centre of the circle, the result follows.

Solution 2. [R. Barrington Leigh] Use the same notation as in Solution 1. Let O be the centre of P. Now, AB is an edge of P, AD is an edge of Q, DO is a radius of the circle and BG a diameter. Let AD and BO intersect at U. Identify in turn the angles $\angle DOU = 72^\circ$, $\angle DAB = 36^\circ$, $\angle ABU = 72^\circ$, $\angle DUO = \angle BUA = 72^\circ$, whence $AU = AB$, $DU = DO$ and $AD - AB = AD - AX = DX = DO$, as desired.

Solution 3. Label the vertices of P as in Solution 1. Let O be the centre of P, and V be a point on EB for which $EV = OE$. We have that $\angle AOB = 36^\circ$, $\angle DOB = \angle OBA = 72^\circ$, $\angle BOE = 108^\circ$ and $\angle OEB = \angle OBE = 36^\circ$. Also, $\angle EOV = \angle EVO = 72^\circ$ and $OE = EV = OA = OB$. Hence, $\Delta DAB = \Delta EVO$ (SAS), so that $OV = AB$. Since $\angle BVO = 108^\circ$ and $\angle BOV = 36^\circ$, $\angle OBV = 36^\circ$, and so $BV = OV = AB$. Hence $BE - AB = EV + BV - AB = EV = OE$, the radius.

Solution 4. Let the circumcircle of P and Q have radius 1. A side of P is the base of an isosceles triangle with equal sides 1 and apex angle 36°, so its length is $2\sin 18^\circ$. Likewise, the length of a side of Q is $2\sin 54^\circ$. The difference between these is

$$2\sin 54^\circ - 2\sin 18^\circ = 2\cos 36^\circ - 2\cos 72^\circ = 2t - 2(2t^2 - 1) = 2 + 2t - 4t^2$$

where $t = \cos 36°$. Now

$$t = \cos 36° = -\cos 144° = 1 - 2\cos^2 72°$$
$$= 1 - 2(2t^2 - 1)^2 = -8t^4 + 8t^2 - 1 ,$$

so that

$$0 = 8t^4 - 8t^2 + t + 1 = (t+1)(8t^3 - 8t^2 + 1)$$
$$= (t+1)(2t-1)(4t^2 - 2t - 1) .$$

Since t is equal to neither -1 nor $\frac{1}{2}$, we must have that $4t^2 - 2t = 1$. Hence

$$2\sin 54° - 2\sin 18° = 2 - (4t^2 - 2t) = 1 ,$$

the radius of the circle.

101. Let a, b, u, v be nonnegative. Suppose that $a^5 + b^5 \leq 1$ and $u^5 + v^5 \leq 1$. Prove that

$$a^2u^3 + b^2v^3 \leq 1 .$$

[*With thanks to Ross Honsberger.*]

Solution. By the arithmetic-geometric means inequality, we have that

$$\frac{2a^5 + 3u^5}{5} = \frac{a^5 + a^5 + u^5 + u^5 + u^5}{5} \geq \sqrt[5]{a^{10}u^{15}} = a^2u^3$$

and, similarly,

$$\frac{2b^5 + 3v^5}{5} \geq b^2v^3 .$$

Adding these two inequalities yields the result.

102. Prove that there exists a tetrahedron $ABCD$, all of whose faces are similar right triangles, each face having acute angles at A and B. Determine which of the edges of the tetrahedron is largest and which is smallest, and find the ratio of their lengths.

Solution 1. Begin with AB, a side of length 1. Now construct a rectangle $ACBD$ with diagonal AB, so that $|AC| = |BD| = s < t = |AD| = |BC|$. The requisite values of s and t will be determined in due course. We want to show that we can fold up D and C from the plane in which AB lies (like folding up the wings of a butterfly) in such a way that we can obtain the desired tetrahedron.

When the triangles ADB and ACB lie flat, we see that C and D are distance 1 apart. Suppose that, when we have folded up C and D to get the required tetrahedron, they are distance r apart. Then ACD should be a right triangle similar to ABC. The hypotenuse of ΔACD cannot be AC as $AC < AD$. Nor can it be CD, for then, we would have $AD = BC$, $AC = AC$, and CD would have to have length 1, possible only when $ABCD$ is coplanar. So the hypotenuse must be AD. The similarity of ΔADC and ΔABC would require that

$$1 : t : s = t : s : r$$

where $r = |CD|$. Thus, $1/t = t/s$ or $s = t^2$ and $t/s = s/r$ or $r = s^2/t = t^3$. So we must fold C and D until they are distance t^3 apart.

Is this possible? Since ΔACB is right, $1 = t^2 + s^2 = t^2 + t^4$, whence $s = t^2 = \frac{1}{2}(-1 + \sqrt{5}) < 1$. Hence $r < 1$. To arrange that we can make the distance between C and D equal to r, we must show that r exceeds the minimum possible distance between C and D, which occurs when ΔADB is folded flat partially covering ΔACB. Suppose this has been done, with $ABCD$ coplanar and C, D both on the same side of AB. Let P and Q be the respective feet of the perpendiculars to AB from C and D. Then

$$|CP| = |DQ| = t^3 , \quad |AP| = |QB| = t^4 , \quad |AQ| = |PB| = t^2 ,$$

and

$$|CD| = |PQ| = t^2 - t^4 = (t^4 + t^6) - t^4 = t^6 < t^3 \ .$$

When C and D are located, we have $|AB| = 1$, $|AD| = |BC| = t$, $|AC| = |BD| = t^2$ and $|CD| = t^3$. Since all faces of the tetrahedron $ABCD$ have sides in the ratio $1 : t : t^2$, all are similar right triangles and $AB : CD = 1 : t^3$.

Solution 2. Let $\alpha = \angle CAB$ and $|AB| = 1$. By the condition on the acute angles of triangles ACB and ACD, $\angle ACB = \angle ADB = 90°$, so that the triangles ΔACD and ΔADB, being similar and sharing a hypotenuse, are congruent.

Suppose, if possible, that $\angle BAD = \alpha$. Then $AC = AD$ and so ΔACD must be isosceles with its right angle at A, contrary to hypothesis. So, $\angle ABD = \alpha$ and $|BD| = |AC| = \cos\alpha$, $|AD| = |BC| = \sin\alpha$.

Consider ΔACD. Suppose that $\angle ACD = 90°$. If $\angle DAC = \alpha$, then $\Delta ABC \equiv \Delta ADC$ and $1 = |AB| = |AD| = \sin\alpha$, yielding a contradiction. Hence $\angle ADC = \alpha$, $|AD| = |AC|/\sin\alpha = \cos\alpha/\sin\alpha$ and $|CD| = |AC|\cot\alpha = \cos^2\alpha/\sin\alpha$. Hence, looking at $|AD|$, we have that

$$\frac{\cos\alpha}{\sin\alpha} = \sin\alpha \Longrightarrow 0 = \cos\alpha - \sin^2\alpha = \cos^2\alpha + \cos\alpha - 1 \ .$$

Therefore, $\cos\alpha = \frac{1}{2}(\sqrt{5} - 1)$ and $\sin^2\alpha = \cos\alpha$.

Observe that $|BC|\sin\alpha = \sin^2\alpha = \cos\alpha = |BD|$ and $|BC|\cos\alpha = \sin\alpha\cos\alpha = \cos^2\alpha/\sin\alpha = |CD|$, so that triangle BCD is right with $\angle CDB = 90°$ and similar to the other three faces.

We need to check that this set-up is feasible. Using spatial coordinates, take

$$C \sim (0, 0, 0) \qquad A \sim (0, \cos\alpha, 0) \qquad B \sim (\sin\alpha, 0, 0) \ .$$

Since $\angle ACD = 90°$, D lies in the plane $y = 0$ and so has coordinates of the form $(x, 0, z)$. Since $\angle CDB = 90°$, $CD \perp DB$, so that

$$0 = (x, 0, z) \cdot (x - \sin\alpha, 0, z) - x^2 + z^2 - x\sin\alpha \ ,$$

Now $|CD| = \cos\alpha\sin\alpha$ forces $\cos^2\alpha\sin^2\alpha = x^2 + z^2$. Hence

$$x\sin\alpha = \cos^2\alpha\sin^2\alpha \Longrightarrow x = \cos^2\alpha\sin\alpha \ .$$

Therefore

$$z^2 = (\cos^2\alpha - \cos^4\alpha)\sin^2\alpha = \cos^2\alpha\sin^4\alpha \Longrightarrow z = \cos\alpha\sin^2\alpha \ ,$$

Hence $D \sim (\cos^2\alpha\sin\alpha, 0, \cos\alpha\sin^2\alpha)$.

Thus, letting $\sin\alpha = t = \frac{1}{2}(\sqrt{5} - 1)$, we have $A \sim (0, t^2, 0)$, $B \sim (t, 0, 0)$, $C \sim (0, 0, 0)$, $D \sim (t^5, 0, t^4)$ with $t^4 + t^2 - 1 = 0$, and $|AB| = 1$, $|AD| = |BC| = t$, $|BD| = |AC| = t^2$ and $|CD| = t^3$. [*Exercise:* Check that the coordinates give the required distances and similar right triangles.] The ratio of largest to smallest edges is $1 : t^3 = 1 : [\frac{1}{2}(\sqrt{5} - 1)]^{3/2} = 1 : \sqrt{2 + \sqrt{5}}$.

We need to dispose of the other possibilities for ΔACD. By the given condition, $\angle DAC \neq 90°$. If $\angle ADC = 90°$, then we have essentially the same situation as before with the roles of α and its complement, and of C and D switched.

Comment. Another way in that was used by several solvers was to note that there are four right angles involved among the four sides, and that at most three angles can occur at a given vertex of the tetrahedron. It is straightforward to argue that it is not possible to have three of the right angles at either C or D. Since all right angles occur at these two vertices, then there must be two at each. As an exercise, you might want to complete the argument from this beginning.

103. Determine a value of the parameter θ so that

$$f(x) \equiv \cos^2 x + \cos^2(x+\theta) - \cos x \cos(x+\theta)$$

is a constant function of x.

Solution 1.

$$\begin{aligned} f(x) &= \cos^2 x + (\cos x \cos\theta - \sin x \sin\theta)^2 - \cos x(\cos x \cos\theta - \sin x \sin\theta) \\ &= \cos^2 x(1+\cos^2\theta - \cos\theta) + (1-\cos^2 x)(\sin^2\theta) - \sin x \cos x \sin\theta(2\cos\theta - 1) \\ &= \sin^2\theta + \cos^2 x(1+\cos^2\theta - \cos\theta - 1 + \cos^2\theta) - \frac{1}{2}\sin 2x \sin\theta(2\cos\theta - 1) \\ &= \sin^2\theta + (2\cos\theta - 1)(\cos^2 x \cos\theta - \sin 2x \sin\theta) \ . \end{aligned}$$

The function $f(x)$ is constant when $2\cos\theta - 1 = 0$, or when $\theta = \pi/3$, and its constant value in this case is $3/4$.

Solution 2.

$$\begin{aligned} f(x) &= \frac{1+\cos 2x}{2} + \frac{1+\cos 2(x+\theta)}{2} - \frac{1}{2}(\cos(2x+\theta) - \cos\theta) \\ &= \frac{1}{2}[2 - \cos\theta + \cos 2x(1+\cos 2\theta - \cos\theta) + \sin 2x(\sin\theta - \sin 2\theta)] \\ &= \frac{1}{2}[2 - \cos\theta + (2\cos\theta - 1)(\cos 2x \cos\theta + \sin 2x \sin\theta] \ . \end{aligned}$$

When $\theta = \pi/3$, $\cos\theta = 1/2$ and the function is the constant $3/4$.

Solution 3. First, note the identity

$$\cos^2 A + \cos^2 B = 1 + \frac{1}{2}(\cos 2A + \cos 2B) = 1 + \cos(A+B)\cos(A-B) \ .$$

Applying this yields

$$\begin{aligned} f(x) &= 1 + \cos(2x+\theta)\cos\theta - \frac{1}{2}(\cos(2x+\theta) + \cos\theta) \\ &= \left(1 - \frac{1}{2}\cos\theta\right) + \cos(2x+\theta)\left[\cos\theta - \frac{1}{2}\right] \ . \end{aligned}$$

Hence, $f(x)$ is the constant $3/4$ when $\theta = \pi/3$.

Solution 4. We have that

$$f(x) = \frac{3}{4}\cos^2 x + [\cos(x+\theta) - \frac{1}{2}\cos x]^2 \ .$$

The function $f(x)$ can be made to take the constant value $3/4$ if we can find a parameter θ for which

$$[\cos(x+\theta) - \frac{1}{2}\cos x]^2 = \frac{3}{4}\sin^2 x$$

for all x. This is equivalent to

$$\cos(x+\theta) = \frac{1}{2}\cos x \pm \frac{\sqrt{3}}{2}\sin x = \cos(x \pm \frac{\pi}{3})$$

for all x. Thus, if $\theta = \pm\pi/3$, then $f(x)$ is constantly equal to $\frac{3}{4}$.

Comment. Some students started by showing that $f(0) = f(\frac{\pi}{2})$ implies that $1 + \cos^2\theta - \cos\theta = \cos^2(\frac{\pi}{2} + \theta) = 1 - \cos^2\theta$, or $0 = 2\cos^2\theta - \cos\theta = \cos\theta(2\cos\theta - 1)$. This tells us that $\theta \equiv \frac{\pi}{2} \pmod{\pi}$ and $\theta \equiv \pm\frac{\pi}{3} \pmod{2\pi}$ are the only possibilities. However, the first of these turns out to be extraneous. It yields $f(x) = 1 \pm \frac{1}{2}\sin 2x$, which is not constant.

104. Prove that there exists exactly one sequence $\{x_n\}$ of positive integers for which

$$x_1 = 1 \ , \qquad x_2 > 1 \ , \qquad x_{n+1}^3 + 1 = x_n x_{n+2}$$

for $n \geq 1$.

Solution. Let $x_2 = u$. Then the first four terms of the sequence are

$$1, \quad u, \quad u^3 + 1, \qquad u^8 + 3u^5 + 3u^2 + (2/u), \cdots$$

so for the whole sequence to consist of positive integers, we must have that $u = 2$. Now for any $n \geq 3$,

$$x_n = \frac{x_{n-1}^3 + 1}{x_{n-2}} = \frac{x_{n-2}^9 + 3x_{n-2}^6 + 3x_{n-2}^3 + 1 + x_{n-3}^3}{x_{n-2}x_{n-3}^3} \tag{1}$$

From the given condition, it can be seen that any consecutive pairs of terms in the sequence, if integers, are coprime. We know that x_1, x_2, x_3 are integers. Let $n \geq 4$. Suppose that it has been shown that x_m is an integer for $1 \leq m \leq n-1$. Then $(x_{n-3}^3 + 1)/x_{n-2} = x_{n-4}$ is an integer, as is $(x_{n-2}^3 + 1)/x_{n-3} = x_{n-1}$ and its cube. Since the numerator of (1) is a multiple of each of x_{n-2} and x_{n-3}^3 separately, and since these two divisors are coprime, x_n must be an integer. The result follows by induction.

Solution 2. [M. Mika] As before, we see that x_2 must be 2. It can be checked that x_3 and x_4 are integers. For any integer $n \geq 4$, we have that

$$\begin{aligned}
x_n^3 + 1 &= \frac{(x_{n-1}^3 + 1)^3}{x_{n-2}^3} + 1 \\
&= \frac{(x_{n-1}^3 + 1)^3}{x_{n-1}x_{n-3} - 1} + 1 \\
&= \frac{x_{n-1}(x_{n-1}^8 + 3x_{n-1}^5 + 3x_{n-1}^2 + x_{n-3})}{x_{n-1}x_{n-3} - 1} \ .
\end{aligned}$$

Therefore

$$(x_n^3 + 1)(x_{n-1}x_{n-3} - 1) = x_{n-1}(x_{n-1}^8 + 3x_{n-1}^5 + 3x_{n-1}^2 + x_{n-3}) \ .$$

Supposing, as an induction hypothesis,that $x_1, \cdots, x_n$ are integers, we see that x_{n-1} and $x_{n-1}x_{n-3} - 1$ are coprime, so we must have that x_{n-1} divides $x_n^3 + 1$. Thus, x_{n+1} is an integer.

105. Prove that within a unit cube, one can place two regular unit tetrahedra that have no common point.

Solution 1. Let $ABCDEFGH$ be the cube, with $ABCD$ the top face, $EFGH$ the lower face and AE, BF, CG, DH edges. Let O be the centre of the cube, and let P, Q, R be the midpoints of AB, DH, FG respectively.

The centre O lies on the diagonal CE, which is the axis of a rotation that takes $B \to D \to G$, $A \to H \to F$, $P \to Q \to R$, so ΔPQR is equilateral with centre O and $CE \perp PQR$.

Using Pythagoras' Theorem, we calculate some lengths:

$$|CP| = |CQ| = |CR| = |RB| = |RH| = \sqrt{1 + (1/4)} = (\sqrt{5})/2 \ ,$$

$$|PQ| = |PR| = |QR| = \sqrt{(5/4) + (1/4)} = \sqrt{3/2} \ ,$$

$$|CO| = (\sqrt{3})/2\ .$$

[As a check that O is the centre of ΔPQR, we can compute $|PO| = |QO| = |RO| = 1/\sqrt{2} = [1/\sqrt{3}]|PQ|$.]

Since the height of a regular tetrahedron with side s is $s\sqrt{2/3}$, we can construct a regular tetrahedron $CUVW$ with apex C, base UVW homothetic to PQR with centre O, height $(\sqrt{3})/2$, and side length $[(\sqrt{3})/2][\sqrt{3/2}] = 3/(2\sqrt{2}) > 1$. Since $3/(2\sqrt{2}) = \sqrt{9/8} < \sqrt{3/2}$, triangle UVW lies within triangle PQR, and so the tetrahedron lies within the cube. Shrink the tetrahedron by a homothety with factor $\sqrt{8/9}$ about its centre to get one of the desired tetrahedra.

The second tetrahedra can be found in a similar way from $EUVW$ (which is congruent to $CUVW$). The two tetrahedra are strictly separated by the plane of PQR.

Solution 2. Let the cube have vertices at the eight points (ϵ, η, ζ) where $\epsilon, \eta, \zeta = 0, 1$. The plane of equation $x + y + z = 3/2$ passes through $(0, 1, \frac{1}{2})$, $(\frac{1}{2}, 1, 0)$, $(1, \frac{1}{2}, 0)$, $(1, 0, \frac{1}{2})$, $(\frac{1}{2}, 0, 1)$ and $(0, \frac{1}{2}, 1)$ at the middle of various edges of the cube, and bisects the cube into two congruent halves. Consider the cube reduced by a homothety of factor $1/\sqrt{2}$ about the origin. Four of its vertices, $(0, 0, 0)$, $(0, 1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, 1/\sqrt{2}, 0)$, $(1/\sqrt{2}, 0, 1/\sqrt{2})$, constitute the four vertices of a regular unit tetrahedron contained in the original cube. Since the sum of the coordinates of all of these points is less than $3/2$, they all lie on the same side of the plane bisecting the cube, as does the whole tetrahedron. Its image reflected in the centre of the cube is a second tetrahedron contained in the upper portion of the cube.

Solution 3. [D. Tseng] Consider unit tetrahedra $CPQR$ and $EUVW$, each sharing a vertex and a face with the cube and directed inwards with the face diagonal AC intersecting PQ in its midpoint S and face diagonal EG intersecting UV in its midpoint X. (Each tetrahedron is carried into the other by a reflection in the centre of the cube.) Planes PQR and UVW are parallel. These tetrahedra intersect the internal plane $ACGE$ in two triangles EXW and CSR. Let SR produced meet EG in M, and let T and N be be points on AC for which $RT \perp AC$ and $MN \perp AC$. Observe that $|RS| = |CS| = \sqrt{3}/2$, $|ST| = 1/(2\sqrt{3})$, $|TC| = 1/\sqrt{3}$ and $|RT| = \sqrt{(2/3)}$. Since $ST : SN = RT : MN$, $|SN| = 1/(2\sqrt{2})$ and $|MG| = |NC| = ((\sqrt{3})/2) - (1/(2\sqrt{2}))$. Hence $|EX| + |MG| = \sqrt{3} - 1/(2\sqrt{2}) < \sqrt{2} = |EG|$. This means that the parallel lines WX and RS have a region of $ACGE$ between them that do not intersect triangles EXW and CSR, and so the two tetrahedra are separated by the slab between the parallel planes PQR and UVW that passes through the centre of the cube.

106. Find all pairs (x, y) of positive real numbers for which the least value of the function

$$f(x, y) = \frac{x^4}{y^4} + \frac{y^4}{x^4} - \frac{x^2}{y^2} - \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x}$$

is attained. Determine that minimum value.

Solution 1. Observe that

$$f(x, y) - 2 = \left(\frac{x^2}{y^2} - 1\right)^2 + \left(\frac{y^2}{x^2} - 1\right)^2 + \left(\frac{x}{y} - \frac{y}{x}\right)^2 + \left(\frac{x}{y} - 2 + \frac{y}{x}\right) \geq \frac{(x-y)^2}{xy} \geq 0$$

with equality if and only if $x = y$. The required minimum is 2.

Solution 2. Let $u = (x^2/y^2) + (y^2/x^2)$ and $v = (x/y) + (y/x)$ Note that $u, v \geq 2$ with equality if and only if $x = y$. Then $f(x, y) = u^2 - u - 2 + v = (u - 2)(u + 1) + v \geq 2$ with equality if and only if $x = y$. The desired minimum is 2.

Solution 3. Let $v = (x/y) + (y/x)$. Then

$$f(x, y) = [v^2 - 2]^2 - 2 - [v^2 - 2] + v = v^4 - 5v^2 + v + 4 = (v - 2)(v^3 + 2v^2 - v - 1) + 2\ .$$

Note that $v \geq 2$ and so $v^3 + 2v^2 - v - 1 = (v^3 - v) + (2v^2 - 1) > 0$. The desired result now follows.

Comment. Several solvers did this problem by calculus. The most important thing you need to know about calculus is when not to use it. Calculus provides very general algorithms for doing optimization problems, and such algorithms often have two undesirable characteristics: (1) they may not provide the quickest and most convenient approach in particular cases; (2) they tend to operate as "black boxes", preventing the solver from appreciating the essence of the problem or the significance of the answer. When you have a problem of this type, you should check to see whether it can be handled without calculus, and use calculus only as a last resort, or at least when it is clear that every other approach is messier.

107. Given positive numbers a_i with $a_1 < a_2 < \cdots < a_n$, for which permutation $(b_1, b_2, \cdots, b_n)$ of these numbers is the product

$$\prod_{i=1}^{n} \left(a_i + \frac{1}{b_i} \right)$$

maximized?

Solution 1. By the arithmetic-geometric means inequality, we have that, for each i, $2a_ib_i \leq a_i^2 + b_i^2$, so that

$$(a_ib_i + 1)^2 = a_i^2b_i^2 + 2a_ib_i + 1 \leq a_i^2b_i^2 + a_i^2 + b_i^2 + 1 = (a_i^2 + 1)(b_i^2 + 1) .$$

Hence

$$\prod_{i=1}^{n}(a_ib_i + 1) \leq \sqrt{\prod_{i=1}^{n}(a_i^2 + 1) \prod_{i=1}^{n}(b_i^2 + 1)} = \prod_{i=1}^{n}(a_i^2 + 1) .$$

Equality occurs if and only if $b_i = a_i$ for each i.

Now

$$\begin{aligned} \prod_{i=1}^{n} \left(a_i + \frac{1}{b_i} \right) &= \frac{\prod_{i=1}^{n}(a_ib_i + 1)}{\prod_{i=1}^{n} b_i} \\ &= \frac{\prod_{i=1}^{n}(a_ib_i + 1)}{\prod_{i=1}^{n} a_i} . \end{aligned}$$

Thus, the given expression is maximized $\Leftrightarrow \prod_{i=1}^{n}(a_ib_i+1)$ is maximized $\Leftrightarrow a_i = b_i$ for each $i \Leftrightarrow (b_1, b_2, \cdots, b_n)$ is obtained from $(a_1, a_2, \cdots, a_n)$ by the identity permutation.

Solution 2. There are finitely many permutations of the numbers, so that there must be a permutation which maximizes the value of the given expression. We show that it is the identity permutation, by showing that, for any other permutation, we can find a permutation that yields a larger value.

Suppose that $(b_1, b_2, \cdots, b_n)$ is a permutation for which there is a pair i, j of indices for which $a_i < a_j$ while $b_i > b_j$. Then

$$\left(a_i + \frac{1}{b_j} \right) \left(a_j + \frac{1}{b_i} \right) - \left(a_i + \frac{1}{b_i} \right) \left(a_j + \frac{1}{b_j} \right) = (a_j - a_i) \left(\frac{1}{b_j} - \frac{1}{b_i} \right) > 0 .$$

with the result that the product can be made larger by interchanging the positions of b_i and b_j. The result follows.

108. Determine all real-valued functions $f(x)$ of a real variable x for which

$$f(xy) = \frac{f(x) + f(y)}{x + y}$$

for all real x and y for which $x + y \neq 0$.

Solution 1. Setting $y = 1$ yields that $(x + 1)f(x) = f(x) + f(1)$ so that $xf(x) = f(1)$ for $x \neq -1$. Set $x = 0$ to obtain $f(1) = 0$, so that, for $x \neq 0$, $(x + 1)f(x) = f(x)$. From this, we deduce that, as long as $x \neq 0, -1$, we have that $f(x) = 0$.

For each nonzero value of x, $xf(0) = f(x) + f(0)$, so that $(x - 1)f(0) = f(x)$. Taking $x = 2$ gives $f(0) = f(2) = 0$. Finally, $2f(-1) = -2f(1) = 0$, so $f(-1) = 0$. Hence, $f(x)$ must be indentically equal to 0.

Solution 2. [S.E. Lu] For all nonzero x, we have that $f(x) = (x - 1)f(0)$. The equality $f(x) + f(y) = (x+y)f(xy)$ leads to either $f(0) = 0$ or $x+y-2 = (xy-1)(x+y)$. The latter simplifies to $(x+y)(2-xy) = 2$ for all nonzero x, y, which is patently false. Hence $f(0) = 0$, so $f(x) \equiv 0$.

Solution 3. Taking $y = 0$ leads to $(x - 1)f(0) = f(x)$ for all $x \neq 0$. Taking $y = 1$ leads to $xf(x) = f(1)$ for all $x \neq -1$. Hence, for $x \neq 0, 1$, we have that $x(x - 1)f(0) = f(1)$. This holds for infinitely many x if and only if $f(0) = f(1) = 0$. It follows that $f(x) = 0$ for all real x.

Comment. Suppose that the given condition is weakened to hold only when *both* x and y are nonzero. Then we get $xf(x) = f(1)$ so that $f(x) = f(1)/x$ for all nonzero x. it can be checked that, for any constant c, $f(x) = c/x$ for is a solution for $x \neq 0$.

109. Suppose that

$$\frac{x^2 + y^2}{x^2 - y^2} + \frac{x^2 - y^2}{x^2 + y^2} = k \ .$$

Find, in terms of k, the value of the expression

$$\frac{x^8 + y^8}{x^8 - y^8} + \frac{x^8 - y^8}{x^8 + y^8} \ .$$

Solution 1. Simplifying, we obtain that

$$k = \frac{2(x^4 + y^4)}{x^4 - y^4} \ ,$$

and, by extension, that

$$\frac{k^2 + 4}{2k} = \frac{k}{2} + \frac{2}{k} = \frac{x^4 + y^4}{x^4 - y^4} + \frac{x^4 - y^4}{x^4 + y^4} = \frac{2(x^8 + y^8)}{x^8 - y^8} \ .$$

Continuing on, we find that

$$\frac{x^8 + y^8}{x^8 - y^8} + \frac{x^8 - y^8}{x^8 + y^8} = \frac{2(x^{16} + y^{16})}{x^{16} - y^{16}} = \frac{k^2 + 4}{4k} + \frac{4k}{k^2 + 4} = \frac{k^4 + 24k^2 + 16}{4k(k^2 + 4)} \ .$$

Comment. R. Barrington Leigh defined a formula

$$R = \frac{a + b}{a - b} + \frac{a - b}{a + b} = 2 \cdot \frac{a^2 + b^2}{a^2 - b^2}$$

for $a \neq \pm b$ which he then applied to $(a, b) = (x^2, y^2), (x^4, y^4)$.

110. Given a triangle ABC with an area of 1. Let $n > 1$ be a natural number. Suppose that M is a point on the side AB with $AB = nAM$, N is a point on the side BC with $BC = nBN$, and Q is a point on the side CA with $CA = nCQ$. Suppose also that $\{T\} = AN \cap CM$, $\{R\} = BQ \cap AN$ and $\{S\} = CM \cap BQ$, where $\cap$ signifies that the singleton is the intersection of the indicated segments. Find the area of the triangle TRS in terms of n.

Solution 1. [R. Furmaniak, Y. Ren] The area of a triangle XYZ will be denoted by $[XYZ]$. Consider the triangle ABQ and the line MC that intersects AB at M, BQ at S and AQ at the external point C. By Menelaus' Theorem for the triangle ABQ and transversal MC,

$$1 = \frac{BM}{MA} \cdot \frac{AC}{QC} \cdot \frac{QS}{SB} = (n-1)n\frac{QS}{SB} \Longrightarrow \frac{SB}{QS} = (n-1)n \ .$$

Observe the triangles BSC and QSC. Since the heights from C to the opposite sides SB and QS coincide, then

$$\frac{[BSC]}{[QSC]} = \frac{SB}{QS} = (n-1)n \Longrightarrow [BSC] = (n-1)n[QSC] \ .$$

Examining triangles QSC and QBC, we similarly find that

$$\frac{[QSC]}{[QBC]} = \frac{QS}{QB} = \frac{QS}{QS+SB} = \frac{1}{(n-1)n+1} = \frac{1}{n^2-n+1}$$

$$\Longrightarrow [QSC] = \frac{1}{n^2-n+1} \cdot [QBC] \ .$$

Since the heights of triangles QBC and ABC from B to QC and AC coincide, it follows that

$$\frac{[QBC]}{[ABC]} = \frac{QC}{AB} = \frac{1}{n} \Longrightarrow [QBC] = \frac{1}{n} \cdot [ABC] = \frac{1}{n}$$

$$\Longrightarrow [QSC] = \frac{1}{n^2-n+1} \cdot \frac{1}{n} = \frac{1}{n(n^2-n+1)}$$

$$\Longrightarrow [BSC] = \frac{(n-1)n}{n(n^2-n+1)} = \frac{n-1}{n^2-n+1} \ .$$

Similarly,

$$[TAC] = [RBA] = \frac{n-1}{n^2-n+1} \ .$$

Now

$$[TSR] = [ABC] - [BSC] - [TAC] - [RBA] = 1 - \frac{3(n-1)}{n^2-n+1} = \frac{(n-2)^2}{n^2-n+1} \ .$$

Solution 2. [M. Butler] Using Menelaus' Theorem with triangle BQC and transversal ARN, we find that

$$\frac{BR}{RQ} \cdot \frac{QA}{AC} \cdot \frac{CN}{NB} = -1$$

so that

$$\frac{BR}{RQ} \cdot \frac{1-n}{n} \cdot \frac{n-1}{1} = -1$$

$$\Longrightarrow BR = \frac{n}{(n-1)^2} \cdot RQ \ .$$

Thus,

$$\begin{aligned} \frac{n-1}{n} &= [ABQ] = [ARB] + [ARQ] \\ &= \left[1 + \frac{(n-1)^2}{n}\right][ARB] \\ &= \left[\frac{n^2-n+1}{n}\right][ARB] \end{aligned}$$

whence

$$[ARB] = \frac{n-1}{n^2-n+1} \ .$$

Therefore

$$[RST] = 1 - \frac{3(n-1)}{n^2-n+1} = \frac{(n-2)^2}{n^2-n+1} \ .$$

Solution 3. Let $a = [AMT]$, $b = [BNR]$, $c = [CQS]$, $x = [BRT]$, $y = [CSR]$, $z = [ATS]$ and $d = [RST]$. Then using $BM = (n-1)AM$, $[BRM] = (n-1)a$,

$$x + d = [BTS] = (n-1)[ATS] = (n-1)z$$

and

$$nb + y = [BRC] + [CSR] = [BSC] = (n-1)[ASC] = (n-1)nc \ .$$

Analogously, from $CN = (n-1)BN$ and $AQ = (n-1)CQ$, we get

$$x + d = (n-1)z, \quad y + d = (n-1)x, \quad z + d = (n-1)y$$

and

$$nb + y = (n^2-n)c, \quad nc + z = (n^2-n)a, \quad na + x = (n^2-n)b \ ,$$

whence

$$x + y + x + 3d = (n-1)(x+y+z) \quad \text{or} \quad 3d = (n-2)(x+y+z)$$

and

$$n(a+b+c) + (x+y+z) = (n^2-n)(a+b+c) \quad \text{or} \quad x+y+z = (n^2-2n)(a+b+c) \ .$$

From $1 = n[na + x + b] = n[nb + y + c] = n[nc + z + a]$, we find that

$$\begin{aligned} 3 &= n^2(a+b+c) + n(x+y+z) + n(a+b+c) \\ &= n(n+1)(a+b+c) + n(x+y+z) \\ &= \left[\frac{n+1}{n-2} + n\right](x+y+z) \\ &= \left[\frac{n^2-n+1}{n-2}\right](x+y+z) \ , \end{aligned}$$

whence

$$d = \frac{(n-2)^2}{n^2-n+1} \ .$$

Solution 4. Since the ratio of areas remains invariant under shear transformations and dilations, we may assume that the triangle is right isosceles. Assign coordinates: $A \sim (0,0)$, $B \sim (0,1)$, $C = (1,0)$, $M \sim (0,1/n)$,

$$N \sim \left(\frac{1}{n}, \frac{n-1}{n}\right)$$

$$Q \sim \left(\frac{n-1}{n}, 0\right) \ .$$

Then

$$R \sim \left(\frac{n-1}{n^2-n+1}, \frac{(n-1)^2}{n^2-n+1}\right)$$

$$S \sim \left(\frac{(n-1)^2}{n^2-n+1}, \frac{1}{n^2-n+1}\right)$$

and

$$T \sim \left(\frac{1}{n^2-n+1}, \frac{n-1}{n^2-n+1}\right) \ .$$

A computation of the area of ΔRST now yields the result.

Comment. Considering how reasonable the result it, H. Lee noted that when $n > 1$, then $(n-2)^2 < n^2 - n + 1$, so that $[TSR] < 1$ as expected, and also noted that when $n = 2$, we get the special case of the medians that intersect in a common point and yield $[TSR] = 0$.

111. (a) Are there four different numbers, not exceeding 10, for which the sum of any three is a prime number?

(b) Are there five different natural numbers such that the sum of every three of them is a prime number?

Solution. [R. Barrington Leigh] (a) Yes, there are four such numbers. The three-member sums of the set $\{1, 3, 7, 9\}$ are the primes 11, 13, 17, 19.

(b) No. We prove the statement by contradiction. Suppose that there are five different natural numbers for which every sum of three is prime. As the numbers are distinct and positive, each such sum must be at least $1+2+3 = 6$, and so cannot be a multiple of 3. Consider the five numbers, modulo 3. If there are three in the same congruent class, their sum is a multiple of 3. If there is one each congruent to 0, 1, 2 modulo 3, then the sum of these three is a multiple of 3. Otherwise, there are only two congruence classes represented with at most two numbers in each, an impossibility. Hence in all cases, there must be three who sum to a multiple of 3.

Comment. Part (b) need not be framed as a contradiction. One could formulate it as follows: Let five positive integers be given. Argue that either each congruent class modulo 3 is represented or that some class is represented by at least three of the numbers. Then note that therefore some three must sum to a multiple of 3. Observe that 3 itself is not a possible sum. Hence, among every five positive integers, there are three who add to a nonprime multiple of 3, and simply say that the answer to the question is "no".

112. Suppose that the measure of angle BAC in the triangle ABC is equal to α. A line passing through the vertex A is perpendicular to the angle bisector of $\angle BAC$ and intersects the line BC at the point M. Find the other two angles of the triangle ABC in terms of α, if it is known that $BM = BA + AC$.

Solution. Let q be the line through A perpendicular to the bisector of angle BAC; this line bisects the external angle at A. The possibility that q is parallel to BC is precluded by the condition that it intersects BC at M. Let $\angle ABC = \beta$ and $\angle ACB = \gamma$, Since p is not parallel to BC, β is not equal to γ.

Case i. Suppose that $\beta > \gamma$. Then M intersects BC so that B lies between M and C. Let BA be produced to D so that $AC = AD$. Since $MB = BA + AC = BA + AD = BD$, $\angle DMB = \angle MDB$ and so $\angle MDB = \frac{1}{2}\angle DBC = \frac{1}{2}\beta$ (exterior angle). Since $AD = AC$, $\angle ADC = \angle ACD = \frac{1}{2}\angle BAC = \frac{1}{2}\alpha$. Since MA produced bisects $\angle DAC$, MA produced right bisects DC and so $MD = MC$. Therefore

$$\gamma + \frac{\alpha}{2} = \angle MCD = \angle MDC = \frac{\beta}{2} + \frac{\alpha}{2} ,$$

whence $\beta = 2\gamma$. Therefore, $180° = \alpha + \beta + \gamma = \alpha + 3\gamma$ and

$$\gamma = \frac{180° - \alpha}{3} \qquad \text{and} \qquad \beta = \frac{360° - 2\alpha}{3} .$$

Case ii. Supppose that $\beta < \gamma$. Then M intersects BC so that C lies between B and M. Let BA be produced to D so that $AD = AC$. Since AM bisects $\angle DAC$, it right bisects CD and so triangle MDC is isosceles. Then $\angle ADM = \angle ADC + \angle MDC = \angle ACD + \angle MCD = \angle ACM$. Since $BM = BA + AC = BD$, $\angle ACM = \angle ADM = \angle BDM = \angle BMD$. Since $\angle ACM = \alpha + \beta$ (exterior angle), $180° = \angle DBM + 2\angle BMD = \beta + 2(\alpha+\beta)$, so that $\beta = \frac{1}{3}(180° - 2\alpha)$ and $\gamma = 180° - \angle ACM = 180° - (\alpha+\beta) = 120° - \frac{1}{3}\alpha = (360° - \alpha)/3$.

Question. Why cannot you just say the second case can be handled as the first case, by symmetry?

113. Find a function that satisfies all of the following conditions:

(a) f is defined for every positive integer n;

(b) f takes only positive values;

(c) $f(4) = 4$;

(d)

$$\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + \cdots + \frac{1}{f(n)f(n+1)} = \frac{f(n)}{f(n+1)} \ .$$

Solution. [R. Barrington Leigh] The function for which $f(n) = n$ for every positive integer n satisfies the condition. [*Exercise:* establish this by induction.] We now show that this is the only example. Substituting $n = 1$ into (d) and noting that $f(2) \neq 0$, we find that $f(1)^2 = 1$, whence $f(1) = 1$. Applying (d) to two consecutive values of the argument yields that

$$\frac{f(n)}{f(n+1)} - \frac{f(n-1)}{f(n)} = \frac{1}{f(n)f(n+1)} \ ,$$

whence

$$[f(n)]^2 - 1 = f(n-1)f(n+1) \ .$$

Substituting $n = 2$ and $n = 3$ into this and noting that $f(1) = 1$ and $f(4) = 4$, we find that

$$[f(2)]^2 - 1 = f(3)$$

and

$$[f(3)]^2 - 1 = 4f(2) \ ,$$

whence

$$0 = [f(2)]^4 - 2[f(2)]^2 - 4f(2) = f(2)[f(2) - 2]([f(2)]^2 + 2f(2) + 2) \ .$$

Since the first and third factors are positive for all postive possibilities for $f(2)$, we must have $f(2) = 2$. As

$$f(n+1) = \frac{[f(n)]^2 - 1}{f(n-1)} \ ,$$

we can prove by induction that $f(n) = n$ for all positive integers n.

114. A natural number is a multiple of 17. Its binary representation (*i.e.*, when written to base 2) contains exactly three digits equal to 1 and some zeros.

(a) Prove that there are at least six digits equal to 0 in its binary representation.

(b) Prove that, if there are exactly seven digits equal to 0 and three digits equal to 1, then the number must be even.

Solution 1. (a) If there are fewer than six digits equal to 0 in its binary representation, then the number must have at most eight digits and be of the form $2^a + 2^b + 2^c$ where $0 \leq a < b < c \leq 7$. The first eight powers of 2 with nonnegative exponent are congruent to $1, 2, 4, 8, -1, -2, -4, -8$ modulo 17, and the sum of any three of these cannot equal to zero and must lie between -14 and 14. Hence it is not possible for three powers of 2 among the first eight to sum to a multiple of 17. Hence, the number must have at least nine digits, including three zeros.

(b) Suppose that the number is equal to $2^a + 2^b + 2^c$ where $0 \leq a < b < c \leq 9$. If this number has exactly 10 digits and is odd, then $a = 0$ and $c = 9$, so that the number is equal to $1 + 2^b + 2^9 = 513 + 2^b \equiv 3 + 2^b \pmod{17}$. But there is no value of b that will make this vanish, modulo 17. Hence, a 10-digit number divisible by 17 must be even. An example is $2 \times 17^2 = 2 \times (1 + 2^5 + 2^8) = (1001000010)_2$.

Solution 2. [R. Furmaniak] (a) Since $17_{10} = 10001_2$, any binary number $abcd_2$ with four or fewer digits multiplied by 17 will yield $abcdabcd_2$. Since the first and last four digits are the same, there must be an even number of 1s. Thus, any multiple of 17 with exactly three binary digits must be a product of 17 and a number that has at least 5 binary digits. Every such product must have at least 9 digits, and so at least three digits equal to 0.

(b) As in Solution 1.

115. Let U be a set of n distinct real numbers and let V be the set of all sums of distinct pairs of them, *i.e.*,

$$V = \{x + y : x, y \in U, x \neq y\} .$$

What is the smallest possible number of distinct elements that V can contain?

Solution. Let $U = \{x_i : 1 \leq i \leq n\}$ and $x_1 < x_2 < \cdots < x_n$. Then

$$x_1 + x_2 < x_1 + x_3 < \cdots < x_1 + x_n < x_2 + x_n < \cdots < x_{n-1} + x_n$$

so that the $2n - 3$ sums $x_1 + x_j$ with $2 \leq j \leq n$ and $x_i + x_n$ with $2 \leq i \leq n - 1$ have distinct values. On the other hand, the set $\{1, 2, 3, 4, \cdots, n\}$ has the smallest pairwise sum $3 = 1 + 2$ and the largest pairwise sum $2n - 1 = (n - 1) + n$, so there are at most $2n - 3$ pairwise sums. Hence V can have as few as, but no fewer than, $2n - 3$ elements.

Comment. This problem was not well done. A set is assumed to be given, and so you must deal with it. Many of you tried to vary the elements in the set, and say that if we fiddle with them to get an arithmetic progression we minimize the number of sums. This is vague and intuitive, does not deal with the given set and needs to be sharpened. To avoid this, the best strategy is to take the given set and try to determine pairwise sums that are sure to be distinct, regardless of what the set is; this suggests that you should look at extreme elements - the largest and the smallest. To make the exposition straightforward, assume with no loss of generality that the elements are in increasing or decreasing order. You want to avoid a proliferation of possibilities.

Secondly, in setting out the proof, note that it should divide cleanly into two parts. First show that, whatever the set, at least $2n-3$ distinct sums occur. Then, by an example, demonstrate that exactly $2n-3$ sums are possible. A lot of solvers got into hot water by combining these two steps.

116. Prove that the equation

$$x^4 + 5x^3 + 6x^2 - 4x - 16 = 0$$

has exactly two real solutions.

Solution. In what follows, we denote the given polynomial by $p(x)$.

Solution 1.

$$p(x) = (x^2 + 3x + 4)(x^2 + 2x - 4) = \left(\left(x + \frac{3}{2}\right)^2 + \frac{7}{4}\right)\left((x + 1)^2 - 5\right) .$$

The first quadratic factor has nonreal roots, and the second two real roots, and the result follows.

Solution 2. Since $p(1) = p(-2) = -8$, the polynomial $p(x) + 8$ is divisible by $x + 2$ and $x - 1$. We find that

$$p(x) = (x + 2)^3(x - 1) - 8 .$$

When $x > 1$, the linear factors are strictly increasing, so $p(x)$ strictly increases from -8 unboundedly, and so vanishes exactly once in the interval $(1, \infty)$. When $x < -2$, the two linear factors are both negative and increasing, so that $p(x)$ strictly decreases from positive values to -8. Thus, it vanishes exactly once in the interval $(-\infty, -2)$. When $-2 < x < 1$, the two linear factors have opposite signs, so that $(x+2)^3(x-1) < 0$ and $p(x) < -8 < 0$. The result follows.

Solution 3. [R. Mong] We have that $p(x) = x^4 + 5x^3 + 6x^2 - 4x - 16 = (x-1)(x+2)^3 - 8$. Let $q(x) = f(-(x+2)) = (-x-3)(-x)^3 - 8 = x^4 + 3x^3 - 8$. By Descartes' Rule of Signs, $p(x)$ and $q(x)$ each have exactly one positive root. (The rule says that the number of positive roots of a real polynomial has the same parity as and no more than the number of sign changes in the coefficients as read in descending order.) It follows that $p(x)$ has exactly one root in each interval $(-\infty, -2)$ and $(0, \infty)$. Since $p(x) \leq -8$ for $-2 \leq x \leq 0$, the desired result follows.

Solution 4. Since the derivative $p'(x) = (x+2)^2(4x-1)$, we deduce that $p'(x) < 0$ for $x < \frac{1}{4}$ and $p'(x) > 0$ for $x > \frac{1}{4}$. It follows that $p(x)$ is strictly decreasing on $(\infty, \frac{1}{4})$ and strictly increasing on $(\frac{1}{4}, \infty)$. Since the leading coefficient is positive and $p(\frac{1}{4}) < 0$, $p(x)$ has exactly one root in each of the two intervals.

117. Let a be a real number. Solve the equation

$$(a-1)\left(\frac{1}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x \cos x}\right) = 2 \ .$$

Solution. First step. When $a = 1$, the equation is always false and there is no solution. Also, the left side is undefined when x is a multiple of $\pi/2$, so we exclude this possibility. Thus, in what follows, we suppose that $a \neq 1$ and that $\sin x \cos x \neq 0$. [*Comment.* This initial clearing away the underbrush avoids nuisance situations later and makes the exposition of the core of the solution go easier.]

Solution 1. Let $u = \sin x + \cos x$. Then $u^2 - 1 = 2 \sin x \cos x$, so that

$$(a-1)(u+1) = u^2 - 1 \Longleftrightarrow$$

$$0 = u^2 - (a-1)u - a = (u+1)(u-a) \ .$$

Since $\sin x \cos x \neq 0$, $u + 1 \neq 0$. Thus, $u = a$, and $\sin x$, $\cos x$ are the roots of the quadratic equation

$$t^2 - at + \frac{a^2-1}{2} = 0 \ .$$

Hence

$$(\sin x, \cos x) = \left(\frac{1}{2}(a \pm \sqrt{2-a^2}), \frac{1}{2}(a \mp \sqrt{2-a^2})\right) \ .$$

For this to be viable, we require that $|a| \leq \sqrt{2}$ and $a \neq 1$.

Solution 2. The given equation (since $\sin x \cos x \neq 0$) is equivalent to

$$(a-1)(\sin x + \cos x + 1) = 2 \sin x \cos x$$

$$\Longrightarrow (a-1)^2(2 + 2\sin x + 2\cos x + 2\sin x \cos x) = 4 \sin^2 x \cos^2 x$$

$$\Longleftrightarrow 4(a-1)(\sin x \cos x) + 2(a-1)^2(\sin x \cos x) = 4 \sin^2 x \cos^2 x$$

$$\Longleftrightarrow 2(a-1) + (a-1)^2 = \sin 2x$$

$$\Longleftrightarrow \sin 2x = a^2 - 1 \ .$$

For this to be viable, we require that $|a| \leq \sqrt{2}$.

For all values of x, we have that

$$2(1 + \sin x + \cos x) + 2 \sin x \cos x = (\sin x + \cos x + 1)^2 \ ,$$

so that

$$(\sin x + \cos x + 1 - 1)^2 = 1 + 2\sin x \cos x = a^2 \ ,$$

whence

$$\sin x + \cos x = \pm a \ .$$

Since we squared the given equation, we may have introduced extraneous roots, so we need to check the solution. Taking $\sin x + \cos x = a$, we find that

$$(a-1)(\sin x + \cos x + 1) = (a-1)(a+1) = a^2 - 1 = 2\sin x \cos x$$

as desired. Taking $\sin x + \cos x = -a$, we find that

$$(a-1)(\sin x + \cos x + 1) = (a-1)(1-a) = -(a-1)^2 \neq a^2 - 1 = 2\sin x \cos x$$

so this does not work. Hence the equation is solvable when $|a| \leq \sqrt{2}$, $a \neq 1$, and the solution is given by $x = \frac{1}{2}\theta$ where $\sin\theta = a^2 - 1$ and $\sin x + \cos x = a$.

Solution 3. We have that

$$\begin{aligned}(a-1)\left(\sqrt{2}\cos\left(x - \frac{\pi}{4}\right) + 1\right) &= (a-1)(\sin x + \cos x + 1) \\ &= 2\sin x \cos x = \sin 2x \\ &= \cos\left(\frac{\pi}{2} - 2x\right) \\ &= \cos 2\left(x - \frac{\pi}{4}\right) \\ &= 2\cos^2\left(x - \frac{\pi}{4}\right) - 1 \ .\end{aligned}$$

Let $t = \cos(x - \pi/4)$. Then

$$\begin{aligned}0 &= 2t^2 - \sqrt{2}(a-1)t - a \\ &= (\sqrt{2}t - a)(\sqrt{2}t + 1) \ .\end{aligned}$$

Since x cannot be a multiple of $\pi/2$, t cannot equal $1/\sqrt{2}$. Hence $\cos(x + \frac{\pi}{4}) = t = a/\sqrt{2}$, so that $x = \frac{\pi}{4} + \phi$ where $\cos\phi = a/\sqrt{2}$. Since the equation at the beginning of this solution is equivalent to the given equation and the quadratic in t, this solution is valid, subject to $|a| \leq \sqrt{2}$ and $a \neq 1$.

Solution 4. Note that $\sin x + \cos x = 1$ implies that $2\sin x \cos x = 0$. Since we are assuming that $\sin x + \cos x \neq 0$, we multiply the equation $(a-1)(\sin x + \cos x + 1) = 2\sin x \cos x$ by $\sin x + \cos x - 1$ to obtain the equivalent equation

$$(a-1)[(\sin x + \cos x)^2 - 1] = 2\sin x \cos x(\sin x + \cos x - 1)$$

$$\Longleftrightarrow (a-1)2\sin x \cos x = 2\sin x \cos x(\sin x + \cos x - 1)$$

$$\Longleftrightarrow \sin x + \cos x = a$$

$$\Longleftrightarrow \sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{a}}$$

$$\Longleftrightarrow x = \theta - \frac{\pi}{4}$$

where $\sin\theta = a/\sqrt{2}$. We have the same conditions on a as before.

Solution 5. The equation is equivalent to

$$(a-1)(\sin x+1)=\cos x(2\sin x+1-a)\ .$$

Squaring, we obtain

$$(a-1)^2(\sin x+1)^2=(1-\sin^2 x)[4\sin^2 x+4(1-a)\sin x+(1-a)^2]\ .$$

Dividing by $\sin x+1$ yields

$$2\sin^2 x-2a\sin x+(a^2-1)=0\Longrightarrow \sin x=\frac{1}{2}(a\pm\sqrt{2-a^2})\ .$$

[Note that the equation $\cos x=a-\sin x$ in Solution 3 leads to the equation here.] Thus

$$\sin^2 x=\frac{1\pm a\sqrt{2-a^2}}{2}\qquad\text{and}\qquad \cos^2 x=\frac{1\mp a\sqrt{2-a^2}}{2}\ .$$

Thus

$$(\sin x,\cos x)=\left(\frac{1}{2}(a\pm\sqrt{2-a^2}),\pm\frac{1}{2}(a\mp\sqrt{2-a^2})\right)\ .$$

Note that $0\le\sin^2 x\le 1$ requires $0\le 1\pm a\sqrt{2-a^2}\le 2$, or equivalently $|a|\le\sqrt{2}$ and $a^2(2-a^2)\le 1\Leftrightarrow |a|\le\sqrt{2}$ and $(a^2-1)^2\ge 0\Leftrightarrow |a|\le\sqrt{2}$.

We need to check for extraneous roots. If

$$(\sin x,\cos x)=((1/2)(a\pm\sqrt{2-a^2}),(1/2)(a\mp\sqrt{2-a^2})\ ,$$

then

$$(a-1)(\sin x+\cos x+1)=(a-1)(a+1)=a^2-1=(1/2)[a^2-(2-a^2)]=2\sin x\cos x$$

as desired. On the other hand, if

$$(\sin x,\cos x)=((1/2)(a\pm\sqrt{2-a^2}),-(1/2)(a\mp\sqrt{2-a^2})\ ,$$

then

$$(a-1)(\sin x+\cos x+1)=(a-1)(\sqrt{2-a^2}+1)$$

while

$$2\sin x\cos x=-(1/2)[a^2-(2-a^2)]=-(a^2-1)\ .$$

These are not equal when $a\neq 1$. Hence

$$(\sin x,\cos x)=((1/2)(a\pm\sqrt{2-a^2}),(1/2)(a\mp\sqrt{2-a^2})\ .$$

Solution 6. Let $u=\sin x+\cos x$, so that $u^2=1+2\sin x\cos x$. Then the equation is equivalent to $(a-1)(u+1)=u^2-1$, whence $u=1$ or $u=a$. We reject $u=1$, so that $u=a$ and we can finish as in Solution 3.

Solution 7. [O. Bormashenko] Since

$$\frac{1}{\sin x}+\frac{1}{\cos x}+\frac{1}{\sin x\cos x}=\frac{2}{a-1}\ ,$$

we have that

$$\left(\frac{1}{\sin x}+\frac{1}{\cos x}\right)^2=\left(\frac{2}{a-1}-\frac{1}{\sin x\cos x}\right)^2$$

$$\Leftrightarrow \frac{1}{\sin^2 x\cos^2 x} + \frac{2}{\sin x\cos x} = \frac{4}{(a-1)^2} - \frac{4}{(a-1)\sin x\cos x} + \frac{1}{\sin^2 x\cos^2 x}$$
$$\Leftrightarrow \frac{1}{\sin x\cos x}\left(2 + \frac{4}{a-1}\right) = \frac{4}{(a-1)^2}$$
$$\Leftrightarrow 2\sin x\cos x = a^2 - 1 \Leftrightarrow \sin 2x = a^2 - 1 \ .$$

We check this solution as in Solution 2.

Solution 8. [S. Patel] Let $z = \cos x + i\sin x$. Note that $z \neq 0, \pm 1, \pm i$. Then

$$\sin x = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2-1}{2iz}$$

and

$$\cos x = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z} \ .$$

The given equation is equivalent to

$$\begin{aligned}
1 &= (a-1)\left[\frac{iz}{z^2-1} + \frac{z}{z^2+1} + \frac{2iz^2}{z^4-1}\right] \\
&= (a-1)\left[\frac{iz}{z^2-1} + \frac{z}{z^2+1} + \frac{i}{z^2-1} + \frac{i}{z^2+1}\right] \\
&= (a-1)\left[\frac{i(z+1)}{z^2-1} + \frac{z+i}{z^2+1}\right] \\
&= (a-1)\left[\frac{i}{z-1} + \frac{1}{z-i}\right] \\
&= (a-1)\left[\frac{(i+1)z}{z^2-(1+i)z+i}\right] \ .
\end{aligned}$$

This is equivalent to

$$z^2 - (1+i)z + i = (a-1)[(1+i)z] \Leftrightarrow z^2 - (1+i)az + i = 0 \ .$$

Hence

$$\begin{aligned}
z &= \frac{(1+i)a \pm \sqrt{2ia^2 - 4i}}{2} \\
&= \frac{(1+i)a \pm \sqrt{2i}\sqrt{a^2-2}}{2} \\
&= \left(\frac{1+i}{2}\right)(a \pm \sqrt{a^2-2}) \ .
\end{aligned}$$

Suppose that $a^2 > 2$. Then $|z|^2 = \frac{1}{2}[a \pm \sqrt{a^2-2}]^2 = (a^2-1) \pm a\sqrt{a^2-2}$. Since $|z| = 1$, we must have $a^2 - 2 = \pm a\sqrt{a^2-2}$, whence $a^4 - 4a^2 + 4 = a^4 - 2a^2$ or $a^2 = 2$, which we do not have. Hence, we must have $a^2 \leq 2$, so that

$$z = \left(\frac{1+i}{2}\right)(a \pm \sqrt{a^2-2}) \ .$$

Therefore

$$\cos x = \Re e z = \frac{a \mp \sqrt{2-a^2}}{2}$$

and

$$\sin x = \Im m z = \frac{a \pm \sqrt{2-a^2}}{2} \ .$$

Comment. R. Barrington Leigh had an interesting approach for solutions with positive values of $\sin x$ and $\cos x$. Consider a right triangle with legs $\sin x$ and $\cos x$, inradius r, semiperimeter s and area Δ. Then

$$\frac{1}{r} = \frac{s}{\Delta} = \frac{1 + \sin x + \cos x}{\sin x \cos x} = \frac{2}{a-1}$$

so that $1 \leq a$. We need to determine right triangles whose inradius is $\frac{1}{2}(a-1)$. Using the formula $r = (s-c)\tan(C/2)$ with $c = 1$ and $C = 90°$, we have that

$$r = (s-1)\tan 45° = s - 1$$

whence

$$\frac{a-1}{2} = \frac{1}{2}(\sin x + \cos x - 1) \Leftrightarrow \sin x + \cos x = a \ .$$

118. Let a, b, c be nonnegative real numbers. Prove that

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc \ .$$

When does equality hold?

Solution 1. Observe that

$$3abc - [a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c)] = abc - (b + c - a)(c + a - b)(a + b - c) \ . \qquad (*)$$

Now

$$2a = (c + a - b) + (a + b - c)$$

with similar equations for b and c. These equations assure us that at most one of $b + c - a$, $c + a - b$ and $a + b - c$ can be negative. If exactly one of these three quantities is negative, than (*) is clearly nonnegative, and is equal to zero if and only if at least one of a, b and c vanishes, and the other two are equal. If all the three quantities are nonnegative, then an application of the arithmetic-geometric means inequality yields that

$$2a \geq 2\sqrt{(c + a - b)(a + b - c)}$$

with similar inequalities for b and c. It follows from this that $(*)$ is nonnegative and vanishes if and only if $a = b = c$, or one of a, b, c vanishes and the other two are equal.

Solution 2. Wolog, suppose that $a \leq b \leq c$. Then

$$\begin{aligned}
&3abc - [a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c)] \\
&\quad = a(bc - ab - ac + a^2) + b(ac - bc - ab + b^2) + c(ab - ac - bc + c^2) \\
&\quad = a(b - a)(c - a) - b(c - b)(b - a) - c(c - a)(c - b) \\
&\quad = a(b - a)(c - a) + (c - b)[ab - b^2 + c^2 - ca] \\
&\quad = a(b - a)(c - a) + (c - b)^2(c + b - a) \geq 0 \ .
\end{aligned}$$

Equality occurs if and only if $a = b = c$ or if $a = 0$ and $b = c$.

Solution 3. [S.-E. Lu] The inequality is equivalent to

$$(a + b - c)(b + c - a)(c + a - b) \leq abc \ .$$

At most one of the three factors on the left side can be negative. If one of them is negative, then the inequality is satisfied, and equality occurs if and only if both sides vanish (*i.e.*, one of the three variables vanishes and the others are equal).

Otherwise, we can square both sides to get the equivalent inequality:

$$[a^2 - (b-c)^2][b^2 - (a-c)^2][c^2 - (a-b)^2] \le a^2b^2c^2 \ .$$

Since $b \le a + c$ and $c \le a + b$, we find that $|b - c| \le a$, whence $(b-c)^2 \le a^2$. Thus, $a^2 - (b-c)^2 \le a^2$, with similar inequalties for the other two factors on the left. It follows that the inequality holds with equality when $a = b = c$ or one variable vanishes and the other two are equal.

Solution 4. [R. Mong] Let $\sum f(a,b,c)$ denote the cyclic sum $f(a,b,c) + f(b,c,a) + f(c,a,b)$. Suppose that $u = a^3 + b^3 + c^3 = \sum a^3$ and $v = \sum (b-c)a^2$. Then

$$u + v = \sum a^3 + (b-c)a^2 = \sum a^2(a+b-c)$$

and

$$u - v = \sum a^3 - (b-c)a^2 = \sum a^2(a-b+c) = \sum a^2(c+a-b) = \sum b^2(a+b-c) \ .$$

Then

$$\sum a^2(a+b-c) \sum b^2(a+b-c) = u^2 - v^2 \le u^2 \ .$$

By the Cauchy-Schwarz Inequality,

$$\sum ab(a+b-c) \le \sqrt{u^2 - v^2} \le a^3 + b^3 + c^3 \ .$$

(Note that the left side turns out to be positive; however, the result would hold anyway even if it were negative, since $a^3 + b^3 + c^3$ is nonnegative.)

Then

$$a^2b + ab^2 - abc + b^2c + bc^2 - abc + a^2c + ac^2 - abc \le a^3 + b^3 + c^3$$
$$\Longrightarrow a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \le 3abc$$

as desired.

Equality holds if and only if

$$a^2(a+b-c) : b^2(b+c-a) : c^2(c+a-b) = b^2(a+b-c) : c^2(b+c-a) : a^2(c+a-b) \ .$$

Suppose, if possible that $a + b = c$, say. Then $b + c - a = 2b$, $c + a - b = 2a$, so that $b^3 : c^2a = c^2b : a^3$ and $c^2 = ab$. This is possible if and only if $a = b = 0$. Otherwise, all terms in brackets are nonzero, and we find that $a^2 : b^2 : c^2 = b^2 : c^2 : a^2$ so that $a = b = c$.

Solution 5. [A. Chan] Wolog, let $a \le b \le c$, so that $b = a + x$ and $c = a + x + y$, where $x, y \ge 0$. The left side of the inequality is equal to

$$3a^3 + 3(2x+y)a^2 + (2x^2 + 2xy - y^2)a - (y^3 + 2xy^2)$$

and the right side is equal to

$$3a^3 + 3(2x+y)a^2 + (3x^2 + 3xy)a \ .$$

The right side minus the left side is equal to

$$(x^2 + xy + y^2)a + (y^3 + 2xy^2) \ .$$

Since each of a, x, y is nonnegative, this expression is nonnegative, and it vanishes if and only if each term vanishes. Hence, the desired inequality holds, with equality, if and only if $y = 0$ and either $a = 0$ or $x = 0$, if and only if either ($a = 0$ and $b = c$) or ($a = b = c$).

119. The medians of a triangle ABC intersect in G. Prove that

$$|AB|^2 + |BC|^2 + |CA|^2 = 3(|GA|^2 + |GB|^2 + |GC|^2) \ .$$

Solution 1. Let the respective lengths of BC, CA, AB, AG, BG and CG be a, b, c, u, v, w. If M is the midpoint of BC, then A, G, M are collinear with $AM = (3/2)AG$. Let $\theta = \angle AMB$. By the law of cosines, we have that

$$c^2 = \frac{9}{4}u^2 + \frac{1}{4}a^2 - \frac{3}{2}au\cos\theta$$

$$b^2 = \frac{9}{4}u^2 + \frac{1}{4}a^2 + \frac{3}{2}au\cos\theta$$

whence

$$b^2 + c^2 = \frac{9}{2}u^2 + \frac{1}{2}a^2 \ .$$

Combining this with two similar equations for the other vertices and opposite sides, we find that

$$2(a^2 + b^2 + c^2) = \frac{9}{2}(u^2 + v^2 + w^2) + \frac{1}{2}(a^2 + b^2 + c^2)$$

which simplifies to $a^2 + b^2 + c^2 = 3(u^2 + v^2 + w^2)$, as desired.

Solution 2. We have that

$$\overrightarrow{AG} = \frac{1}{3}(\overrightarrow{AB} + \overrightarrow{AC}) \ .$$

Hence

$$|\overrightarrow{AG}|^2 = \frac{1}{9}[|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 + 2(\overrightarrow{AB} \cdot \overrightarrow{AC})] \ .$$

Similarly

$$|\overrightarrow{BG}|^2 = \frac{1}{9}[|\overrightarrow{BA}|^2 + |\overrightarrow{BC}|^2 + 2(\overrightarrow{BA} \cdot \overrightarrow{BC})] \ ,$$

and

$$|\overrightarrow{CG}|^2 = \frac{1}{9}[|\overrightarrow{CA}|^2 + |\overrightarrow{CB}|^2 + 2(\overrightarrow{CA} \cdot \overrightarrow{CB})] \ .$$

Therefore

$$\begin{aligned}
&9[|\vec{A}G|^2 + |\overrightarrow{BG}|^2 + |\overrightarrow{CG}|^2] \\
&\quad = 2[|\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2] + \overrightarrow{AB} \cdot (\overrightarrow{AC} + \overrightarrow{CB}) + \overrightarrow{AC} \cdot (\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{BC} \cdot (\overrightarrow{BA} + \overrightarrow{AC}) \\
&\quad = 3[|\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2 \ ,
\end{aligned}$$

as desired.

120. Determine all pairs of nonnull vectors $\mathbf{x}$, $\mathbf{y}$ for which the following sequence $\{a_n : n = 1, 2, \cdots\}$ is (a) increasing, (b) decreasing, where

$$a_n = |\mathbf{x} - n\mathbf{y}| \ .$$

Solution 1. By the triangle inequality, we obtain that

$$|\mathbf{x} - n\mathbf{y}| + |\mathbf{x} - (n-2)\mathbf{y}| \geq 2|\mathbf{x} - (n-1)\mathbf{y}| \ ,$$

whence

$$|\mathbf{x} - n\mathbf{y}| - |\mathbf{x} - (n-1)\mathbf{y}| \geq |\mathbf{x} - (n-1)\mathbf{y}| - |\mathbf{x} - (n-2)\mathbf{y}| \ ,$$

for $n \geq 3$. This establishes that the sequence is never decreasing, and will increase if and only if

$$|\mathbf{x} - 2\mathbf{y}| \geq |\mathbf{x} - \mathbf{y}| \geq 0 \ .$$

This condition is equivalent to

$$\mathbf{x} \cdot \mathbf{x} - 4\mathbf{x} \cdot \mathbf{y} + 4\mathbf{y} \cdot \mathbf{y} \geq \mathbf{x} \cdot \mathbf{x} - 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y}$$

or $3|\mathbf{y}|^2 \geq 2\mathbf{x} \cdot \mathbf{y}$.

Solution 2.

$$\begin{aligned} a_n^2 &= |\mathbf{x}|^2 - 2n(\mathbf{x} \cdot \mathbf{y}) + n^2|\mathbf{y}|^2 \\ &= |\mathbf{y}|^2 \left[\left(n - \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{y}|^2} \right)^2 \right] + \left[|\mathbf{x}|^2 - \frac{(\mathbf{x} \cdot \mathbf{y})^2}{|\mathbf{y}|^4} \right] \ . \end{aligned}$$

This is a quadratic whose nonconstant part involves the form $(n - c)^2$. This is an increasing function of n, for positive integers n, if and only if $c \leq 3/2$. Hence, the sequence increases if and only if

$$\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{y}|^2} \leq \frac{3}{2} \ .$$

121. Let n be an integer exceeding 1. Let $a_1, a_2, \cdots, a_n$ be posive real numbers and $b_1, b_2, \cdots, b_n$ be arbitrary real numbers for which

$$\sum_{i \neq j} a_i b_j = 0 \ .$$

Prove that $\sum_{i \neq j} b_i b_j < 0$.

Solution 1. For the result to hold, we need to assume that at least one of the b_i is nonzero. The condition is that

$$(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \ .$$

Now

$$\begin{aligned} 2\sum_{i \neq j} b_i b_j &= (b_1 + b_2 + \cdots + b_n)^2 - (b_1^2 + b_2^2 + \cdots + b_n^2) \\ &= \frac{(a_1 b_1 + \cdots + a_n b_n)^2}{(a_1 + \cdots + a_n)^2} - (b_1^2 + \cdots + b_n^2) \\ &\leq \frac{(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)}{(a_1 + \cdots + a_n)^2} - (b_1^2 + \cdots + b_n^2) \\ &= \frac{(b_1^2 + \cdots + b_n^2)}{(a_1 + \cdots + a_n)^2} [(a_1^2 + \cdots + a_n^2) - (a_1 + \cdots + a_n)^2] < 0 \ , \end{aligned}$$

from which the desired result follows. The inequality is due to the Cauchy-Schwarz Inequality.

Solution 2. [R. Barrington Leigh] Suppose that not all the b_i vanish and that $b_1 \geq b_2 \geq \cdots \geq b_n$ (wolog). Since $\sum_{i \neq j} a_i b_j = 0$, not all the b_i have the same sign, and so $b_1 > 0 > b_n$. Wolog, we may assume that $B \equiv b_1 + b_2 + \cdots + b_n \geq 0$. (If $B < 0$, we can change the signs of all the b_i which alters neither the hypothesis nor the conclusion.) We have that

$$0 = a_1(b_1 - B) + a_2(b_2 - B) + \cdots + a_n(b_n - B) < (a_1 + a_2 + \cdots + a_n)(b_1 - B) \ ,$$

so that $b_1 > B$. Hence

$$2\sum_{i \neq j} b_i b_j = B^2 - \sum b_i^2 < B^2 - b_1^2 < 0 \ ,$$

as desired.

122. Determine all functions f from the real numbers to the real numbers that satisfy

$$f(f(x)+y) = f(x^2-y) + 4f(x)y$$

for any real numbers x, y.

Solution 1. Let $y = \frac{1}{2}(x^2 - f(x))$. Then

$$f\left(\frac{f(x)+x^2}{2}\right) = f\left(\frac{f(x)+x^2}{2}\right) + 2f(x)[x^2 - f(x)] ,$$

from which it follows that, for each x, either $f(x) = 0$ or $f(x) = x^2$. [*Note:* this does **not** imply yet that the same option holds for all x.] In particular, $f(0) = 0$, so that $f(y) = f(-y)$ for all y.

Suppose that $f(c) = 0$. Then, for each real y, $f(y) = f(c^2 - y)$, whence $f(c^2) = f(0) = 0$. Thus, for each real y, $f(y) = f(c^4 - y)$. Suppose that $f(y) \neq 0$. Then

$$f(y) = y^2 \Longrightarrow y^2 = (c^2-y)^2 = (c^4-y)^2 \Longrightarrow 0 = c^2(c^2-2y) = c^4(c^4-2y) .$$

If c were nonzero, then we would have $c^2/2 = y = c^4/2$, so $c = 1$ and $y = \frac{1}{2}$. But then $f(-y) = f(y) = f(1-y)$; substituting $y = -\frac{1}{2}$ yields $\frac{1}{4} = f(\frac{1}{2}) = f(\frac{3}{2})$, which is false. Hence $c = 0$. It follows that, either $f(x) \equiv 0$ (for all x) or else that $f(x) \equiv x^2$ (for all x). These solutions can (and should be) checked.

Solution 2. Let $y = x^2 - f(x)$. Then

$$f(x^2) = f(f(x) + x^2 - f(x)) = f(f(x)) + 4f(x)[x^2 - f(x)] .$$

Taking $y = 0$, we see that $f(f(x)) = f(x^2)$, so that $4f(x)[x^2 - f(x)] = 0$. Hence, for each x, either $f(x) = 0$ or $f(x) = x^2$.

Suppose, if possible, that there are two nonzero reals u and v for which $f(u) = 0$ and $f(v) = v^2$. Setting $(x, y) = (u, v)$ yields that $v^2 = f(u^2 - v)$. Since $v \neq 0$, we must have that

$$v^2 = f(u^2 - v) = u^4 - 2u^2v + v^2 \Rightarrow 0 = u^2(u^2 - 2v) \Rightarrow v = \frac{1}{2}u^2 .$$

This would mean that we could find only one such pair (u, v), which is false. Hence this case is not possible, so that, either $f(x) = 0$ for all x or else that $f(x) = x^2$ for all x.

Solution 3. From $(x, y) = (0, 0)$, we have that $f(f(0)) = f(0)$. From $(x, y) = (0, -f(0))$, we have that $f(0) = f(f(0)) - 4f(0)^2$, whence $f(0) = 0$. From $x = 0$, we have that $f(y) = f(-y)$ for all y. Finally, taking $y = x^2$ and $y = -f(x)$, we get

$$f(x^2 + f(x)) = f(0) + 4f(x)x^2 = f(x^2 + f(x)) - 4f(x)^2 + 4f(x)x^2 .$$

so that $0 = 4f(x)[x^2 - f(x)]$. We can finish as in the other solutions.

Solution 4. [R. Barrington Leigh] Taking $y = x - f(x)$ and then $y = x^2 - x$ yields that

$$\begin{aligned} f(x) &= f(f(x) + x - f(x)) = f(x^2 - x + f(x)) + 4f(x)(x - f(x)) \\ &= f(x^2 - (x^2 - x)) + 4f(x)(x^2 - x) + 4f(x)(x - f(x)) \\ &= f(x)[1 + 4x^2 - 4x + 4x - 4f(x)] = f(x) + 4f(x)[x^2 - f(x)] , \end{aligned}$$

so that for each x, either $f(x) = 0$ or $f(x) = x^2$. The solution can be completed as before.

123. Let a and b be the lengths of two opposite edges of a tetrahedron which are mutually perpendicular and distant d apart. Determine the volume of the tetrahedron.

Solution 1. Construct parallel planes distant d apart that contain the edges of lengths a and b. In the planes, congruent parallelograms can be constructed whose diagonals are of lengths a and b and right bisect each other, and each of which has an edge of the tetrahedron as a diagonal. Each parallelogram can be obtained from the other by a translation relating their centres, so the two parallelograms bound a prism with opposite faces distant d apart. The volume of this prism is $\frac{1}{2}abd$.

The prism is the disjoint union of the given tetrahedron and four tetrahedra, all of height d, two having as base a triangle with base a and height $\frac{1}{2}b$ and two having as base a triangle with base b and height $\frac{1}{2}a$. Each of these latter four tetrahedra have volume $\frac{1}{3}(\frac{1}{2}\cdot\frac{ab}{2})d = \frac{abd}{12}$. Hence, the volume of the given tetrahedron is

$$\frac{abd}{2} - 4\left(\frac{abd}{12}\right) = \frac{1}{6}(abd)\ .$$

Solution 2. Suppose that $ABCD$ is the tetrahedron with opposite edges AB of length a and CD of length b orthogonal and at distance d from each other.

Case (i). Suppose that AB and CD are oriented so that there are points E and F on AB and CD respectively for which EF is perpendicular to both AB and CD. Then $|EF| = d$ and $[ABF] = \frac{1}{2}ad$. The tetrahedron $ABCD$ is the union of the nonoverlapping tetrahedra $ABFC$ and $ABFD$, each with ΔABF as "base" and perpendicular height along CD. Hence the volume of $ABCD$ is equal to

$$\frac{1}{3}[ABF](|FC| + |FD|) = \frac{1}{3}\left(\frac{1}{2}ad\right)|CD| = \frac{1}{6}abd\ .$$

Case (ii). Suppose that E and F are on AB possibly produced and on CD produced, say, with EF perpendicular to AB and CD. Then we can argue in a way similar to that in Case (i) that the volume of $ABCD$ is equal to the volume of $ABFC$ less the volume of $ABFD$ to obtain the answer $(1/6)abd$.

Solution 3. [C. Lau; H. Lee] Let $ABCD$ be the given tetrahedron with $|BC| = a$ and $|AD| = b$. Suppose E lies on BC, possibly produced, with $AE \perp BC$. Then AD must lie in the plane containing AE and perpendicular to BC. Let F lie on AD produced with $EF \perp AD$. Note that $|EF| = d$. Let G be the foot of the perpendicular from D to AE produced. Then

$$[ADE] = \frac{1}{2}|AD||EF| = \frac{1}{2}bd = \frac{1}{2}|AE||GD|\ .$$

It follows that the volume of $ABCD$ is equal to

$$\frac{1}{3}[ABC]|GD| = \frac{1}{6}|AE||BC||GD| = \frac{1}{6}abd\ .$$

124. Prove that

$$\frac{(1^4+\frac{1}{4})(3^4+\frac{1}{4})(5^4+\frac{1}{4})\cdots(11^4+\frac{1}{4})}{(2^4+\frac{1}{4})(4^4+\frac{1}{4})(6^4+\frac{1}{4})\cdots(12^4+\frac{1}{4})} = \frac{1}{313}\ .$$

Solution. The left side can be written as

$$\prod\{4x^4+1 : x \ \text{ odd}, \quad 1 \le x \le 11\} \div \prod\{4x^4+1 : x \ \text{ even}, \quad 2 \le x \le 12\}\ .$$

Now

$$\begin{aligned}4x^4+1 &= 4x^4+4x^2+1-4x^2 = (2x^2+1)^2-(2x)^2\\ &= (2x^2-2x+1)(2x^2+2x+1) = [(x-1)^2+x^2][x^2+(x+1)^2]\ .\end{aligned}$$

From this, we see that the left side is equal to

$$\frac{[1^2(1^2+2^2)][(2^2+3^2)(3^2+4^2)]\cdots[(10^2+11^2)(11^2+12^2)]}{[(1^2+2^2)(2^2+3^2)][(3^2+4^2)(4^2+5^2)]\cdots[(11^2+12^2)(12^2+13^2)]} = \frac{1^2}{12^2+13^2} = \frac{1}{313} \ .$$

Comment. In searching for factors, note that any common divisor of $4n^4+1$ and $4(n+1)^4+1$ must divide the difference

$$[(n^4+1)^4 - n^4] = [(n+1)^2 - n^2][(n+1)^2 + n^2] = (2n+1)(2n^2+2n+1) \ ,$$

so that we can try either $2n+1$ (which does not work) or $2n^2+2n+1$ to find that $4n^4+1 = (2n^2+2n+1)(2n^2-2n+1)$ and

$$4(n+1)^4 + 1 = (2n^2+2n+1)(2n^2+6n+5) = (2n^2+2n+1)[2(n+2)^2 - 2(n+2)+1] \ .$$

125. Determine the set of complex numbers z which satisfy

$$\text{Im } (z^4) = (\text{Re } (z^2))^2 \ ,$$

and sketch this set in the complex plane. (*Note:* Im and Re refer respectively to the imaginary and real parts.)

Solution 1. Let $z = x + yi$ and $z^2 = u + vi$. Then $u = x^2 - y^2$, $v = 2xy$ and $z^4 = (u^2 - v^2) + 2uvi$. Im (z^4) = (Re $(z^2))^2$ implies that $2uv = u^2$. Thus, $u = 0$ or $u = 2v$. These reduce to $x^2 = y^2$ or $(x-2y)^2 = 5y^2$, so that the locus consists of the points z on the lines determined by the equations $y = x$, $y = -x$, $y = (\sqrt{5}-2)x$, $y = (-\sqrt{5}-2)x$.

Solution 2. Let $z = r(\cos\theta + i\sin\theta)$; then $z^2 = r^2(\cos 2\theta + i \sin 2\theta)$ and $z^4 = r^4(\cos 4\theta + i\sin 4\theta)$. The condition is equivalent to

$$r^4 \sin 4\theta = (r^2 \cos 2\theta)^2 \Leftrightarrow 2 \sin 2\theta \cos 2\theta = \cos^2 2\theta \ .$$

Hence $\cos 2\theta = 0$ or $\tan 2\theta = \frac{1}{2}$. The latter possibility leads to $\tan^2\theta + 4\tan\theta - 1 = 0$ or $\tan\theta = -2 \pm \sqrt{5}$. This yields the same result as in Solution 1.

Solution 3. Let $z = x + yi$. Then $z^2 = x^2 - y^2 + 2xyi$ and $z^4 = (x^4 - 6x^2y^2 + y^4) + 4xy(x^2-y^2)i$. Then the condition in the problem is equivalent to

$$4xy(x^2 - y^2) = (x^2 - y^2)^2 \ ,$$

which in turn is equivalent to $y = \pm x$ or $y^2 + 4xy - x^2 = 0$, *i.e.*, $y = (-2 \pm \sqrt{5})x$.

126. Let n be a positive integer exceeding 1, and let n circles (*i.e.*, circumferences) of radius 1 be given in the plane such that no two of them are tangent and the subset of the plane formed by the union of them is connected. Prove that the number of points that belong to at least two of these circles is at least n.

Solution. Let Γ be the set of circles and S be the set of points belonging to at least two of them. For $C \in \Gamma$ and $s \in S \cap C$, define $f(s, C) = 1/k$, where k is the number of circles passing through s, including C. For $C \in \Gamma$ and $s \notin C$, define $f(s, C) = 0$. Observe that, for each $s \in S$,

$$\sum_{C \in \Gamma} f(s, C) = 1 \ .$$

Let $C \in \Gamma$; select $s \in S \cap C$ for which $f(s, C) = 1/k$ is minimum. Let $C = C_1, C_2, \cdots, C_k$ be the circles that contain s. These circles (apart from C) meet C in distinct points, so that

$$\sum_{s \in S} f(s, C) \geq \frac{1}{k} + \frac{k-1}{k} = 1 \ .$$

Hence the number of points in S is equal to

$$\sum_{s \in S} \sum_{C \in \Gamma} f(s, C) = \sum_{C \in \Gamma} \sum_{s \in S} f(s, C) \geq n \ .$$

Comment. The full force of the connectedness condition is not needed. It is required only that each circle intersect with at least one other circle.