A sequence of squares

I know a middle school teacher who has his students memorize the squares of numbers up to 1000. (A square of a number is that number multiplied by itself; for example, the square of 24, written 24^2, is $24 \times 24 = 576$.) This is not just an idle task for at least two reasons. Knowing the squares can be a great help in mental arithmetic (a topic I may take up in a later column). Squares of numbers are involved in a lot of interesting mathematics, much of which is accessible to the young and can be used to liven up exercises.

For example, let us begin with 16, the square of 4, and build up a sequence of numbers by inserting into the middle of each entry the digit pair 15 to get the next entry: 16, 1156, 111556, 11115556, 1111155556, and so on. These turn out to be the squares of 4, 34, 334, 3334, 33334. Does the pattern continue?

You can formulate this question in terms of algebra (and I would invite any reader whose algebra is not too rusty to give it a shot), but there are more natural ways to understand what is going on. Find the products 34×34, 334×334, 3334×3334 using long multiplication; pay attention to how the intermediate numbers interact; then you will see what happens in general.

Another way to get a handle on the pattern is to note that $4 = 12 \div 3$, $34 = 102 \div 3$, $334 = 1002 \div 3$, $3334 = 10002 \div 3$ and so on. The square of 3334, for example, is equal to the square of 10002 divided by 9. The square of the numbers 102, 1002, 10002 are easy to calculate: 10404, 1004004, 100040004. Now see what happens when you divide these squares by 9.

There is another similar sequence of squares, this time obtained by starting with 49 and continually inserting 48: 49, 4489, 444889, 44448889, and so on. These are the squares of 7, 67, 667, 6667 and so on. By squaring the latter numbers using long multiplication, you can once again see what is going on. You can also use the fact that these numbers are $21 \div 3$, $201 \div 3$, $2001 \div 3$, $20001 \div 3$. Again, you can square the numerators and then divide them by 9.

The two sequences are related. Notice that $7 - 4 = 3$ while $7^2 - 4^2 = 33$; $67 - 34 = 33$ while $67^2 - 34^2 = 3333$; $667 - 334 = 333$ while $667^2 - 334^2 = 333333$. Again, we can study the way the numbers work to understand why this pattern emerges. There is a helpful rule for finding the difference of two squares described by the equation:

$$a^2 - b^2 = (a + b) \times (a - b).$$

This simply says that the difference between the squares of two given numbers is given by the product of their sum and their difference. Thus:

$$667^2 - 334^2 = (667 + 334) \times (667 - 334) = (1001) \times (333) = 333,333.$$

Check this out with other pairs of squares in the sequences.