CHAPTER FOUR

INTERPOLATION AND REPRESENTATIONS

§1. INTRODUCTION

Polynomials are useful in approximating more general functions. Not only are polynomials numerous and easy to describe, but the family of polynomials has structure that can be exploited, that of a linear space over the underlying field or of a ring. The question of approximation raises a variety of issues, as one needs to decide, for example, whether the approximation should be pointwise or some kind of average approximation, and one is often faced with having to make a tradeoff between desirable features that may be incompatible.

The most basic approach is to specify the values of the function at a set of n points where the approximation is desired and then simply construct a polynomial to produce these values. There is a naive approach for this.

Suppose that we are given a collection of $2n$ complex numbers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ (with the a_i all distinct) and wish to determine a function f for which $f(a_i) = b_i$ for each index i. Such a function is said to *interpolate* the ordered pairs (a_i, b_i). There are n restrictions, so we want a function that can be described with n degrees of freedom. One such function is a polynomial of degree at most $n-1$ with n undetermined coefficients:

$$f(x) = c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1 x + c_0 \ .$$

Plugging in $x = a_i$ for the n values of i leads to a system of n linear equations in the unknowns c_{n-1}, c_{n-2}, $\cdots$, c_1, c_0 whose matrix of coefficients is

$$\begin{pmatrix} a_1^{n-1} & a_1^{n-2} & \cdots & a_1 & 1 \\ a_2^{n-1} & a_2^{n-2} & \cdots & a_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n^{n-1} & a_n^{n-2} & \cdots & a_n & 1 \end{pmatrix}$$

Its determinant is a polynomial of degree $(n-1) + (n-2) + \cdots + 2 + 1 = \binom{n}{2}$, with the $\binom{n}{2}$ factors $a_i - a_j$ for $1 \leq i < j \leq n$. Accordingly, the determinant of the coefficients is

$$\prod_{1 \leq i \leq n} (a_i - a_j) \ .$$

As this is nonzero, there is a unique determination of the coefficients c_i of the polynomial.

There are more sophisticated and insightful ways to approach this polynomial matching situation. However, even if we can match the values of an unknown function at a finite set of points, we have no control over the matching polynomial anywhere else and it may prove to be a poor match for the function elsewhere. We will take this up in the next chapter.

§2. THE LAGRANGE POLYNOMIAL

The fitting of a polynomial to a finite set of values can be effected by another route with two stages: (1) construct a polynomial f of degree less than n for which $f(a_i) = b_i$ $(1 \leq i \leq n)$; (2) show that the polynomial is unique.

The uniqueness is a straightforward consequence of the fundamental theorem of algebra. If f_1 and f_2 are two polynomials that suit, then $f_1 - f_2$ is a polynomial of degree less than n that is divisible by $\prod_i (x - a_i)$, and so must be the zero polyomial.

To develop a strategy for (1), we note that there is an element of linearity that can be exploited. Suppose that we have two vectors $(u_1, u_2, \cdots, u_n)$ and $(v_1, v_2, \cdots, v_n)$, and that we can obtain two polynomials g and h of degrees less than n for which $g(a_i) = u_i$ and $h(a_i) = v_i$ $(0 \leq i \leq n)$. Then, for any constant r and s we have that

$$(rg + sh)(a_i) = rg(a_i) + sh(a_i) = ru_i + sv_i$$

for each i. Thus, the mapping $(u_1, u_2, \cdots, u_n) \longrightarrow g$ of a set of values to be assumed at n points to the polynomial of degree less than n that delivers these values is a linear mapping on $\mathbf{R}^n$ or $\mathbf{C}^n$ as appropriate into the vector space of polynomials of degree less than n.

Thus, to find the polynomial f, it suffices to find the images of the basis vectors $(0, 0, \cdots, 1, \cdots, 0)$ under this linear mapping. Indeed, for each i, the polynomial defined by

$$f_i(x) = \frac{(x - a_1)\cdots\widehat{(x - a_i)}\cdots(x - a_n)}{(a_i - a_1)\cdots(a_i - a_{i-1})(a_i - a_{i+1})\cdots(a_i - a_n)}$$

satisfies $f_i(a_j) = \delta_{i,j}$, where, as usual, the hat denotes a deleted term and $\delta_{i.j}$ is the Kronecker delta that is 1 when $i = j$ and 0 otherwise. Then

$$f(x) = \sum_{i=1}^{n} b_i f_i(x)$$

is the desired polynomial.

We can look at the Lagrange polynomial in another way. Suppose that we are given a polynomial $f(x)$ of degree less than n whose values at n distinct points a_1, a_2, $\cdots$, a_n are known. There is only one such polynomial, so any expression that delivers these values of the required form is the polynomial. Let $g(x) = (x - a_1)(x - a_2)\cdots(x - a_n)$. Then

$$g'(x) = \sum_{i=1}^{n}(x - a_1)(x - a_2)\cdots\widehat{(x - a_i)}\cdots(x - a_n) \ ,$$

so that

$$f(x) = \sum_{i=1}^{n} \frac{f(a_i)}{g'(a_i)}(x - a_1)(x - a_2)\cdots\widehat{(x - a_i)}\cdots(x - a_n)) \ .$$

This can be regarded as an identity for any polynomials f and g, where the zeros of g are given and the degree of f is less than that of g. For example, we can take $n \geq 2$ and $f(x)$ the constant polynomial equal to 1. Then

$$1 = \sum_{i=1}^{n} \frac{(x - a_1)(x - a_2)\cdots\widehat{(x - a_i)}\cdots(x - a_n)}{g'(a_i)} \ .$$

Comparing the leading and constant coefficients of the two sides of this polynomial identity, we obtain this result:

Let $g(x)$ be a polynomial of degree at least 2 with n distinct zeros ζ_1, ζ_2, $\cdots$, ζ_n. Then

$$0 = \sum_{i=1}^{n} \frac{1}{g'(\zeta_i)} \ ,$$

and, when $\zeta_1\zeta_2\cdots\zeta_n \neq 0$,

$$1 = (-1)^{n-1}\zeta_1\zeta_2\cdots\zeta_n \sum_{i=1}^{n} \frac{1}{\zeta_i g'(\zeta_i)} \ .$$

§3. THE NEWTON POLYNOMIAL

The Lagrange polynomial treats the interpolation points symmetrically and is created "all at once" as it were. However, another possibility is to build up the desired polynomial in stages, at each stage keeping the degree as small as you can. The polynomial f of lowest degree for which $f(a_1) = b_1$ is the constant polynomial $f(x) \equiv b_1$.

We make an adjustment to achieve also $f(a_2) = b_2$, but without disturbing what has already been achieved. So, keeping the factor theorem in mind, we try

$$f(x) = b_1 + k_1(x - a_1)$$

where k_1 is to be determined by the condition that $f(a_2) = b_2$. This leads to a linear equation for k_1 and we find that

$$f(x) = b_1 + \frac{b_2 - b_1}{a_2 - a_1}(x - a_1)$$

does the job.

To satisfy the condition $f(a_3) = b_3$, we try

$$f(x) = b_1 + k_1(x - a_1) + k_2(x - a_1)(x - a_2)$$

and find that k_2 must assume the value

$$\begin{aligned} k_2 &= \frac{1}{(a_3 - a_1)(a_3 - a_2)}\left[(b_3 - b_1) - \frac{b_2 - b_1}{a_2 - a_1}(a_3 - a_1)\right] \\ &= \frac{(b_3 - b_1)(a_2 - a_1) - (b_2 - b_1)(a_3 - a_1)}{(a_3 - a_1)(a_3 - a_2)(a_2 - a_1)} \\ &= \frac{(b_3 - b_2)(a_2 - a_1) - (b_2 - b_1)(a_3 - a_2)}{(a_3 - a_1)(a_3 - a_2)(a_2 - a_1)} \\ &= \frac{1}{a_3 - a_1}\left[\frac{b_3 - b_2}{a_3 - a_2} - \frac{b_2 - b_1}{a_2 - a_1}\right] . \end{aligned}$$

We can continue on in this way, our task made easier by calculating *divided differences.* If $f(a_1) = b_1$ and $f(a_2) = b_2$, then the divided difference of f for these two points is given by

$$\Delta(a_1, a_2; b_1, b_2) = \frac{b_2 - b_1}{a_2 - a_1} .$$

For brevity, we may write this as $\Delta(a_1, a_2)$ when the values of b_1 and b_2 are understood. Observe that $\Delta(a_1, a_2) = \Delta(a_2, a_1)$.

The second order divided difference is defined with reference to three points in the domain of f:

$$\Delta^2(a_1, a_2, a_3; b_1, b_2, b_3) = \frac{1}{a_3 - a_1}[\Delta(a_2, a_3; b_2, b_3) - \Delta(a_1, a_2; b_1, b_2)] .$$

There are many interesting equivalences for divided differences. For example, we see that

$$\Delta^2(a_1, a_2, a_3) = \frac{1}{a_3 - a_2}[\Delta(a_1, a_3) - \Delta(a_1, a_2)] .$$

To check that this is useful, consider the polynomial:

$$f(x) = b_1 + \Delta(a_1, a_2)(x - a_1) + \Delta^2(a_1, a_2, a_3)(x - a_1)(x - a_2) .$$

We have already noted that $f(a_1) = b_1$ and $f(a_2) = b_2$. Also,

$$\begin{aligned}
f(a_3) &= b_1 + \Delta(a_1, a_2)(a_3 - a_1) + \Delta^2(a_1, a_2, a_3)(a_3 - a_1)(a_3 - a_2) \\
&= b_1 + (a_3 - a_1)\left[\Delta(a_1, a_2) + \frac{(\Delta(a_2, a_3) - \Delta(a_1, a_2))(a_3 - a_2)}{a_3 - a_1}\right] \\
&= b_1 + [\Delta(a_1, a_2)(a_3 - a_1) + \Delta(a_2, a_3)(a_3 - a_2) - \Delta(a_1, a_2)(a_3 - a_2)] \\
&= b_1 + [\Delta(a_1, a_2)(a_2 - a_1) + \Delta(a_2, a_3)(a_3 - a_2)] = b_1 + (b_2 - b_1) + (b_3 - b_2) = b_3 \ .
\end{aligned}$$

If we want to interpolate a fourth point (a_4, b_4), we can define

$$\Delta^3(a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4) = \frac{1}{a_4 - a_1}[\Delta^2(a_2, a_3, a_4; b_2, b_3, b_4) - \Delta^2(a_1, a_2, a_3; b_1, b_2, b_3)] \ .$$

Let

$$\begin{aligned}
f(x) = b_1 + \Delta(a_1, a_2)(x - a_1) + \Delta^2(a_1, a_2, a_3)(x - a_1)(x - a_2) \\
+ \Delta^3(a_1, a_2, a_3, a_4)(x - a_1)(x - a_2)(x - a_3) \ .
\end{aligned}$$

Then

$$\begin{aligned}
f(a_4) &= b_1 + \Delta(a_1, a_2)(a_4 - a_1) + \Delta^2(a_1, a_2, a_3)(a_4 - a_1)(a_4 - a_2) \\
&\quad + \Delta^3(a_1, a_2, a_3)(a_4 - a_1)(a_4 - a_2)(a_4 - a_3) \\
&= b_1 + \Delta(a_1, a_2)(a_4 - a_1) + (a_4 - a_1)(a_4 - a_2) \\
&\quad \left[\Delta^2(a_1, a_2, a_3) + \left(\frac{\Delta^2(a_2, a_3, a_4) - \Delta^2(a_1, a_2, a_3)}{a_4 - a_1}\right)(a_4 - a_3)\right] \\
&= b_1 + \Delta(a_1, a_2)(a_4 - a_1) + (a_4 - a_2)[\Delta^2(a_2, a_3, a_4)(a_4 - a_3) + \Delta^2(a_1, a_2, a_3)(a_3 - a_1)] \\
&= b_1 + \Delta(a_1, a_2)(a_4 - a_1) + (a_4 - a_2) \\
&\quad \left[\frac{\Delta(a_3, a_4) - \Delta(a_2, a_3)}{a_4 - a_2}(a_4 - a_3) + \Delta(a_2, a_3) - \Delta(a_1, a_2)\right] \\
&= b_1 + \Delta(a_1, a_2)(a_4 - a_1) + \Delta(a_3, a_4)(a_4 - a_3) - \Delta(a_2, a_3)(a_4 - a_3) \\
&\quad + \Delta(a_2, a_3)(a_4 - a_2) - \Delta(a_1, a_2)(a_4 - a_2) \\
&= b_1 + \Delta(a_1, a_2)(a_2 - a_1) + \Delta(a_2, a_3)(a_3 - a_2) + \Delta(a_3, a_4)(a_4 - a_3) \\
&= b_1 + (b_2 - b_1) + (b_3 - b_2) + (b_4 - b_3) = b_4 \ .
\end{aligned}$$

Thus, the function interpolates the four points (a_i, b_i) for $1 \leq i \leq 4$.

In an analogous way, we can define divided differences of higher order and extend the interpolating polynomial indefinitely to accommodate as many interpolation points as we require.

As you might expect, the limiting case of the Newton polynomial when the n points coalesce into one is the Taylor expansion. Consider the special case where we let the difference between adjacent points be h and let h tend to 0:

$$a_1 = a, a_2 = a + h, \cdots, a_n = a + (n-1)h, a_{n+1} = a + nh \ .$$

Let f be a polynomial of degree n and adopt the notation:

$$\Delta_n^r(f; h) \equiv \Delta^r(a_1, a_2, \cdots, a_r; f(a_1), f(a_2), \cdots, f(a_r))$$

for $1 \leq r \leq n + 1$.

We depend upon a generalization of the Mean Value Theorem, to wit that

$$\Delta_h^r(f; h) = \frac{1}{r!} f^{(r)}(a + \xi h)$$

where $1 < \xi < r$ and $1 \leq r \leq n$. When $r = 1$, this is the standard Mean Value Theorem. When $r = 2$, we obtain

$$\begin{aligned}\Delta_h^2(f;h) &= \frac{[f(a+2h)-f(a+h)]-[f(a+h)-f(a)]}{h}\cdot\frac{1}{2h} \\ &= \frac{g(a+h)-g(a)}{h}\cdot\frac{1}{2h} = g'(a+\xi_1 h)\cdot\frac{1}{2h} \\ &= \frac{f'(a+\xi_1 h+h)-f'(a+\xi_1 h)}{2h} = \frac{f''(a+\xi_1 h+\xi_2 h)}{2}\ ,\end{aligned}$$

where $g(x) = f(x+h) - f(x)$ and $0 < \xi_1, \xi_2 < 1$.

We can continue by induction. Assuming the result for $r = k$, we have that

$$\begin{aligned}\Delta_h^{k+1}(f;a) &= \frac{\Delta_h^k(f;a+h)-\Delta_h^k(f;a)}{(k+1)h} = \frac{\Delta_h^k(f';a+\xi_1 h)}{k+1} \\ &= \frac{1}{(k+1)k!}(f')^k(a+\xi_1 h+\xi_2 h)\ ,\end{aligned}$$

where $0 < \xi_1 < 1$, $0 < \xi_2 < k$, by the standard Mean Value Theorem and the induction hypothesis. ♣

From this,we see that

$$\lim_{h\to 0}\Delta_h^r(f;a) = \frac{1}{r!}f^{(r)}(a)$$

and the Newton polynomial representation tends to the Taylor representation.

§4. THE DIFFERENCE OPERATOR

A special instance of this process occurs when we want to interpolate function values at consecutive integers, say 0, 1, 2, 3, *etc.*. In this case, we define the ordinary (undivided) differences by

$$\Delta^0 f(n) \equiv If(n) = f(n)$$

$$\Delta^1 f(n) = \Delta f(n) = f(n+1) - f(n)$$

$$\Delta^2 f(n) = \Delta f(n+1) - \Delta f(n) = f(n+2) - 2f(n+1) + f(n)$$

and generally

$$\Delta^k f(n) = \Delta^{k-1} f(n+1) - \Delta^{k-1} f(n)$$

for $k \geq 2$. There is a useful operational calculus that we can bring into play. Define $Ef(n) = f(n+1)$. Then $E^k f(n) = f(n+k)$ and $\Delta = (E - I)$, so that $\Delta^k = (E - I)^k$. From the above, we see that

$$\Delta^2 f(n) = f(n+2) - 2f(n+1) + f(n) = E^2 f(n) - 2Ef(n) + If(n) = (E-I)^2 f(n)\ .$$

By means of this *operational calculus*, we have the formula

$$\Delta^k f(n) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}f(n+k-j)\ .$$

We can treat polynomials in I, E and Δ as satisfying the regular rules of algebra. In particular

$$f(n) = E^n f(0) = (I+\Delta)^n f(0) = \sum_{i=0}^{n}\binom{n}{i}\Delta^i f(0)\ ,$$

for each positive integer k, facts that can be verified directly by unpacking the operational notation.

The $\Delta-$ operator is analogous to differentiation. We can define the *factorial power* of x by

$$x^{(r)} = x(x-1)(x-2)\cdots(x-r+1)$$

for each integer $r \geq 1$, with $x^{(0)} = 1$. Then, it turns out that $\Delta x^{(r)} = rx^{(r-1)}$ for each nonnegative integer r. Thus, f is a polynomial of degree n if and only if $\Delta^n f(x)$ is a nonzero constant and $\Delta^{n+1} f(x)$ is identically 0. This yields the identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x+k) = 0$$

for all x whenever $f(x)$ is a polynomial of degree strictly less than n.

For a polynomial,

$$f(x) = (I+\Delta)^x f(0) = \sum_{j=0}^{\infty} \binom{x}{j} \Delta^j f(0) \ ,$$

where $\binom{x}{j} = x(x-1)\cdots(x-j+1)/j! = x^{(j)}/j!$, and the series terminates after a finite number of terms. When the differences at some point are known, this gives us another way of evaluating a polynomial. For a nonpolynomial function defined at all of the positive integers, the series is infinite, and like an ordinary power series may or may not converge.

If a polynomial of degree n takes integer values at $n+1$ consecutive integers, it takes integers values at every integer and is a linear combination of polynomials of the form $\binom{x}{k}$. Thus, every polynomial taking integer values of $\mathbf{Z}$ has the form $\sum_{k=0}^{n} a_k \binom{x}{k}$.

We can consider infinite series involving factorial powers. For example, the Binomial Series for $(1+t)^x$,

$$(1+t)^x = \sum_{k=0}^{\infty} \frac{t^k}{k!} x^{(k)}$$

is such a factorial series for each numerical value of t. As for regular series, factorial series have a domain of convergence. They converge for complex values of x whose real parts exceed a certain value α and coverge absolutely when the real parts exceed a value β with $0 \leq \beta - \alpha \leq 1$. Another novelty of factorial series is that the representation of a function through them is not unique; for example the identity function 0 is given by

$$0 = (1-1)^x = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{(r)}$$

when the real part of x exceeds 0.

More generally, we have the binomial expansion

$$(1+z)^x = \sum_{n=0}^{\infty} \binom{x}{n} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!} x^{(n)} \ .$$

When $|z| = 1$, but $z \neq -1$, the series on the right diverges for $x \leq -1$, converges conditionally for $-1 < x < 0$ and converges absolutely for $x \geq 0$. When $z = -1$, it diverges for $x < 0$ and converges for $x \geq 0$. ([**1**], p. 486)

§5. HERMITE INTERPOLATION

Returning to the interpolation question, we can generalize Lagrange interpolation to Hermite interpolation, where we look for the polynomial of minimum degree that interpolates values of polynomials and certain derivatives at each point. Suppose that n is a positive integer and that $k \leq n$. Let $a_1, a_2, \cdots a_k$ be complex numbers and suppose that m_i $(1 \leq i \leq k)$ are positive integers for which $m_1+m_2+\cdots+m_k = n+1$. The problem is to determine a polynomial of degree n for which

$$p^{(j)}(a_i) = b_{i.j}$$

for $1 \leq i \leq k$, $0 \leq j \leq m_i - 1$, where the $b_{i,j}$ are assigned values and $p^{(j)}(a)$ denotes the jth derivative of $p(x)$ evaluated at $x = a$.

This can be established by induction. We already know the result for $n = 0$ and $n = 1$. Suppose that it is known for all values of the degree up to $n - 1$. Let the problem be given as stated above. Suppose that $q^{(j)}(a_i) = b_{i,j}$ for $1 \leq i \leq k-1$ and relevant values of j, and that the degree of q does not exceed $m_1 + m_2 + \cdots + m_{k-1} - 1$. Let

$$r(x) = (x - a_1)^{m_1}(x - a_2)^{m_2} \cdots (x - a_{k-1})^{m_{k-1}} \ .$$

Observe that the deg $r(x) = m_1+m_2+\cdots+m_{k-1}$ and that $r^{(j)}(a_i) = 0$ for $1 \leq i \leq k-1$ and $0 \leq j \leq m_{i-1}$. Define $p(x) = q(x) + s(x)r(x)$, where $s(x)$ is a polynomial to be determined. The condition $p(a_k) = b_{k,1}$ allows us to determine the values of $s(a_k)$. By differentiating and looking at the derivatives of $p(x)$ in turn, we can find the desired values of $s^{(j)}(a_k)$ for $0 \leq j \leq m_k - 1$. There exists a polynomial $s(x)$ of degree $m_k - 1$ that fills the bill. The polynomial $p(x)$ satisfies the desired conditions. ♠.

Example. Suppose we desire a polynomial $p(x)$ of degree 5 that satisfies the conditions $p(0) = 5$, $p(1) = 3$, $p'(1) = 0$, $p''(1) = 6$, $p(2) = 3$, $p'(2) = 1$. The conditions for $x = 0$ and $x = 1$ are satisfied by the polynomial $q(x) = x^3 - 3x + 5$ which is of minimum degree. Let

$$p(x) = (x^3 - 3x + 5) + s(x)x(x - 1)^3$$

so that

$$p'(x) = (3x^2 - 3) + s(x)[(x - 1)^3 + 3x(x - 1)^2] + s'(x)[x(x - 1)^3] \ .$$

We need to have $s(2) = -2$ and $s'(2) = 3$, so we take $s(x) = 3x - 8$. Hence, the desired polynomial is

$$p(x) = (x^3 - 3x + 5) + (3x - 8)x(x^3 - 3x^2 + 3x - 1) = 3x^5 - 17x^4 + 34x^3 - 27x^2 + 5x + 5 \ .$$

Note that in Hermite interpolation, where we have specified the value of a derivative at a point, we have also specified the values of all derivatives of lower order. This raises the question as to when, given $n + 1$ pieces of information about the value of a function and its derivatives at various points, we can interpolate it with a polynomial of degree n.

For example, there is no polynomial p of degree 1 for which $p'(0) = 0$ and $p'(1) = 1$, and no polynomial q of degree 2 for which $q(0) = q(2) = 0$ and $q'(1) = 1$. The best we can do in the first case is $p(x) = \frac{1}{2}x^2 + c$, and in the second,

$$q(x) = (c - x)x(x - 2) = -x^3 + (c + 2)x^2 - 2cx \ ,$$

where c is an arbitrary constant. If we generalize the specification of q to $q(0) = q(2) = 0$ and $q'(1) = \lambda$, then a polynomial of minimum degree that fits these data is

$$q(x) = (c - \lambda x)x(x - 2) \ .$$

More generally, it seems that if we have assigned $n + 1$ pieces of information, then we can match a polynomial of degree d with c interdeterminate coefficients, where $d - c = n$. But this is clear. Suppose that we are given $n + 1$ equations of the type $p^{(j)}(a_i) = b_{ij}$ to be satisfied by a polynomial. For each i, let $m_i - 1$ be the order of the highest derivative specified at a_i and suppose that d_i lower derivatives (possibly including the zeroth) are unspecified. We can assign these parametric values, and then match a polynomial of degree $(\sum m_i) - 1$ to the data. We have $n+1 = \sum(m_i - d_i)$ pieces of information and $\sum d_i$ parameters, so the difference between the degree of the polynomial and n is

$$[(\sum m_i) - 1] - [\sum(m_i - d_i) - 1] = \sum d_i \ .$$

§6. PROBLEMS AND INVESTIGATIONS

1. Let n be a positive integer and let $p(x)$ be that polynomial of degree $n-1$ for which $p(k) = 2^k$ for $1 \leq k \leq n$. Determine $p(0)$ and $p(n+1)$.

2. Let m and n be positive integers, and let $p(x)$ be that polynomial of degree $n-1$ for which $p(k) = k^{-m}$ for $1 \leq k \leq n$. Determine $p(n+1)$.

3. Let $f(t) = t^2 + t + 1$. Determine a polynomial $g(t)$ of degree less than 5 for which $f(t)g(t) \equiv 1$ (mod 5) for every integer t.

Hints and comments.

1. Consider the expansion of $(1+t)^{x-1}$.

2. Consider $q(x) = x^m p(x) - 1$ and its values at $0, 1, \cdots, n$.

References

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